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The Matching Uniqueness of A Graphs

Shichang Shen

School of Mathematics and Statistics, Qinghai Nationalities University, Xining, China
Email: 13909785766@163.com

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Abstract

In the paper, We discussed the matching uniqueness of graphs with degree sequence $(1^3, 2^{s-4}, 3)$. The necessary and sufficient conditions for $T(1,5,n)\cup\left(\bigcup_{i=0}^{s} C_{p_i}\right)$ and its complement are matching unique are given.

Keywords

Graph, Matching Polynomial, Matching Uniqueness

1. Introduction

All graphs considered in the paper are simple and undirected. The terminology not defined here can be found in [1]. Let *G* be a graph with *n* vertices. An *r*-matching in a graph *G* is a set of *r* edges, no two of which have a vertex in common. The number of *r*-matching in *G* will be denoted by $p(G,r)$. We set $p(G,0)=1$ and define the matching polynomial of *G* by

$$\mu(G,x)=\sum_{r\geq 0}(-1)^r p(G,r)x^{n-2r}$$

For any graph *G*, the roots of $\mu(G,x)$ are all real numbers. Assume that $\gamma_1(G)\geq\gamma_2(G)\geq\cdots\geq\gamma_n(G)$, the largest root $\gamma_1(G)$ is referred to as the largest mathing root of *G*.

Throughout the paper, we denote by P_n and C_n the path and the cycle on *n* vertices, respectively. $T(a,b,c)(a\leq b\leq c)$ denotes the tree with a vertex *v* of degree 3 such that $T(a,b,c)-v=P_a\cup P_b\cup P_c$, and $T(a,b,c,1,1)(a\leq b\leq c)$ denotes the tree obtained by appending a pendant vertex of the path P_c in $T(a,b,c)$ to a vertex with degree 2 of P_3. $Q(s_1,s_2)$ is obtained by appending a cycle C_{s_1+1} to a pendant vertex of a path P_{s_2}. Two graphs are matching equivalency if they share the same matching polynomial. A graph *G* is said to be matching unique if for any graph *H*, $\mu(G,x)=\mu(H,x)$ implies that *H* is isomorphic to *G*. The study in this ares has made great progress. For details, the reader is referred to the surveys [2]-[6]. In the paper, we prove $T(1,5,n)\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right)(n\geq 5)$ and its complement are matching unique if and only if $n\neq 5,8,15$ or $n=6$, $p_i\neq 6$.

2. Basic Results

Lemma 1 [1] The matching polynomial $\mu(G,x)$ satisfies the following identities:

1) $\mu(G\cup H,x)=\mu(G,x)\mu(H,x)$.

2) $\mu(G,x)=\mu(G\setminus e,x)-\mu(G\setminus u,v,x)$ if $e=\{u,v\}$ is an edge of *G*.

Lemma 2 [1] Let *G* be a connected graph, and let *H* be a proper subgraph *G*.
Then $\gamma_1(G)>\gamma_1(H)$.

Lemma 3 [2] Let $G=T(a,b,c)\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right)$, if $H\sim G$, then *H* are precisely the graphs of the following types:

$$T(s_1,s_2,s_3)\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right),\ Q(s_1,s_2)\cup P_l\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right),\ K_1\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right).$$

Lemma 4 1) [1] $\gamma_1(P_n)=2\cos\left(\frac{\pi}{n+1}\right),\gamma_1(C_n)=2\cos\left(\frac{\pi}{2n}\right)$.

2) [2] $\gamma_1(T(m,m,n))=\gamma_1(Q(m,n))=\gamma_1(Q(n+1,m-1))$.

3) [2] $\gamma_1(Q(2,m-1))\leq\gamma_1(T(1,m,n))\,(2<m\leq n)<\gamma_1(Q(2,m+1))$.

