

The Multi-Interval-Valued Fuzzy Soft Set with Application in Decision Making

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Abstract

In 1999, Molodtsov introduced the concept of soft set theory as a general mathematical tool for dealing with uncertainty. By combining the multi-fuzzy set and soft set models, Y. Yang, X. Tan and C. Meng introduced the concept of multi-fuzzy soft sets and studied some of its operations, such as complement, “AND”, “OR”, Union and Intersection. They also gave an algorithm to analyze a decision problem using multi-fuzzy soft set. In this paper, we introduce the concept of multi-interval-valued fuzzy soft set (M-IVFSS). We also define its basic operations, namely complement, union, intersection, AND and OR. Finally, we give an application of this concept in decision-making problem.

Keywords

Soft Set, Fuzzy Soft Set, Multi-Fuzzy Soft Set, Multi-Interval-Valued Fuzzy Soft Set

1. Introduction

Most of the problems in engineering, medical science, economics, environments etc. have various uncertainties. Molodtsov [1] initiated the concept of soft set theory as a mathematical tool for dealing with uncertainties. Chen *et al.* [2] and Maji *et al.* in [3] and [4] studied some different operations and application of soft sets. Furthermore, Maji *et al.* [5] presented the definition of fuzzy soft set and Roy *et al.* [6] presented the applications of this notion to decision making problems. By using fuzzy sets, Ahmad and Kharal in 2009 [7] studied this theory and defined arbitrary fuzzy soft union and fuzzy soft intersection and proved DeMorgan’s Inclusions and DeMorgan’s Laws in fuzzy soft set theory. In 2010, Feng *et al.* [8] gave deeper insights into decision making based on fuzzy soft sets. They discussed the validity of the Roy-Maji method and showed its true limitations. By means of level soft sets, they presented an adjustable approach to fuzzy soft set based decision making and gave some illustrative examples. Moreover, the weighted fuzzy soft set is introduced and its application to decision making is also

investigated.

The concept of soft fuzzy set and some properties of soft fuzzy set are discussed in 2008 by Yao *et al.* [9] and the relations of fuzzy soft sets and soft fuzzy sets are compared by instances. Kharal and Ahmad in 2009 [10] defined the concept of a mapping on classes of fuzzy soft sets and studied the properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets, and supported them with examples and counterexamples.

Chaudhuri and K. De in 2009 [11] defined the concepts of soft relation and fuzzy soft relation and applied them to solve a number of decision making problems. The advantages of fuzzy soft relation compared to other paradigms are discussed. Jiang *et al.* [12] presented in 2011 an extended fuzzy soft set theory by using the concepts of fuzzy description logics to act as the parameters of fuzzy soft sets. They also defined some operations for the extended fuzzy soft sets. Moreover, they proved that certain DeMorgan's laws hold in the extended fuzzy soft set theory with respect to these operations. In 2010, Majumdar and Samanta [13] defined generalised fuzzy soft sets and studied some of their properties. They also gave applications of generalised fuzzy soft sets in decision making problem and medical diagnosis problem.

Also in 2010 Xiao *et al.* [14] proposed a combined forecasting approach based on fuzzy soft sets (CFFSS) by using an export dataset of Chongqing Municipality China from 1993 to 2006 and compares a CFFSS with the combined forecasting approach based on the rough sets theory (CFRS). This approach constructed the fuzzy membership function and the tabular form of the fuzzy soft sets model. Ça ğ man *et al.* in 2011 [15] introduced fuzzy parameterized (*FP*)-soft sets and their related properties and proposed a decision making method based on *FP*-soft set theory. Ça ğ man *et al.* in 2010 [16] defined fuzzy parameterized fuzzy soft sets (*fpfs*-sets) and their operations. They then presented the decision making method on the *fpfs*-set theory and provided an example that demonstrated that this method can work successfully. It can be applied to problems on many fields that contain uncertainty.

Alkhazaleh *et al.* [17] generalized the concept of fuzzy soft set to possibility fuzzy soft set and they gave some applications of this concept in decision making and medical diagnosis. They also introduced the concept of fuzzy parameterized interval-valued fuzzy soft set [18], where the mapping is defined from the fuzzy set parameters to the interval-valued fuzzy subsets of the universal set, and gave an application of this concept in decision making. Alkhazaleh and Salleh [19] introduced the concept of soft expert sets where the user can know the opinion of all experts in one model and also gave an application of this concept in decision-making problem. Alkhazaleh and Salleh [20] generalized the concept of a soft expert set to fuzzy soft expert set, which is a more effective and useful. They also defined its basic operations, namely complement, union, intersection, AND and OR, and gave an application of this concept in decision-making problem. They also studied a mapping on fuzzy soft expert classes and its properties. As a generalization of Molodtsov's soft set, Alkhazaleh *et al.* [21] presented the definition of a soft multiset and its basic operations such as complement, union and intersection. In 2012 Alkhazaleh and Salleh [22] introduced the concept of fuzzy soft multiset as a combination of soft multiset and fuzzy set and studied its properties and operations. They presented the applications of this concept to decision-making problems. In 2012 Salleh *et al.* [23] introduced the notion of multiparameterized soft set and studied its properties. Yang *et al.* [24] presented the concept of interval-valued fuzzy soft set by combining the interval-valued fuzzy set [25] [26] and soft set models. In 2011 Sebastian and Ramakrishnan [27] proposed the concept of the multi-fuzzy set which is a more general fuzzy set using ordinary fuzzy sets as building blocks, its membership function is an ordered sequence of ordinary fuzzy membership functions. The notion of multi-fuzzy sets provides a new method to represent some problems which are difficult to explain in other extensions of fuzzy set theory, such as the color of pixels. Yang *et al.* [28] in 2012 introduced the concept of multi-fuzzy soft set which is a combination of multi-fuzzy set and soft set and studied and its basic operations such as complement, union and intersection. They also introduced the application of this concept in decision making. For more information on soft set and fuzzy soft set, see Abdul Razak Salleh 2011 [29]. In this paper, we introduce the concept of multi-interval-valued fuzzy soft set (M-IVFSS). We also define its basic operations, namely complement, union, intersection, AND and OR. Finally, we give an application of this concept in decision-making problem.

2. Preliminaries

In this section we recall some definitions and properties required in this paper.

Definition 1 [26] An interval-valued fuzzy set \tilde{X} on a universe U is a mapping such that

$$\tilde{X} : U \rightarrow \text{Int}([0,1]),$$

where $Int([0,1])$ stands for the set of all closed subintervals of $[0,1]$, the set of all interval-valued fuzzy sets on U is denoted by $\tilde{P}(U)$.

Suppose that $\tilde{X} \in \tilde{P}(U), \forall x \in U, \mu_x(x) = [\mu_x^-(x), \mu_x^+(x)]$ is called the degree of membership of an element x to X . $\mu_x^-(x)$ and $\mu_x^+(x)$ are referred to as the lower and upper degrees of membership of x to X where $0 \leq \mu_x^-(x) \leq \mu_x^+(x) \leq 1$.

