Applied Mathematics, 2015, 6, 1174-1181
Published Online June 2015 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2015.67107

Extended Jacobian Elliptic Function Expansion Method and Its Applications in Biology

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Received 26 January 2015; accepted 22 June 2015; published 25 June 2015

Abstract

In this work, an extended Jacobian elliptic function expansion method is proposed for constructing the exact solutions of nonlinear evolution equations. The validity and reliability of the method are tested by its applications to Dynamical system in a new Double-Chain Model of DNA and a diffusive predator-prey system which play an important role in biology.

Keywords

Extended Jacobian Elliptic Function Expansion Method, Dynamical System in a New Double-Chain Model of DNA, A Diffusive Predator-Prey System, Traveling Wave Solutions, Solitary Wave Solutions

1. Introduction

The nonlinear partial differential equations of mathematical physics are major subjects in physical science [1]. Exact solutions for these equations play an important role in many phenomena in physics such as fluid mechanics, hydrodynamics, Optics, Plasma physics and so on. Recently many new approaches for finding these solutions have been proposed, for example, tanh-sech method [2]-[4], extended tanh-method [5]-[7], $\exp(-\varphi(\xi))$ [8]-[11], homogeneous balance method [12], F-expansion method [13]-[15], exp-function method [16] [17], trigonometric function series method [18], $\left(\frac{G'}{G}\right)$-expansion method [19]-[22], Jacobi elliptic function method [23]-[26] and so on.

How to cite this paper: Zahran, E.H.M. and Khater, M.M.A. (2015) Extended Jacobian Elliptic Function Expansion Method and Its Applications in Biology. *Applied Mathematics*, **6**, 1174-1181. http://dx.doi.org/10.4236/am.2015.67107

The objective of this article is to apply the extended Jacobian elliptic function expansion method for finding the exact traveling wave solution of Dynamical system in a new Double-Chain Model of DNA and a diffusive predator-prey system which play an important role in biology and mathematical physics.

The rest of this paper is organized as follows: In Section 2, we give the description of the extended Jacobi elliptic function expansion method In Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 4, conclusions are given.

2. Description of Method

Consider the following nonlinear evolution equation

$$F\left(u,u_t,u_x,u_{tt},u_{xx},\cdots\right)=0, \tag{1}$$

where F is polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [23]-[26]

Step 1. Using the transformation

$$u=u(\xi),\ \ \xi=x-ct, \tag{2}$$

where k and c are the wave number and wave speed, to reduce Equation (1) to the following ODE:

$$P\left(u,u',u'',u''',\cdots\right)=0, \tag{3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $'=\frac{\mathrm{d}}{\mathrm{d}\xi}'$.

Step 2. Making good use of ten Jacobian elliptic functions, we assume that (3) has the solutions in these forms:

$$u(\xi)=a_0+\sum_{j=1}^{N} f_i^{\,j-1}(\xi)\left[a_j f_i(\xi)+b_j g_i(\xi)\right],\ \ i=1,2,3,\cdots, \tag{4}$$

With

$$\begin{aligned}
&f_1(\xi)=sn\xi, \quad g_1(\xi)=cn\xi,\\
&f_2(\xi)=sn\xi, \quad g_2(\xi)=dn\xi,\\
&f_3(\xi)=ns\xi, \quad g_3(\xi)=cs\xi,\\
&f_4(\xi)=ns\xi, \quad g_4(\xi)=ds\xi,\\
&f_5(\xi)=sc\xi, \quad g_5(\xi)=nc\xi,\\
&f_6(\xi)=sd\xi, \quad g_6(\xi)=nd\xi,
\end{aligned} \tag{5}$$

where $sn\xi$, $cn\xi$, $dn\xi$, are the Jacobian elliptic sine function, The jacobian elliptic cosine function and the Jacobian elliptic function of the third kind and other Jacobian functions which is denoted by Glaisher's symbols and are generated by these three kinds of functions, namely

$$\begin{aligned}
&ns\xi=\frac{1}{sn\xi},\ nc\xi=\frac{1}{cn\xi},\ nd\xi=\frac{1}{dn\xi},\ sc\xi=\frac{cn\xi}{sn\xi},\\
&cs\xi=\frac{sn\xi}{cn\xi},\ ds\xi=\frac{dn\xi}{sn\xi},\ sd\xi=\frac{sn\xi}{dn\xi},
\end{aligned} \qquad 6)$$