4) [3] $\gamma_1(T(m,m,n))>\gamma_1(Q(m-1,n))\,(m\geq 3),\gamma_1(Q(m+1,m))=\gamma_1(Q(m,2m+2))$, $\gamma_1(Q(m,m-1))>\gamma_1(Q(m-1,m))$.

5) [4] $\gamma_1(T(1,3,n))<\gamma_1(T(1,4,6)),\gamma_1(T(1,4,n))<\gamma_1(T(1,5,7))$.

6) [5] $2<\gamma_1(T(1,m,n))(2<m<n)<\left(2+\sqrt{5}\right)^{\frac{1}{2}}<\gamma_1(T(s_1,s_2,s_3))\,(2\leq s_1<s_2<s_3)$.

Lemma 5 [5] Let *G* be a tree and let $G_{u,v}$ be obtained from *G* by subdividing the edge *uv* of *G*, then

1) $\gamma_1(G_{u,v})>\gamma_1(G)$, if *uv* not lies on an internal path of *G*.

2) $\gamma_1(G_{u,v})<\gamma_1(G)$, if *uv* lies on an internal path of *G*, and if *G* is not isomorphic to $T(1,1,n,1,1)$.

Lemma 6 [6] $\bigcup_{i=0}^{s}C_{p_i}$ are matching unique.

Lemma 7 $\gamma_1(T(1,5,n))<\gamma_1(T(1,6,8))$.

Proof. Direct computation (using Matlab 8.0), we immediately have the following:

$$\mu(T(1,5,9,1,1),x)=x^{19}-18x^{17}+134x^{15}-533x^{13}+122x^{11}-1617x^{9}+1176x^{7}-413x^{5}+50x^{3},$$

$$\mu(T(1,6,8),x)=x^{16}-15x^{14}+90x^{12}-276x^{10}+458x^{8}-400x^{6}+164x^{4}-24x^{2}+1.$$

$$\gamma_1(T(1,5,9,1,1))=2.0518,\ \gamma_1(T(1,6,8))=2.0522.$$

By Lemma 2, 5, we get $\gamma_1(T(1,5,5))<\gamma_1(T(1,5,6))<\gamma_1(T(1,5,7))<\cdots<\gamma_1(T(1,5,n))$
$<\gamma_1(T(1,5,n-2,1,1))<\gamma_1(T(1,5,n-3,1,1))<\cdots<\gamma_1(T(1,5,9,1,1))<\gamma_1(T(1,6,8))$.

3. Main Results

Theorem 1 Let $G=T(1,5,n)\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right)(n\geq 5)$, then *G* are matching unique if and only if $n\neq 5,8,15$ or

$n=6, p_i \neq 6$.

Proof. The necessary condition follows immediately from Lemma 1. We have

$$\mu\left(T(1,5,5)\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right),x\right)=\mu\left(Q(5,1)\cup P_5\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right),x\right)$$

$$\mu\left(T(1,5,8)\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right),x\right)=\mu\left(Q(2,5)\cup P_7\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right),x\right)$$

$$\mu\left(T(1,5,15)\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right),x\right)=\mu\left(T(1,6,7)\cup C_7\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right),x\right)$$

$$\mu\left(T(1,5,6)\cup C_6\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right),x\right)=\mu\left(T(1,4,13)\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right),x\right)$$

Now suppose that $n\neq 5,8,15$ or $n=6, p_i\neq 6$, H is a graph being matching equivalency with G. We proceed to prove that H must be isomorphic to G. By Lemma 3

$$H\in\left\{T(s_1,s_2,s_3)\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right),Q(s_1,s_2)\cup P_l\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right),K_1\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right)\right\}$$