Definition 2 [25] The subset, complement, intersection and union of the interval-valued fuzzy sets are defined as follows: Let $\tilde{X}, \tilde{Y} \in \tilde{P}(U)$ then

1) The complement of \tilde{X} is denoted by \tilde{X}^c where

$$\mu_{\tilde{X}^c}(x) = 1 - \mu_{\tilde{X}}(x) = [1 - \mu_{\tilde{X}}^+(x), 1 - \mu_{\tilde{X}}^-(x)];$$

2) The intersection of \tilde{X} and \tilde{Y} is denoted by $\tilde{X} \cap \tilde{Y}$ where

$$\mu_{\tilde{X} \cap \tilde{Y}}(x) = \inf[\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(x)] = [\inf(\mu_{\tilde{X}}^-(x), \mu_{\tilde{Y}}^-(x)), \inf(\mu_{\tilde{X}}^+(x), \mu_{\tilde{Y}}^+(x))];$$

3) The union of \tilde{X} and \tilde{Y} is denoted by $\tilde{X} \cup \tilde{Y}$ where

$$\mu_{\tilde{X} \cup \tilde{Y}}(x) = \sup[\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(x)] = [\sup(\mu_{\tilde{X}}^-(x), \mu_{\tilde{Y}}^-(x)), \sup(\mu_{\tilde{X}}^+(x), \mu_{\tilde{Y}}^+(x))];$$

4) X is a subset of Y denoted by $X \subseteq Y$ if $\mu_x^-(x) \leq \mu_y^-(x)$ and $\mu_x^+(x) \leq \mu_y^+(x)$.

Molodtsov defined soft set in the following way. Let U be a universe and E be a set of parameters. Let $P(U)$ denote the power set of U and $A \subseteq E$.

Definition 3 [1] A pair (F, A) is called a soft set over U , where F is a mapping

$$F : A \rightarrow P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U . For $\varepsilon \in A, F(\varepsilon)$ may be considered as the set of ε -approximate elements of the soft set (F, A) .

Definition 4 [5] Let U be an initial universal set and let E be a set of parameters. Let I^U denote the power set of all fuzzy subsets of U . Let $A \subseteq E$. A pair (F, E) is called a fuzzy soft set over U where F is a mapping given by

$$F : A \rightarrow I^U.$$

Definition 5 [24] Let U be an initial universe and E be a set of parameters. $\tilde{P}(U)$ denotes the set of all interval-valued fuzzy sets of U . Let $A \subseteq E$. A pair (\tilde{F}, A) is an interval-valued fuzzy soft set over U , where \tilde{F} is a mapping given by $\tilde{F} : A \rightarrow \tilde{P}(U)$.

Definition 6 [27] Let k be a positive integer, a multi-fuzzy set \tilde{A} in U is a set of ordered sequences

$$\tilde{A} = \left\{ \frac{u}{(\mu_1(u), \mu_2(u), \dots, \mu_i(u), \dots, \mu_k(u))} : u \in U \right\}$$

where $\mu_i(u) \in \tilde{P}(U), i = 1, 2, \dots, k$. The function $\mu_{\tilde{A}} = \{\mu_1, \mu_2, \dots, \mu_k\}$ is called the multi-membership function of multi-fuzzy set \tilde{A} ; k is called the dimension of \tilde{A} . The set of all multi-fuzzy sets of dimension k in U is denoted by $M^k FS(U)$.

Remark 7 [27] Clearly, a multi-fuzzy set of dimension 1 is a Zadeh fuzzy set, and a multi-fuzzy set of dimension 2 with $\mu_1(u) + \mu_2(u) \leq 1$ is an Atanassov intuitionistic fuzzy set.

Remark 8 [27] If $\sum_{i=1}^k \mu_i(u) \leq 1, \forall u \in U$, then the multi-fuzzy set of dimension k is called a normalized multi-fuzzy set. If $\sum_{i=1}^k \mu_i(u) = l > 1$, for some $u \in U$, we redefine the multi-membership degree

$(\mu_1(u), \mu_2(u), \dots, \mu_k(u))$ as $\frac{1}{l}(\mu_1(u), \mu_2(u), \dots, \mu_k(u))$ then the non-normalized multi-fuzzy set can be changed into a normalized multi-fuzzy set.

Definition 9 [27] Let $\tilde{A} \in M^k FS(U)$. If $\tilde{A} = \left\{ \frac{u}{(0,0,\dots,0)} : u \in U \right\}$, then \tilde{A} is called the null multi-fuzzy set of dimension k , denoted by $\tilde{\phi}_k$. If $\tilde{A} = \left\{ \frac{u}{(1,1,\dots,1)} : u \in U \right\}$, then \tilde{A} is called the absolute multi-fuzzy set of dimension k , denoted by $\tilde{1}_k$.

Definition 10 [27] Let $\tilde{A} = \left\{ \frac{u}{(\mu_1(u), \mu_2(u), \dots, \mu_k(u))} : u \in U \right\}$ and

$\tilde{B} = \left\{ \frac{u}{(v_1(u), v_2(u), \dots, v_k(u))} : u \in U \right\}$ be two multi-fuzzy sets of dimension k in U . We define the following relations and operations:

ing relations and operations:

- 1) $\tilde{A} \sqsubseteq \tilde{B}$ iff $\mu_i(u) \leq v_i(u), \forall u \in U$ and $1 \geq i \leq k$.
- 2) $\tilde{A} = \tilde{B}$ iff $\mu_i(u) = v_i(u), \forall u \in U$ and $1 \geq i \leq k$.
- 3) $\tilde{A} \sqcup \tilde{B} = \left\{ \frac{u}{(\mu_1(u) \vee v_1(u), \mu_2(u) \vee v_2(u), \dots, \mu_k(u) \vee v_k(u))} : u \in U \right\}$
- 4) $\tilde{A} \sqcap \tilde{B} = \left\{ \frac{u}{(\mu_1(u) \wedge v_1(u), \mu_2(u) \wedge v_2(u), \dots, \mu_k(u) \wedge v_k(u))} : u \in U \right\}$
- 5) $\tilde{A}^c = \left\{ \frac{u}{(\mu_1^c(u), \mu_2^c(u), \dots, \mu_k^c(u))} : u \in U \right\}$

Definition 11 [28] A pair (\tilde{F}, A) is called a multi-fuzzy soft set of dimension k over U , where \tilde{F} is a mapping given by

$$\tilde{F} : A \rightarrow M^k FS(U).$$

A multi-fuzzy soft set is a mapping from parameters to $M^k FS(U)$. It is a parameterized family of multi-fuzzy subsets of U . For $e \in A$; $\tilde{F}(e)$ may be considered as the set of e -approximate elements of the multi-fuzzy soft set (\tilde{F}, A) .