That have the relations

$$\begin{aligned}
&sn^2\xi+cn^2\xi=1,\ dn^2\xi+m^2sn^2\xi=1,\ ns^2\xi=1+cs^2\xi,\\
&ns^2\xi=m^2+ds^2\xi,\ sc^2\xi+1=nc^2\xi,\ m^2sd^2+1=nd^2\xi,
\end{aligned} \tag{7}$$

With the modulus m $(0<m<1)$. In addition we know that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}sn\xi=cn\xi dn\xi,\ \frac{\mathrm{d}}{\mathrm{d}\xi}cn\xi=-sn\xi dn\xi,\ \frac{\mathrm{d}}{\mathrm{d}\xi}dn\xi=-m^2sn\xi cn\xi. \tag{8}$$

The derivatives of other Jacobian elliptic functions are obtained by using Equation (8). To balance the highest order linear term with nonlinear term we define the degree of u as $D[u]=n$ which gives rise to the degrees of other expressions as

$$D\left[\frac{\mathrm{d}^q u}{\mathrm{d}\xi^q}\right]=n+q,\quad D\left[u^p\left(\frac{\mathrm{d}^q u}{\mathrm{d}\xi^q}\right)^s\right]=np+s(n+q). \tag{9}$$

According the rules, we can balance the highest order linear term and nonlinear term in Equation (3) so that n in Equation (4) can be determined.

In addition we see that when $m\to 1$, $sn\xi$, $cn\xi$, and $dn\xi$ degenerate as $\tanh\xi$, $\mathrm{sech}\xi$, $\mathrm{sech}\xi$, respectively, while when therefore Equation (5) degenerate as the following forms

$$u(\xi)=a_0+\sum_{j=1}^{N}\tanh^{j-1}(\xi)\left[a_j\tanh(\xi)+b_j\mathrm{sech}(\xi)\right], \tag{10}$$

$$u(\xi)=a_0+\sum_{j=1}^{N}\coth^{j-1}(\xi)\left[a_j\coth(\xi)+b_j\coth(\xi)\right], \tag{11}$$

$$u(\xi)=a_0+\sum_{j=1}^{N}\tan^{j-1}(\xi)\left[a_j\tan(\xi)+b_j\sec(\xi)\right], \tag{12}$$

$$u(\xi)=a_0+\sum_{j=1}^{N}\cot^{j-1}(\xi)\left[a_j\cot(\xi)+b_j\csc(\xi)\right]. \tag{13}$$

Therefore the extended Jacobian elliptic function expansion method is more general than sine-cosine method, the tan-function method and Jacobian elliptic function expansion method.

3. Application

3.1. Example 1: Dynamical System in a New Double-Chain Model of DNA

An attractive nonlinear model for the nonlinear science in the deoxyribonucleic acid (DNA). The dynamics of DNA molecules is one of the most fascinating problems of modern biophysics because it is at the basis of life. The DNA structure has been studied during last decades. The investigation of DNA dynamics has successfully predicted the appearance of important nonlinear structures. It has been shown that the non linearity is responsible for forming localized waves. These localized waves are interesting because they have the capability to transport energy without dissipation [27]-[35]. In Ref. [34] [35], it is given that a new double-chain model of DNA consists of two long elastic homogeneous strands which represent two poly nucleotide chains of the DNA molecule, connected with each other by an elastic membrane representing the hydrogen bonds between the base pair of the two chains. Under some appropriate approximation, the new double-chain model of DNA can be described by the following two general nonlinear dynamical system:

$$u_{tt}-c_1^2u_{xx}=\lambda_1 u+\gamma_1 uv+\mu_1 u^3+\beta_1 uv^2, \tag{14}$$

$$v_{tt}-c_2^2v_{xx}=\lambda_2 v+\gamma_2 u^2+\mu_2 u^2 v+\beta_2 v^3+c_0, \tag{15}$$