Case 1. If $H=Q(s_1,s_2)\cup P_l\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right)$. By $n>5$, we know that $\gamma_1(H)>2$. Hence, the component of $\gamma_1(H)>2$ in H may be only $Q(s_1,s_2)$. By Lemma 4, $\gamma_1(Q(2,4))\leq\gamma_1(T(1,5,n))<\gamma_1(Q(2,6))$ and $\gamma_1(T(1,5,8))=\gamma_1(Q(2,5))$. Let $s_1=2$, then $\gamma_1(T(1,5,n))\neq\gamma_1(Q(2,s_2))$, a contradiction. Let $s_1=3$. If $s_2=1$, then $\gamma_1(Q(3,1))=\gamma_1(Q(2,2))$, a contradiction. If $s_2=2$, then $\gamma_1(Q(3,2))=\gamma_1(Q(2,6))$, a contradiction. If $s_2\geq 3$, then $\gamma_1(Q(3,s_2))\geq\gamma_1(Q(3,3))>\gamma_1(Q(2,n))$, a contradiction. Let $s_1=4$. If $s_2=1$, then $\gamma_1(Q(4,1))=\gamma_1(Q(2,3))$, a contradiction. If $s_2\geq 2$, then $\gamma_1(Q(4,s_2))\geq\gamma_1(Q(4,2))=\gamma_1(Q(3,3))>\gamma_1(Q(2,n))$, a contradiction. Let $s_1=5$. If $s_2=1$, then $\gamma_1(Q(5,1))=\gamma_1(Q(2,4))$, a contradiction. If $s_2\geq 2$, then $\gamma_1(Q(5,s_2))\geq\gamma_1(Q(5,2))=\gamma_1(Q(3,4))>\gamma_1(Q(3,3))>\gamma_1(Q(2,n))$, a contradiction. Let $s_1\geq 6$, then $\gamma_1(Q(s_1,s_2))\geq\gamma_1(Q(6,1))>\gamma_1(Q(2,5))$, a contradiction.

Case 2 If $H=T(s_1,s_2,s_3)\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right)$. By $\gamma_1(H)>2$, hence the component of $\gamma_1(H)>2$ in H may be only $T(s_1,s_2,s_3)$. Let $s_1=1$. If $s_2=1$, then $\gamma_1(T(1,1,s_3))<2$, a contradiction. If $s_2=2,3$, then $\gamma_1(T(1,2,s_3))<\gamma_1(Q(2,4))$, a contradiction. If $s_2=4$, then $\gamma_1(T(1,4,s_3))=\gamma_1(T(1,5,n))$, by Lemma 4, we get $n=6$, thus $s_3=13$. That is,

$$\mu\left(T(1,5,6)\cup C_6\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right),x\right)=\mu\left(T(1,4,13)\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right),x\right)=\mu\left(T(1,5,6)\cup\left(\bigcup_{i=0}^{s}C_{p_i}\right),x\right),\text{ then}$$

$$\mu\left(C_6\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right),x\right)=\mu\left(\bigcup_{i=0}^{s}C_{p_i},x\right),$$ by Lemma 6, p_i has at least one equal to 6, a contradiction. If $s_2=5$, by Lemma 4, 6, we have $s_3=n, s=m, p_i=q_i$, thus H be isomorphic to G. Let $s_1=2$. If $s_2=2$, $\gamma_1(T(2,2,s_3))=\gamma_1(Q(2,s_3))\neq\gamma_1(T(1,5,n))$, a contradiction. If $s_2\geq 3, s_3\geq 3$, a contradiction. Let $s_1\geq 3$, by Lemma 4, $\gamma_1(T(1,5,n))<\gamma_1(T(s_1,s_2,s_3))$, a contradiction.

Case 3 If $H=K_1\cup\left(\bigcup_{i=0}^{m}C_{q_i}\right)$, by $\gamma_1(G)>2$, a contradiction. Combing cases 1 - 3, H is isomorphic to G.

The proof is complete. For a graph, its matching polynomial determine the matching polynomial of its Complement [6], so the complement of $G = T(1,5,n) \cup \left(\bigcup_{i=0}^{s} C_{p_i} \right) (n \geq 5)$ are matching unique if and only if $n \neq 5,8,15$ or $n = 6, p_i \neq 6$.

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