Example 12 [28] Suppose that $U = \{c_1; c_2; c_3; c_4; c_5\}$ is the set of color cloths under consideration, $A = \{e_1; e_2; e_3\}$ is the set of parameters, where e_1 stands for the parameter? color? which consists of red, green and blue, e_2 stands for the parameter? ingredient? which is made from wool, cotton and acrylic, and e_3 stands for the parameter? price? which can be various: high, medium and low. We define a multi-fuzzy soft set of dimension 3 as follows:

$$\begin{aligned} \tilde{F}(e_1) &= \left\{ \frac{c_1}{(0.4;0.2;0.3)}; \frac{c_2}{(0.2;0.1;0.6)}; \frac{c_3}{(0.1;0.3;0.4)}; \frac{c_4}{(0.3;0.1;0.3)}; \frac{c_5}{(0.7;0.1;0.2)} \right\}; \\ \tilde{F}(e_2) &= \left\{ \frac{c_1}{(0.1;0.2;0.6)}; \frac{c_2}{(0.3;0.2;0.4)}; \frac{c_3}{(0.5;0.3;0.1)}; \frac{c_4}{(0.6;0.1;0.3)}; \frac{c_5}{(0.6;0.2;0.1)} \right\}; \\ \tilde{F}(e_3) &= \left\{ \frac{c_1}{(0.3;0.2;0.1)}; \frac{c_2}{(0.4;0.1;0.2)}; \frac{c_3}{(0.2;0.2;0.5)}; \frac{c_4}{(0.7;0.1;0.2)}; \frac{c_5}{(0.5;0.2;0.3)} \right\}. \end{aligned}$$

Definition 13 [28] Let $A; B \in E$. Let (\tilde{F}, A) and (\tilde{G}, B) be two multi-fuzzy soft sets of dimension k over U . (\tilde{F}, A) is said to be a multi-fuzzy soft subset of (\tilde{G}, B) if

- 1) $A \subseteq B$;
 - 2) $\forall e \in A, \tilde{F}(e) \sqsubseteq \tilde{G}(e)$.
- In this case, We write $(\tilde{F}, A) \sqsubseteq (\tilde{G}, B)$.

Definition 14 [28] The complement of a multi-fuzzy soft set (\tilde{F}, A) of dimension k over U is denoted by $(\tilde{F}, A)^c$ and is defined by

$$(\tilde{F}, A)^c = (\tilde{F}^c, A)$$

where $\tilde{F}^c : A \rightarrow M^k FS(U)$ is a mapping given by $\tilde{F}^c(e) = (\tilde{F}(e))^c, \forall e \in A$.

Definition 15 [28] If (\tilde{F}, A) and (\tilde{G}, B) are two multi-fuzzy soft sets of dimension k over U the “ (\tilde{F}, A) AND (\tilde{G}, B) ”, denoted by $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B)$, where $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta), \forall (\alpha, \beta) \in A \times B$.

Definition 16 [28] If (\tilde{F}, A) and (\tilde{G}, B) are two multi-fuzzy soft sets of dimension k over U the “ (\tilde{F}, A) OR (\tilde{G}, B) ”, denoted by $(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{H}, A \times B)$, where $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \sqcup \tilde{G}(\beta), \forall (\alpha, \beta) \in A \times B$.

Definition 17 [28] Union of two multi-fuzzy soft sets (\tilde{F}, A) and (\tilde{G}, B) of dimension k over U , is the multi-fuzzy soft set (\tilde{H}, C) where $C = A \cup B$, and $\forall \varepsilon \in C$,

$$\tilde{H}(\varepsilon) = \begin{cases} \tilde{F}(\varepsilon), & \text{if } \varepsilon \in A - B \\ \tilde{G}(\varepsilon), & \text{if } \varepsilon \in B - A \\ \tilde{F}(\varepsilon) \sqcup \tilde{G}(\varepsilon), & \text{if } \varepsilon \in A \cap B \end{cases}$$

Definition 18 [28] Intersection of two multi-fuzzy soft sets (\tilde{F}, A) and (\tilde{G}, B) of dimension k over U , is the multi-fuzzy soft set (\tilde{H}, C) where $C = A \cup B$, and $\forall \varepsilon \in C$,

$$\tilde{H}(\varepsilon) = \begin{cases} \tilde{F}(\varepsilon), & \text{if } \varepsilon \in A - B \\ \tilde{G}(\varepsilon), & \text{if } \varepsilon \in B - A \\ \tilde{F}(\varepsilon) \cap \tilde{G}(\varepsilon), & \text{if } \varepsilon \in A \cap B \end{cases}$$

3. Multi-Interval-Valued-Fuzzy Soft Sets

In this section we introduce the concept of multi-interval-valued-fuzzy soft sets as generalisation of definition given by [28]. We also give basic properties of this concept.

Before we define the concept of multi-interval-valued-fuzzy soft sets, we define the concept of multi-interval-valued-fuzzy sets as follows:

Definition 19 Let k be a positive integer, a multi-interval-valued fuzzy set \tilde{A} in U is a set of ordered sequences

$$\tilde{A} = \left\{ \frac{u}{(\mu_1(u), \mu_2(u), \dots, \mu_i(u), \dots, \mu_k(u))} : u \in U \right\}$$

where $\mu_i(u) \in \widetilde{Int}(U), i = 1, 2, \dots, k$. The function $\mu_{\tilde{A}} = \{\mu_1, \mu_2, \dots, \mu_k\}$ is called the multi-membership function of multi-interval-valued-fuzzy set \tilde{A} ; k is called the dimension of \tilde{A} . The set of all multi-interval-valued-fuzzy sets of dimension k in U is denoted by $M^k IVS(U)$.

Definition 20 A pair (\tilde{F}, A) is called a multi-interval-valued-fuzzy soft set of dimension k over U , where \tilde{F} is a mapping given by

$$\tilde{F} : A \rightarrow M^k Int(U).$$

A multi-interval-valued-fuzzy soft set is a mapping from parameters to $M^k Int(U)$. It is a parameterized family of multi-interval-valued-fuzzy subsets of U . For $e \in A; \tilde{F}(e)$ may be considered as the set of e -approximate elements of the multi-interval-valued fuzzy soft set (\tilde{F}, A) .

Example 21 Suppose that $U = \{c_1; c_2; c_3\}$ is the set of color cloths under consideration, $A = \{e_1; e_2; e_3\}$ is the set of parameters, where e_1 stands for the parameter “color” which consists of red, green and blue, e_2 stands for the parameter “ingredient” which is made from wool, cotton and acrylic, and e_3 stands for the parameter “price” which can be various: high, medium and low. We define a multi-interval-valued-fuzzy soft set of dimension 3 as follows:

$$\begin{aligned} \tilde{F}(e_1) &= \left\{ \frac{c_1}{([0.4, 0.6]; [0.1, 0.4]; [0.4, 0.5])}; \frac{c_2}{([0.5, 0.8]; [0.4, 0.6]; [0, 0.2])}; \frac{c_3}{([0.1, 0.3]; [0.2, 0.5]; [0.5, 0.6])} \right\}; \\ \tilde{F}(e_2) &= \left\{ \frac{c_1}{([0.1, 0.3]; [0.3, 0.4]; [0.5, 0.6])}; \frac{c_2}{([0.2, 0.5]; [0.1, 0.4]; [0.3, 0.6])}; \frac{c_3}{([0.3, 0.5]; [0.1, 0.2]; [0, 0.1])} \right\}; \\ \tilde{F}(e_3) &= \left\{ \frac{c_1}{([0.6, 0.9]; [0.4, 0.6]; [0.7, 0.8])}; \frac{c_2}{([0.3, 0.5]; [0.1, 0.5]; [0, 0.2])}; \frac{c_3}{([0.3, 0.7]; [0.2, 0.5]; [0.4, 0.7])} \right\}. \end{aligned}$$

Definition 22 Let $A, B \in E$. Let (\tilde{F}, A) and (\tilde{G}, B) be two multi-interval-valued-fuzzy soft sets of dimension k over U . (\tilde{F}, A) is said to be a multi-interval-valued-fuzzy soft subset of (\tilde{G}, B) if

- 1) $A \subseteq B$;
- 2) $\forall e \in A, \tilde{F}(e) \sqsubseteq \tilde{G}(e)$

In this case, We write $(\tilde{F}, A) \sqsubseteq (\tilde{G}, B)$.