where

$$c_1=\pm\frac{Y}{\rho};\ c_2=\pm\frac{F}{\rho};\ \lambda_1=\frac{-2\mu}{\rho\sigma h}(c-l_0);$$

$$\lambda_2=\frac{-2\mu}{\rho\sigma};\ \gamma_1=2\gamma_2=\frac{2\sqrt{2}\mu l_0}{\rho\sigma h^2};\ \mu_1=\mu_2=\frac{-2\mu l_0}{\rho\sigma h^3};$$

$$\beta_1=\beta_2=\frac{4\mu l_0}{\rho\sigma h^3};\ c_0=\frac{\sqrt{2}\mu(h-l_0)}{\rho\sigma}, \tag{16}$$

where ρ, σ, Y and F denote respectively the mass density, the area of transverse cross-section, the Young's modulus and tension density of each strand; μ is the rigidity of the elastic membrane; *h* is the distance between the two strands, and l_0 is the height of the membrane in the equilibrium positive. In Equations (14) and (15), *u* is the difference of the longitudinal displacements of the bottom and top strands, while *v* is the difference of the transverse displacements of the bottom and top strands.

we first introduce the transformation

$$v = au + b, \tag{17}$$

where *a* and *b* are constants, to reduce Equations (14) and (15) to the following system of equations:

$$u_{tt} - c_1^2 u_{xx} = u^3\left(\mu_1 + \beta_1 a^2\right) + u^2\left(2\beta_1 ab + a\gamma_1\right) + u\left(\lambda_1 + b\gamma_1 + \beta_1 b^2\right), \tag{18}$$

and

$$u_{tt} - c_2^2 u_{xx} = u^3\left(\mu_2 + \beta_2 a^2\right) + u^2\left(\frac{\gamma_2}{a} + \frac{\mu_2 b}{a} + 3\beta_2 ab\right) + u\left(\lambda_2 + 3\beta_2 b^2\right) + \frac{\lambda_2 b}{a} + \frac{\beta_2 b^3}{a} + \frac{c_0}{a}. \tag{19}$$

Comparing Equations (18) and (19) and using (17) we deduce that $b = \frac{h}{\sqrt{2}}$ and $F = Y$. Now Equations (18) and (19) can be written as

$$u_{tt} - c_1^2 u_{xx} - Au^3 - Bu^2 - Cu = 0, \tag{20}$$

where

$$A = \frac{\alpha}{h^3}\left(-2 + 4a^2\right);\ B = \frac{6\sqrt{2}a\alpha}{h^2};\ C = \left(\frac{-2\alpha}{l_0} + \frac{6\alpha}{h}\right);\ \alpha = \frac{\mu l_0}{\rho\sigma};\ c_1^2 = \frac{Y}{\rho}. \tag{21}$$

The wave transformation $u(x,t) = u(\xi)$, $\xi = kx + \omega t$, reduce Equation (20) to the following ODE:

$$\left(\omega^2 - k^2 c_1^2\right)u'' - Au^3 - Bu^2 - Cu = 0, \tag{22}$$

where $\omega^2 - k^2 c_1^2 \neq 0$. Balancing u'' and u^3 yields, $N + 2 = 3N \to N = 1$. Consequently, we have the formal solution:

$$u = a_0 + a_1 sn + b_1 cn, \tag{23}$$

where a_0, a_1 and b_1 are constant such that $a_1 \neq 0$ or $b_1 \neq 0$. From (23), it is easy to see that

$$u' = a_1 cndn - b_1 sndn, \tag{24}$$

$$u'' = -m^2 sna_1 + 2a_1 sn^3 m^2 + 2m^2 sn^2 cnb_1 - a_1 sn - b_1 cn. \tag{25}$$