Example 23 Consider Example 21 where $A = \{e_1, e_2\}$ and $B = \{e_1, e_2, e_3\}$. Clearly, $A \subset B$. Suppose (\tilde{F}, A) and (\tilde{G}, B) are two multi-interval-valued-fuzzy soft sets of dimension 3 defined as follows

$$\begin{aligned} \tilde{F}(e_1) &= \left\{ \frac{c_1}{([0.4, 0.6]; [0.1, 0.4]; [0.4, 0.5])}; \frac{c_2}{([0.5, 0.8]; [0.4, 0.6]; [0, 0.2])}; \frac{c_3}{([0.1, 0.3]; [0.2, 0.5]; [0.5, 0.6])} \right\}; \\ \tilde{F}(e_2) &= \left\{ \frac{c_1}{([0.1, 0.3]; [0.3, 0.4]; [0.5, 0.6])}; \frac{c_2}{([0.2, 0.5]; [0.1, 0.4]; [0.3, 0.6])}; \frac{c_3}{([0.3, 0.5]; [0.1, 0.2]; [0, 0.1])} \right\}; \end{aligned}$$

and

$$\begin{aligned} \tilde{G}(e_1) &= \left\{ \frac{c_1}{([0.5, 0.7]; [0.3, 0.4]; [0.5, 0.8])}; \frac{c_2}{([0.6, 0.9]; [0.5, 0.9]; [0.2, 0.4])}; \frac{c_3}{([0.4, 0.5]; [0.3, 0.7]; [0.6, 0.8])} \right\}; \\ \tilde{G}(e_2) &= \left\{ \frac{c_1}{([0.3, 0.5]; [0.6, 0.7]; [0.5, 0.7])}; \frac{c_2}{([0.3, 0.6]; [0.2, 0.5]; [0.4, 0.6])}; \frac{c_3}{([0.3, 0.5]; [0.3, 0.4]; [0, 0.1])} \right\}; \\ \tilde{G}(e_3) &= \left\{ \frac{c_1}{([0.6, 1]; [0.5, 0.6]; [0.7, 0.9])}; \frac{c_2}{([0.4, 0.6]; [0.3, 0.5]; [0.1, 0.3])}; \frac{c_3}{([0.5, 0.7]; [0.4, 0.6]; [0.5, 0.8])} \right\}. \end{aligned}$$

It is clear that (\tilde{F}, A) is multi-interval-valued-fuzzy soft subset of (\tilde{G}, B) .

Definition 24 The complement of a multi-interval-valued-fuzzy soft set (\tilde{F}, A) of dimension k over U is denoted by $(\tilde{F}, A)^c$ and is defined by

$$(\tilde{F}, A)^c = (\tilde{F}^c, A)$$

where $\tilde{F}^c : A \rightarrow M^k \text{Int}(U)$ is a mapping given by $\tilde{F}^c(e) = (\tilde{F}(e))^c, \forall e \in A$.

Example 25 Consider Example 21 where

$$\begin{aligned} \tilde{F}(e_1) &= \left\{ \frac{c_1}{([0.4, 0.6]; [0.1, 0.4]; [0.4, 0.5])}; \frac{c_2}{([0.5, 0.8]; [0.4, 0.6]; [0, 0.2])}; \frac{c_3}{([0.1, 0.3]; [0.2, 0.5]; [0.5, 0.6])} \right\}; \\ \tilde{F}(e_2) &= \left\{ \frac{c_1}{([0.1, 0.3]; [0.3, 0.4]; [0.5, 0.6])}; \frac{c_2}{([0.2, 0.5]; [0.1, 0.4]; [0.3, 0.6])}; \frac{c_3}{([0.3, 0.5]; [0.1, 0.2]; [0, 0.1])} \right\}; \end{aligned}$$

$$\tilde{F}(e_3) = \left\{ \frac{c_1}{\left([0.6, 0.9]; [0.4, 0.6]; [0.7, 0.8] \right)}; \frac{c_2}{\left([0.3, 0.5]; [0.1, 0.5]; [0, 0.2] \right)}; \frac{c_3}{\left([0.3, 0.7]; [0.2, 0.5]; [0.4, 0.7] \right)} \right\}.$$

By using interval-valued fuzzy complement for $\tilde{F}(e)$ we have

$$\tilde{F}^c(e_1) = \left\{ \frac{c_1}{\left([0.4, 0.6]; [0.6, 0.9]; [0.5, 0.6] \right)}; \frac{c_2}{\left([0.2, 0.5]; [0.4, 0.6]; [0.8, 1] \right)}; \frac{c_3}{\left([0.7, 0.9]; [0.5, 0.8]; [0.4, 0.5] \right)} \right\};$$

$$\tilde{F}^c(e_2) = \left\{ \frac{c_1}{\left([0.7, 0.9]; [0.6, 0.7]; [0.4, 0.5] \right)}; \frac{c_2}{\left([0.5, 0.8]; [0.6, 0.9]; [0.4, 0.7] \right)}; \frac{c_3}{\left([0.5, 0.7]; [0.8, 0.9]; [0.9, 1] \right)} \right\};$$

$$\tilde{F}^c(e_3) = \left\{ \frac{c_1}{\left([0.1, 0.4]; [0.4, 0.6]; [0.2, 0.3] \right)}; \frac{c_2}{\left([0.5, 0.7]; [0.5, 0.9]; [0.8, 1] \right)}; \frac{c_3}{\left([0.3, 0.7]; [0.5, 0.8]; [0.3, 0.6] \right)} \right\}.$$

Proposition 26 Let (\tilde{F}, A) be a MIVFSS of dimension k over U . Then the following holds:

$$\left(\tilde{F}^c(e)\right)^c = \tilde{F}(e), \forall e \in A.$$

Proof.