Substituting Equations (23) and (25) into Equation (22) and equating all coefficients of sn^3, sn^2cn, sn^2, $sncn$, sn, cn, sn^0 to zero, we obtain

$$2\left(\omega^2 - k^2 c_1^2\right)m^2 a_1 - A\left(-3a_1 b_1^2 + a_1^3\right) = 0, \tag{26}$$

$$2\left(\omega^2 - k^2 c_1^2\right)m^2 b_1 - A\left(3a_1^2 b_1 - b_1^3\right) = 0, \tag{27}$$

$$-A\left(-3a_0 b_1^2 + 3a_0 a_1^2\right) - B\left(a_1^2 - b_1^2\right) = 0, \tag{28}$$

$$-6Aa_0 a_1 b_1 - 2Ba_1 b_1 = 0, \tag{29}$$

$$\left(\omega^2 - k^2 c_1^2\right)\left(-a_1 - m^2 a_1\right) - A\left(3a_0^2 a_1 + 3a_1 b_1^2\right) - 2Ba_0 a_1 - Ca_1 = 0, \tag{30}$$

$$-\left(\omega^2 - k^2 c_1^2\right)b_1 - A\left(3a_0^2 b_1 - b_1^3\right) - 2Ba_0 b_1 - Cb_1 = 0, \tag{31}$$

$$-A\left(a_0^3 + 3a_0 b_1^2\right) - B\left(a_0^2 + b_1^2\right) - Ca_0 = 0. \tag{32}$$

Solving the above system with the aid of Maple or Mathematica, we have the following solitary wave solution:

Case 1.

$$A=\frac{2m^2\left(-\omega^2+k^2c_1^2\right)}{b_1^2},\ B=0,\ C=-\omega^2+k^2c_1^2-2m^2\omega^2+2m^2k^2c_1^2,\ a_0=0,\ a_1=0,\ b_1=b_1, \tag{33}$$

Case 2.

$$\begin{aligned}&A=\frac{2m^2\left(-\omega^2+k^2c_1^2\right)}{b_1^2},\ B=\pm\sqrt{\frac{-\left(1+2m^2\right)}{2}}\frac{-6m\left(-\omega^2+k^2c_1^2\right)}{b_1},\\ &C=-2\left(1+2m^2\right)\left(-\omega^2+k^2c_1^2\right),\ a_0=\pm\sqrt{\frac{-\left(1+2m^2\right)}{2}}\frac{b_1}{m},\ a_1=0,\ b_1=b_1,\end{aligned} \tag{34}$$

Case 3.

$$\begin{aligned}&A=\frac{-2m^2\left(-\omega^2+k^2c_1^2\right)}{a_1^2},\ B=\pm\sqrt{\frac{\left(1+m^2\right)}{2}}\frac{6m\left(-\omega^2+k^2c_1^2\right)}{a_1},\\ &C=-2\left(1+m^2\right)\left(-\omega^2+k^2c_1^2\right),a_0=\pm\sqrt{\frac{\left(1+m^2\right)}{2}}\frac{a_1}{m},\ a_1=a_1,\ b_1=0,\end{aligned} \tag{35}$$

Sothat solution of Equation (22) has the form

Case 1.

$$u(\xi)=\pm\sqrt{\frac{2m^2\left(-\omega^2+k^2c_1^2\right)}{A}}cn. \tag{36}$$

Case 2.

$$u(\xi)=\pm\sqrt{\frac{-\left(1+2m^2\right)}{2}}\frac{b_1}{m}\pm\sqrt{\frac{2m^2\left(-\omega^2+k^2c_1^2\right)}{A}}cn \tag{37}$$

Case 3.

$$u(\xi)=\pm\sqrt{\frac{\left(1+m^2\right)}{2}}\frac{a_1}{m}\pm\sqrt{\frac{-2m^2\left(-\omega^2+k^2c_1^2\right)}{A}}sn. \tag{38}$$

3.2. Example 2. A Diffusive Predator-Prey System

Consider a system of two coupled nonlinear partial differential equations describing the spatio-temporal dynamics of a predator-prey system [36],

$$\begin{aligned}&u_t=u_{xx}-\beta u+(1+\beta)u^2-u^3-uv,\\ &v_t=v_{xx}+\kappa uv-mv-\delta v^3.\end{aligned} \tag{39}$$

where κ, δ, m and β are positive parameters. The solutions of predator-prey system have been studied in various aspects [36]-[38]. The dynamics of the diffusive predator-prey system have assumed the following relations between the parameters, namely $m=\beta$ and $\kappa+\frac{1}{\sqrt{\delta}}=\beta+1$. Under there assumptions, Equation (39) can be rewritten in the form:

$$\begin{aligned}&u_t=u_{xx}-\beta u+\left(\kappa+\frac{1}{\sqrt{\delta}}\right)u^2-u^3-uv,\\ &v_t=v_{xx}+\kappa uv-\beta v-\delta v^3.\end{aligned} \tag{40}$$