Let $\tilde{F}^c(e) = \tilde{G}(e)$ then

$$\left(\tilde{F}^c(e)\right)^c = \tilde{G}^c(e)$$

But from Definition 24 $\tilde{G}(e) = (\tilde{F}^c(e))$ then

$$\tilde{G}(e)^c = \left(\tilde{F}^c(e)\right)^c = \tilde{F}(e)$$

Definition 27 Union of two multi-interval-valued-fuzzy soft sets (\tilde{F}, A) and (\tilde{G}, B) of dimension k over U , is the multi-interval-valued-fuzzy soft set (\tilde{H}, C) where $C = A \cup B$, and $\forall \varepsilon \in C$,

$$\tilde{H}(\varepsilon) = \begin{cases} \tilde{F}(\varepsilon), & \text{if } \varepsilon \in A - B \\ \tilde{G}(\varepsilon), & \text{if } \varepsilon \in B - A \\ \tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon), & \text{if } \varepsilon \in A \cap B \end{cases}$$

Example 28 Consider Example 21 where $A = \{e_1; e_2; e_3\}$ and $B = \{e_1; e_2; e_3\}$. Suppose (\tilde{F}, A) and (\tilde{G}, B) are two multi-interval-valued-fuzzy soft sets of dimension 3 defined as follows

$$\tilde{F}(e_1) = \left\{ \frac{c_1}{\left([0.4, 0.6]; [0.1, 0.4]; [0.4, 0.5] \right)}; \frac{c_2}{\left([0.5, 0.8]; [0.4, 0.6]; [0, 0.2] \right)}; \frac{c_3}{\left([0.1, 0.3]; [0.2, 0.5]; [0.5, 0.6] \right)} \right\};$$

$$\tilde{F}(e_2) = \left\{ \frac{c_1}{\left([0.1, 0.3]; [0.3, 0.4]; [0.5, 0.6] \right)}; \frac{c_2}{\left([0.2, 0.5]; [0.1, 0.4]; [0.3, 0.6] \right)}; \frac{c_3}{\left([0.3, 0.5]; [0.1, 0.2]; [0, 0.1] \right)} \right\};$$

$$\tilde{F}(e_3) = \left\{ \frac{c_1}{\left([0.3, 0.5]; [0.7, 0.8]; [0.2, 0.4] \right)}; \frac{c_2}{\left([0.3, 0.4]; [0.2, 0.5]; [0.3, 0.6] \right)}; \frac{c_3}{\left([0.4, 0.5]; [0.4, 0.6]; [0.5, 0.8] \right)} \right\};$$

and

$$\tilde{G}(e_1) = \left\{ \frac{c_1}{\left([0.2, 0.4]; [0.1, 0.3]; [0.1, 0.5] \right)}; \frac{c_2}{\left([0.4, 0.5]; [0.5, 0.9]; [0.2, 0.4] \right)}; \frac{c_3}{\left([0.4, 0.5]; [0.3, 0.7]; [0.6, 0.8] \right)} \right\};$$

$$\tilde{G}(e_2) = \left\{ \frac{c_1}{\left([0.3, 0.5]; [0.6, 0.7]; [0.5, 0.7] \right)}; \frac{c_2}{\left([0.3, 0.6]; [0.2, 0.5]; [0.5, 0.7] \right)}; \frac{c_3}{\left([0.4, 0.7]; [0.1, 0.3]; [0, 0.1] \right)} \right\};$$

$$\tilde{G}(e_3) = \left\{ \frac{c_1}{\left([0.4, 0.6]; [0.6, 0.8]; [0.8, 0.9] \right)}, \frac{c_2}{\left([0.7, 0.9]; [0.3, 0.5]; [0.3, 0.5] \right)}, \frac{c_3}{\left([0.2, 0.4]; [0.4, 0.6]; [0.3, 0.5] \right)} \right\};$$

By using the interval-valued fuzzy union we have $(\tilde{F}, A) \cup (\tilde{G}, B) = (\tilde{H}, C)$, where

$$\tilde{H}(e_1) = \left\{ \frac{c_1}{\left([0.4, 0.6]; [0.1, 0.4]; [0.4, 0.5] \right)}, \frac{c_2}{\left([0.5, 0.8]; [0.5, 0.9]; [0.2, 0.4] \right)}, \frac{c_3}{\left([0.4, 0.5]; [0.3, 0.7]; [0.6, 0.8] \right)} \right\};$$

$$\tilde{H}(e_2) = \left\{ \frac{c_1}{\left([0.3, 0.5]; [0.6, 0.7]; [0.5, 0.7] \right)}, \frac{c_2}{\left([0.3, 0.6]; [0.2, 0.5]; [0.5, 0.7] \right)}, \frac{c_3}{\left([0.4, 0.7]; [0.1, 0.3]; [0, 0.1] \right)} \right\};$$

$$\tilde{H}(e_3) = \left\{ \frac{c_1}{\left([0.4, 0.6]; [0.7, 0.8]; [0.8, 0.9] \right)}, \frac{c_2}{\left([0.7, 0.9]; [0.3, 0.5]; [0.3, 0.6] \right)}, \frac{c_3}{\left([0.4, 0.5]; [0.4, 0.7]; [0.5, 0.9] \right)} \right\}.$$

Proposition 29 Let $(\tilde{F}, A), (\tilde{G}, A)$ and (\tilde{H}, A) be any three MIVFSSs of dimension k over U . Then the following results hold:

- 1) $(\tilde{F}, A) \cup (\tilde{G}, A) = (\tilde{G}, A) \cup (\tilde{F}, A)$.
- 2) $(\tilde{F}, A) \cup ((\tilde{G}, A) \cup (\tilde{H}, A)) = ((\tilde{F}, A) \cup (\tilde{G}, A)) \cup (\tilde{H}, A)$.
- 3) $(\tilde{F}, A) \cup (\tilde{F}, A) = (\tilde{F}, A)$.

Proof.

$$1) (\tilde{F}, A) \cup (\tilde{G}, A) = (\tilde{R}, A)$$

From Definition 27 and by consider the case when $\varepsilon \in B \cap C$ as the other cases are trivial, we have $\tilde{R}(\varepsilon) = \tilde{F}(\varepsilon) \cup \tilde{G}(\varepsilon) = \tilde{G}(\varepsilon) \cup \tilde{F}(\varepsilon)$ (since the union of interval-valued fuzzy sets is commutative).

2) The proof is straightforward from Definition 27.

3) The proof is straightforward from Definition 27.

Definition 30 Intersection of two multi-interval-valued-fuzzy soft sets (\tilde{F}, A) and (\tilde{G}, B) of dimension k over U , is the multi-interval-valued-fuzzy soft set (\tilde{H}, C) where $C = A \cup B$, and $\forall \varepsilon \in C$,

$$\tilde{H}(\varepsilon) = \begin{cases} \tilde{F}(\varepsilon), & \text{if } \varepsilon \in A - B \\ \tilde{G}(\varepsilon), & \text{if } \varepsilon \in B - A \\ \tilde{F}(\varepsilon) \cap \tilde{G}(\varepsilon), & \text{if } \varepsilon \in A \cap B \end{cases}$$

Example 31 Consider Example 28. By using the interval-valued fuzzy intersection we have

$$(\tilde{F}, A) \cap (\tilde{G}, B) = (\tilde{H}, C), \text{ where}$$

$$\tilde{H}(e_1) = \left\{ \frac{c_1}{\left([0.2, 0.4]; [0.1, 0.3]; [0.1, 0.5] \right)}, \frac{c_2}{\left([0.4, 0.5]; [0.4, 0.6]; [0, 0.2] \right)}, \frac{c_3}{\left([0.1, 0.3]; [0.2, 0.5]; [0.5, 0.6] \right)} \right\};$$

$$\tilde{H}(e_2) = \left\{ \frac{c_1}{\left([0.1, 0.3]; [0.3, 0.4]; [0.5, 0.6] \right)}, \frac{c_2}{\left([0.2, 0.5]; [0.1, 0.4]; [0.3, 0.6] \right)}, \frac{c_3}{\left([0.3, 0.5]; [0.1, 0.2]; [0, 0.1] \right)} \right\};$$

$$\tilde{H}(e_3) = \left\{ \frac{c_1}{\left([0.3, 0.5]; [0.6, 0.8]; [0.2, 0.4] \right)}, \frac{c_2}{\left([0.3, 0.4]; [0.2, 0.5]; [0.3, 0.5] \right)}, \frac{c_3}{\left([0.2, 0.4]; [0.4, 0.6]; [0.3, 0.5] \right)} \right\}.$$

Proposition 32 Let $(\tilde{F}, A), (\tilde{G}, A)$ and (\tilde{H}, A) be any three MIVFSSs of dimension k over U . Then the following results hold:

- 1) $(\tilde{F}, A) \cap (\tilde{G}, A) = (\tilde{G}, A) \cap (\tilde{F}, A)$.
- 2) $(\tilde{F}, A) \cap ((\tilde{G}, A) \cap (\tilde{H}, A)) = ((\tilde{F}, A) \cap (\tilde{G}, A)) \cap (\tilde{H}, A)$.

$$3) (\tilde{F}, A) \cap (\tilde{F}, A) = (\tilde{F}, A).$$

Proof.

$$1) (\tilde{F}, A) \cap (\tilde{G}, A) = (\tilde{R}, A)$$

From Definition 30 and by consider the case when $\varepsilon \in B \cap C$ as the other cases are trivial, we have $\tilde{R}(e) = \tilde{F}(e) \cap \tilde{G}(e) = \tilde{G}(e) \cap \tilde{F}(e)$ (since intersection of interval-valued fuzzy sets is commutative).

2) The proof is straightforward from Definition 30.

3) The proof is straightforward from Definition 30.

Proposition 33 Let (\tilde{F}, A) and (\tilde{G}, A) be any two MIVFSSs of dimension k over U . Then the DeMorgan's Laws hold:

$$1) \left((\tilde{F}, A) \cup (\tilde{G}, A) \right)^c = (\tilde{F}, A)^c \cap (\tilde{G}, A)^c .$$

$$2) \left((\tilde{F}, A) \cap (\tilde{G}, A) \right)^c = (\tilde{F}, A)^c \cup (\tilde{G}, A)^c .$$

Proof.

$$1) (\tilde{F}, A)^c \cap (\tilde{G}, A)^c = (\tilde{F}^c(e) \cap \tilde{G}^c(e)) = \left((\tilde{F}(e) \cup (\tilde{G}(e)))^c \right) = \left((\tilde{F}, A) \cup (\tilde{G}, A) \right)^c .$$

2) The proof is similar to the above progress.

Proposition 34 Let $(\tilde{F}, A), (\tilde{G}, A)$ and (\tilde{H}, A) be any three MIVFSSs of dimension k over U . Then the following results hold:

$$1) (\tilde{F}, A) \cup \left((\tilde{G}, A) \cap (\tilde{H}, A) \right) = \left((\tilde{F}, A) \cup (\tilde{G}, A) \right) \cap \left((\tilde{F}, A) \cup (\tilde{H}, A) \right).$$

$$2) (\tilde{F}, A) \cap \left((\tilde{G}, A) \cup (\tilde{H}, A) \right) = \left((\tilde{F}, A) \cap (\tilde{G}, A) \right) \cup \left((\tilde{F}, A) \cap (\tilde{H}, A) \right).$$

Proof. a) For all $x \in E$,

$$\begin{aligned} & \lambda_{\tilde{F}(x) \cup (\tilde{G}(x) \cap \tilde{H}(x))}^-(x) \\ &= \left[\sup_{i \rightarrow k} \left(\lambda_{\tilde{F}(x)}^-(x), \lambda_{(\tilde{G}(x) \cap \tilde{H}(x))}^-(x) \right), \sup_{i \rightarrow k} \left(\lambda_{\tilde{F}(x)}^+(x), \lambda_{(\tilde{G}(x) \cap \tilde{H}(x))}^+(x) \right) \right] \\ &= \left[\sup_{i \rightarrow k} \left(\lambda_{\tilde{F}(x)}^-(x), \inf_{i \rightarrow k} \left(\lambda_{\tilde{G}(x)}^-(x), \lambda_{\tilde{H}(x)}^-(x) \right) \right), \sup_{i \rightarrow k} \left(\lambda_{\tilde{F}(x)}^+(x), \inf_{i \rightarrow k} \left(\lambda_{\tilde{G}(x)}^+(x), \lambda_{\tilde{H}(x)}^+(x) \right) \right) \right] \\ &= \left[\inf_{i \rightarrow k} \left(\sup_{i \rightarrow k} \left(\lambda_{\tilde{F}(x)}^-(x), \lambda_{\tilde{G}(x)}^-(x) \right), \sup_{i \rightarrow k} \left(\lambda_{\tilde{F}(x)}^-(x), \lambda_{\tilde{H}(x)}^-(x) \right) \right), \right. \\ & \quad \left. \inf_{i \rightarrow k} \left(\sup_{i \rightarrow k} \left(\lambda_{\tilde{F}(x)}^+(x), \lambda_{\tilde{G}(x)}^+(x) \right), \sup_{i \rightarrow k} \left(\lambda_{\tilde{F}(x)}^+(x), \lambda_{\tilde{H}(x)}^+(x) \right) \right) \right] \\ &= \lambda_{(\tilde{F}(x) \cup \tilde{G}(x)) \cap (\tilde{F}(x) \cup \tilde{H}(x))}^-(x) \end{aligned}$$

b) Similar to the proof of a.

Definition 35 If (\tilde{F}, A) and (\tilde{G}, B) are two multi-interval-valued-fuzzy soft sets of dimension k over U the “ (\tilde{F}, A) AND (\tilde{G}, B) ”, denoted by $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B)$, where $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta)$, $\forall (\alpha, \beta) \in A \times B$.

Example 36 Consider Example 21. By using the interval-valued fuzzy union we have (\tilde{F}, A) AND $(\tilde{G}, B) = (\tilde{H}, A \times B)$, where

$$\begin{aligned} \tilde{H}(e_1, e_1) &= \left\{ \frac{c_1}{\left(([0.4, 0.6]; [0.1, 0.4]; [0.4, 0.5]) \right)} ; \frac{c_2}{\left(([0.5, 0.8]; [0.5, 0.9]; [0.2, 0.4]) \right)} ; \frac{c_3}{\left(([0.4, 0.5]; [0.3, 0.7]; [0.6, 0.8]) \right)} \right\}; \\ \tilde{H}(e_1, e_2) &= \left\{ \frac{c_1}{\left(([0.4, 0.6]; [0.6, 0.7]; [0.5, 0.7]) \right)} ; \frac{c_2}{\left(([0.5, 0.8]; [0.4, 0.6]; [0.5, 0.7]) \right)} ; \frac{c_3}{\left(([0.4, 0.7]; [0.2, 0.5]; [0.5, 0.6]) \right)} \right\}; \end{aligned}$$