We use the wave transformation $u(x,t)=u(\xi),\ \xi=x-ct$ to reduce Equation (40) to the following nonlinear system of ordinary differential equations:

$$\begin{cases} u'' + cu' - \beta u + \left(\kappa + \frac{1}{\sqrt{\delta}} \right) u^2 - u^3 - uv = 0, \\ v'' + cv' + \kappa uv - \beta v - \delta v^3 = 0, \end{cases} \tag{41}$$

where c is a nonzero constant.

In order to solve Equation (41), let us consider the following transformation

$$v = \frac{1}{\sqrt{\delta}} u, \tag{42}$$

Substituting the transformation (42) into Equation (41), we get

$$u'' + cu' - \beta u + \kappa u^2 - u^3 = 0. \tag{43}$$

Balancing u'' with u^3 in Equation (43) yields, $N + 2 = 3N \Rightarrow N = 1$. Consequently, we get the same formal solution (23). Substituting (23)-(25) into (43), setting the coefficients of (sn^3, $sn^2 cn$, sn^2, $sncn$, cn, sn, sn^0) to zero, we obtain the following under determined system of algebraic equations for (a_0, a_1, b_1).

$$2a_1 m^2 - a_1^3 + 3a_1 b_1^2 = 0, \tag{44}$$

$$2m^2 b_1 - 3a_1^2 b_1 + b_1^3 = 0, \tag{45}$$

$$\kappa\left(a_1^2 - b_1^2\right) - 3a_0 a_1^2 + 3a_0 b_1^2 = 0, \tag{46}$$

$$2\kappa a_1 b_1 - 6a_0 a_1 b_1 = 0, \tag{47}$$

$$-a_1 m^2 - a_1 - \beta a_1 + 2\kappa a_0 a_1 - 3a_0^2 a_1 - 3a_1 b_1^2 = 0, \tag{48}$$

$$-b_1 - \beta b_1 + 2k a_0 b_1 - 3a_0^2 b_1 - b_1^3 = 0, \tag{49}$$

$$-\beta a_0 + \kappa\left(a_0^2 + b_1^2\right) - a_0^3 - 3a_0 b_1^2 = 0, \tag{50}$$

solving Equations (44)-(50) using the maple or mathematica program to get solitary wave solution of equations we get

$$m = \pm\sqrt{\frac{\beta}{2} - 1}, \kappa = \pm 3\sqrt{\frac{\beta}{2}}, a_0 = \sqrt{\frac{\beta}{2}}, a_1 = \pm\sqrt{\beta - 2}, \omega = \omega, b_1 = 0.$$

So we get

$$u(\xi) = a_0 + a_1 sn \tag{51}$$

$$u(\xi) = \pm\sqrt{m^2 + 1} \pm \sqrt{2} msn, \tag{52}$$

when $m = 1$ hyperbolic solution

$$u(\xi) = \pm\sqrt{2}\left(1 + \tanh(\xi)\right) \tag{53}$$

4. Conclusion

We establish exact solutions for the dynamics of DNA molecules which is one of the most fascinating problems of modern biophysics because it is at the basis of life. The DNA structure has been studied during last decades. The investigation of DNA dynamics has successfully predicted the appearance of important nonlinear structures and a system of two coupled nonlinear partial differential equations describing the spatio-temporal dynamics of a predator-prey system where the prey per capita growth rate is subject to the All effect. The extended Jacobian elliptic function expansion method has been successfully used to find the exact traveling wave solutions of some nonlinear evolution equations. As an application, the traveling wave solutions for Dynamical system in a new Double-Chain Model of DNA and a diffusive predator-prey system, which have been constructed using the extended Jacobian elliptic function expansion method. Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of the system of shallow water wave equations and a diffusive predator-prey system, are new and different from those obtained in [34]-[38]. It can be concluded that this method is reliable and proposes a variety of exact solutions NPDEs. The performance of this method is effective and can be applied to many other nonlinear evolution equations.

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