$$\begin{aligned} \tilde{H}(e_1, e_3) &= \left\{ \frac{c_1}{\left([0.4, 0.6]; [0.6, 0.8]; [0.8, 0.9] \right)}; \frac{c_2}{\left([0.7, 0.9]; [0.4, 0.6]; [0.3, 0.5] \right)}; \frac{c_3}{\left([0.2, 0.4]; [0.4, 0.6]; [0.5, 0.6] \right)} \right\}; \\ \tilde{H}(e_2, e_1) &= \left\{ \frac{c_1}{\left([0.2, 0.4]; [0.3, 0.4]; [0.5, 0.6] \right)}; \frac{c_2}{\left([0.4, 0.5]; [0.5, 0.9]; [0.3, 0.6] \right)}; \frac{c_3}{\left([0.4, 0.5]; [0.3, 0.7]; [0.6, 0.8] \right)} \right\}; \\ \tilde{H}(e_2, e_2) &= \left\{ \frac{c_1}{\left([0.3, 0.5]; [0.6, 0.7]; [0.5, 0.7] \right)}; \frac{c_2}{\left([0.3, 0.6]; [0.2, 0.5]; [0.5, 0.7] \right)}; \frac{c_3}{\left([0.4, 0.7]; [0.1, 0.3]; [0, 0.1] \right)} \right\}; \\ \tilde{H}(e_2, e_3) &= \left\{ \frac{c_1}{\left([0.4, 0.6]; [0.7, 0.8]; [0.8, 0.9] \right)}; \frac{c_2}{\left([0.7, 0.9]; [0.3, 0.5]; [0.3, 0.6] \right)}; \frac{c_3}{\left([0.3, 0.5]; [0.4, 0.6]; [0.3, 0.5] \right)} \right\}; \\ \tilde{H}(e_3, e_1) &= \left\{ \frac{c_1}{\left([0.3, 0.5]; [0.7, 0.8]; [0.2, 0.5] \right)}; \frac{c_2}{\left([0.4, 0.5]; [0.5, 0.9]; [0.3, 0.6] \right)}; \frac{c_3}{\left([0.4, 0.5]; [0.4, 0.7]; [0.6, 0.8] \right)} \right\}; \\ \tilde{H}(e_3, e_2) &= \left\{ \frac{c_1}{\left([0.3, 0.5]; [0.7, 0.8]; [0.5, 0.7] \right)}; \frac{c_2}{\left([0.3, 0.6]; [0.2, 0.5]; [0.5, 0.7] \right)}; \frac{c_3}{\left([0.4, 0.7]; [0.4, 0.6]; [0.5, 0.8] \right)} \right\}; \\ \tilde{H}(e_3, e_3) &= \left\{ \frac{c_1}{\left([0.4, 0.6]; [0.7, 0.8]; [0.8, 0.9] \right)}; \frac{c_2}{\left([0.7, 0.9]; [0.3, 0.5]; [0.3, 0.6] \right)}; \frac{c_3}{\left([0.4, 0.5]; [0.4, 0.6]; [0.5, 0.8] \right)} \right\}. \end{aligned}$$

Definition 37 If (\tilde{F}, A) and (\tilde{G}, B) are two multi-fuzzy soft sets of dimension k over U the “ (\tilde{F}, A) OR (\tilde{G}, B) ”, denoted by $(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{H}, A \times B)$, where $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cup \tilde{G}(\beta), \forall (\alpha, \beta) \in A \times B$.

Example 38 Consider Example 21. By using the interval-valued fuzzy intersection we have (\tilde{F}, A) OR $(\tilde{G}, B) = (\tilde{H}, A \times B)$, where

$$\begin{aligned} \tilde{H}(e_1, e_1) &= \left\{ \frac{c_1}{\left([0.2, 0.4]; [0.1, 0.3]; [0.1, 0.5] \right)}; \frac{c_2}{\left([0.4, 0.5]; [0.4, 0.6]; [0, 0.2] \right)}; \frac{c_3}{\left([0.1, 0.3]; [0.2, 0.5]; [0.5, 0.6] \right)} \right\}; \\ \tilde{H}(e_1, e_2) &= \left\{ \frac{c_1}{\left([0.3, 0.5]; [0.1, 0.4]; [0.4, 0.5] \right)}; \frac{c_2}{\left([0.3, 0.6]; [0.2, 0.5]; [0, 0.2] \right)}; \frac{c_3}{\left([0.1, 0.3]; [0.1, 0.3]; [0, 0.1] \right)} \right\}; \\ \tilde{H}(e_1, e_3) &= \left\{ \frac{c_1}{\left([0.4, 0.6]; [0.1, 0.4]; [0.4, 0.5] \right)}; \frac{c_2}{\left([0.5, 0.8]; [0.3, 0.5]; [0, 0.2] \right)}; \frac{c_3}{\left([0.1, 0.3]; [0.2, 0.5]; [0.3, 0.5] \right)} \right\}; \\ \tilde{H}(e_2, e_1) &= \left\{ \frac{c_1}{\left([0.1, 0.3]; [0.1, 0.3]; [0.1, 0.5] \right)}; \frac{c_2}{\left([0.2, 0.5]; [0.1, 0.4]; [0.2, 0.4] \right)}; \frac{c_3}{\left([0.3, 0.5]; [0.1, 0.2]; [0, 0.1] \right)} \right\}; \\ \tilde{H}(e_2, e_2) &= \left\{ \frac{c_1}{\left([0.1, 0.3]; [0.3, 0.4]; [0.5, 0.6] \right)}; \frac{c_2}{\left([0.2, 0.5]; [0.1, 0.4]; [0.3, 0.6] \right)}; \frac{c_3}{\left([0.3, 0.5]; [0.1, 0.2]; [0, 0.1] \right)} \right\}; \\ \tilde{H}(e_2, e_3) &= \left\{ \frac{c_1}{\left([0.1, 0.3]; [0.3, 0.4]; [0.5, 0.6] \right)}; \frac{c_2}{\left([0.2, 0.5]; [0.1, 0.4]; [0.3, 0.5] \right)}; \frac{c_3}{\left([0.2, 0.4]; [0.1, 0.2]; [0, 0.1] \right)} \right\}; \\ \tilde{H}(e_3, e_1) &= \left\{ \frac{c_1}{\left([0.2, 0.4]; [0.1, 0.3]; [0.1, 0.4] \right)}; \frac{c_2}{\left([0.3, 0.4]; [0.2, 0.5]; [0.2, 0.4] \right)}; \frac{c_3}{\left([0.4, 0.5]; [0.3, 0.6]; [0.5, 0.8] \right)} \right\}; \end{aligned}$$

$$\tilde{H}(e_3, e_2) = \left\{ \frac{c_1}{\left([0.3, 0.5]; [0.6, 0.7]; [0.2, 0.4] \right)}, \frac{c_2}{\left([0.3, 0.4]; [0.2, 0.5]; [0.3, 0.6] \right)}, \frac{c_3}{\left([0.4, 0.5]; [0.1, 0.2]; [0, 0.1] \right)} \right\};$$

$$\tilde{H}(e_3, e_3) = \left\{ \frac{c_1}{\left([0.4, 0.6]; [0.6, 0.8]; [0.2, 0.4] \right)}, \frac{c_2}{\left([0.3, 0.4]; [0.2, 0.4]; [0.2, 0.5] \right)}, \frac{c_3}{\left([0.2, 0.4]; [0.4, 0.6]; [0.3, 0.5] \right)} \right\}.$$

Proposition 39 Let (\tilde{F}, A) and (\tilde{G}, A) are two MIVFSSs of dimension k over U . Then the following results hold:

- 1) $\left((\tilde{F}, A) \wedge (\tilde{G}, A) \right)^c = (\tilde{F}, A)^c \vee (\tilde{G}, A)^c,$
- 2) $\left((\tilde{F}, A) \vee (\tilde{G}, A) \right)^c = (\tilde{F}, A)^c \wedge (\tilde{G}, A)^c,$

Proof.

a) Suppose that $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{O}, A \times B)$.

Therefore, $\left((\tilde{F}, A) \wedge (\tilde{G}, B) \right)^c = (\tilde{O}, A \times B)^c = \left((\tilde{O}^c(\alpha, \beta)), (A \times B) \right)$. Now,

$$\left((\tilde{F}, A) \vee (\tilde{G}, B) \right)^c = \left(\left((\tilde{F}^c(\alpha)), A \right) \wedge \left((\tilde{G}^c(\alpha)), B \right) \right) = (\tilde{J}, (A \times B)),$$

where $\tilde{J}(x, y) = (F^c(x)) \cap (G^c(x))$.

Now, take $(\alpha, \beta) \in (A \times B)$.

Therefore,

$$\begin{aligned} (\tilde{O})^c(\alpha, \beta) &= \left(\tilde{c} \left[\inf_{i \rightarrow k} (\tilde{F}^-(\alpha), \tilde{G}^-(\beta)), \inf_{i \rightarrow k} (\tilde{F}^+(\alpha), \tilde{G}^+(\beta)) \right] \right) \\ &= \left(\left[1 - \inf_{i \rightarrow k} (\tilde{F}^+(\alpha), \tilde{G}^+(\beta)), 1 - \inf_{i \rightarrow k} (\tilde{F}^-(\alpha), \tilde{G}^-(\beta)) \right] \right) \\ &= \left(\left[\sup_{i \rightarrow k} (1 - \tilde{F}^+(\alpha), 1 - \tilde{G}^+(\beta)), \sup_{i \rightarrow k} (1 - \tilde{F}^-(\alpha), 1 - \tilde{G}^-(\beta)) \right] \right) \\ &= \left(\left[\tilde{c}(\tilde{F}(\alpha)) \tilde{U} \tilde{c}(\tilde{G}(\beta)) \right] \right) = \tilde{J}(\alpha, \beta) \end{aligned}$$

Then \tilde{O}^c and \tilde{J} are the same. Hence, proved.

b) Similar to the proof of a.

4. mivfs-Aggregation Operator

In this section, we define an aggregate interval-valued fuzzy set of an MIVFS-set. We also define MIVFS-aggregation operator that produces an aggregate interval-valued fuzzy set from an MIVFS-set and its parameter set. Also we give an application of this operator in decision making problem.

Definition 40 Let $(\tilde{F}, A) \in MIVFSS(U)$. Then a MIVFS-aggregation operator, denoted by $MIVFS_{agg}$, is defined by

$$MIVFS_{agg} : \tilde{F}(E) \times MIVFSS(U) \rightarrow Int(U),$$

$$MIVFS_{agg}(\tilde{F}, A) = (\tilde{F}, A)^*$$

where

$$(\tilde{F}, A)^* = \left\{ u / \mu_{(\tilde{F}, A)^*}(u) : u \in U \right\}$$

Is an interval-valued fuzzy set over U . The value $(\tilde{F}, A)^*$ is called an aggregate interval-valued fuzzy set of (\tilde{F}, A) . Here, the membership degree $\mu_{(\tilde{F}, A)^*}(u)$ of u is defined as follows:

$$\mu_{(\tilde{F}, A)^*}(u) = \left[c^- = \frac{1}{k|A|} \sum_{x \in A1 \rightarrow k} \sum \mu_{\tilde{F}_k(x)^-}(u), c^+ = \frac{1}{k|A|} \sum_{x \in A1 \rightarrow k} \sum \mu_{\tilde{F}_k(x)^+}(u) \right],$$

where $|A|$ is the cardinality of A .

In the following example, we present an application of *MIVFS-aggregation operator* to solve a decision making problem.

Example 41

Step 1 Let the constructed MIVFS-set, (\tilde{F}, A) , be given as follows:

$$\begin{aligned} \tilde{F}(e_1) &= \left\{ \frac{u_1}{\left([0.4, 0.6]; [0.1, 0.4]; [0.4, 0.5] \right)}, \frac{u_2}{\left([0.5, 0.8]; [0.4, 0.6]; [0, 0.2] \right)}, \frac{u_3}{\left([0.1, 0.3]; [0.2, 0.5]; [0.5, 0.6] \right)} \right\}; \\ \tilde{F}(e_2) &= \left\{ \frac{u_1}{\left([0.1, 0.3]; [0.3, 0.4]; [0.5, 0.6] \right)}, \frac{u_2}{\left([0.2, 0.5]; [0.1, 0.4]; [0.3, 0.6] \right)}, \frac{u_3}{\left([0.3, 0.5]; [0.1, 0.2]; [0, 0.1] \right)} \right\}; \\ \tilde{F}(e_3) &= \left\{ \frac{u_1}{\left([0.6, 0.9]; [0.4, 0.6]; [0.7, 0.8] \right)}, \frac{u_2}{\left([0.3, 0.5]; [0.1, 0.5]; [0, 0.2] \right)}, \frac{u_3}{\left([0.3, 0.7]; [0.2, 0.5]; [0.4, 0.7] \right)} \right\}. \end{aligned}$$

Step 2 The aggregate interval-valued fuzzy set can be found as

$$(\tilde{F}, A)^* = \left\{ \frac{u_1}{[0.388, 0.566]}, \frac{u_2}{[0.211, 0.477]}, \frac{u_3}{[0.233, 0.455]} \right\}$$

Step 3 $\forall u_i \in U$, compute the score r_i of u_i such that

$$r_i = \sum_{u_j \in U} \left((c_i^- - c_j^-) + (c_i^+ - c_j^+) \right)$$

Thus, we have

$$r_1 = 0.537, r_2 = -0.266, r_3 = -0.266$$

Step 4 The decision is any one of the elements in S where $S = \max_{u_i \in U} \{r_i\}$. In our example, u_1 is the best choice because $\max_{u_i \in U} \{r_i\} = r_1$.

5. Conclusion

As a generalisation of multi-fuzzy soft set and by combining this concept and interval-value fuzzy set, the concept of the multi-interval-valued fuzzy soft set is introduced and some of its properties studied. The complement, union and intersection, operations have been defined on the multi-interval-valued fuzzy soft set. An application of this theory is given in solving a decision making problem. We hope that our work would help enhancing this study on multi-fuzzy soft sets for the researchers.

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