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Stationary Solutions of a Mathematical Model for Formation of Coral Patterns

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Abstract

A reaction-diffusion type mathematical model for growth of corals in a tank is considered. In this paper, we study stationary problem of the model subject to the homogeneous Neumann boundary conditions. We derive some existence results of the non-constant solutions of the stationary problem based on Priori estimations and Topological Degree theory. The existence of non-constant stationary solutions implies the existence of spatially variant time invariant solutions for the model.

Keywords

Reaction-Diffusion Equations, Stationary Solutions, Priori Estimates, Topological Degree Theory

1. Introduction

Most of the corals consist of colony of polyps reside in cups like skeletal structures on stony corals called calices. Polyps of hard corals produce a stony skeleton of calcium carbonate which causes the growth of the coral reefs. Polyps' maximum diameter is a species-specific characteristic. Once they reach this maximum diameter they divide [1]. In this way, if survive, they divide over and over and form a colony. If the coral colony does not break off, it grows as its individual polyps divide to form new polyps [2]. As new polyps are formed they build new calices to reside. This causes the growth of solid matrix of the stony corals.

Various modeling approaches on coral morphogenesis processes have been reported in [1] [3]-[9]. Morphogenesis of branching corals has been described by Diffusion-Limited Aggregation (DLA) type models in [1] [6] [10].

A reaction diffusion type mathematical model for growth of corals in a tank is proposed in [11] [12] considering the nutrient polyps interaction. This model is derived based on the model appear in [8]. The nondimensionalized version of this mathematical model takes the form:

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}=\Delta u+1-u-\alpha^2v^2u,\quad x\in\Omega\subset\mathbb{R}^2,\quad t>0\\&\frac{\partial v}{\partial t}=d\Delta v-\lambda v+\alpha^2v^2u,\quad x\in\Omega\subset\mathbb{R}^2,\quad t>0\\&\frac{\partial w}{\partial t}=\lambda_1 v,\quad x\in\Omega\subset\mathbb{R}^2,\quad t>0.\end{aligned}\right\}\tag{1}$$

Here, u and v are vertically averaged nondimensionalized concentrations of dissolved nutrients (foods of coral polyps) and aggregating solid material (calcium carbonate) on the coral reefs respectively. α, d, λ and λ_1 are positive constants. The local and global stabilities of the solutions of the corresponding system of ordinary differential equations

$$\left.\begin{aligned}&\frac{\mathrm{d}u}{\mathrm{d}t}=1-u-\alpha^2uv^2\\&\frac{\mathrm{d}v}{\mathrm{d}t}=-\lambda v+\alpha^2uv^2\\&\frac{\mathrm{d}w}{\mathrm{d}t}=\lambda_1 v\end{aligned}\right\},\tag{2}$$

are discussed in [11]. Turing type instability analysis and patterns formation behavior of the model (1) subject to the boundary conditions

$$\left.\begin{aligned}&\nabla u\cdot\boldsymbol{n}=0,\quad x\in\partial\Omega,\\&\nabla v\cdot\boldsymbol{n}=0,\quad x\in\partial\Omega,\end{aligned}\right\}\tag{3}$$

are discussed in [12]. Here ∇ denotes the gradient operator and $\boldsymbol{n}$ denotes the outward unit normal vector to the domain boundary $\partial\Omega$.

1.1. Constant Solutions (Steady States)

There are three constant solutions (homogeneous steady sates) $S_1\equiv(u_{s1},v_{s1})$, $S_2\equiv(u_{s2},v_{s2})$ and $S_3\equiv(u_{s3},v_{s3})$ for the system (1). Here $u_{s1}=1$, $v_{s1}=0$, $u_{s2}=\frac{\alpha-\sqrt{\alpha^2-4\lambda^2}}{2\alpha}$, $v_{s2}=\frac{\alpha+\sqrt{\alpha^2-4\lambda^2}}{2\alpha\lambda}$, $u_{s3}=\frac{\alpha+\sqrt{\alpha^2-4\lambda^2}}{2\alpha}$ and $v_{s3}=\frac{\alpha-\sqrt{\alpha^2-4\lambda^2}}{2\alpha\lambda}$ for $\alpha>2\lambda$.

1.2. Stationary Problem

In this article, the existence of the stationary solutions of the system (stationary problem corresponding to the system (1)):

$$\left.\begin{aligned}&\Delta u+\overbrace{\left(1-u-\alpha^2uv^2\right)}^{f(u(x),v(x))}=0,\ x\in\Omega\\&d\Delta v+\overbrace{\left(-\lambda v+\alpha^2uv^2\right)}^{g(u(x),v(x))}=0,\ x\in\Omega\end{aligned}\right\}\tag{4}$$

subject to no-flux boundary conditions (3), is discussed.

The main result presented in this article is the existence of non-constant positive solutions. These existence results are proved based on the Priori estimates and Topological Degree theory [13]-[15].

2. Priori Estimates

In this section we obtain estimates for the upper and lower bounds for the stationary solutions of the system (4).

This boundedness property can be expressed as the following theorem:

Theorem 1. *Let* (u,v) *be any solution of* (4) *except* S_1. *Then there exists a constant C such that*

$$\frac{1}{C} \le u(x), v(x) \le C$$

for $x \in \bar{\Omega}$, where $\bar{\Omega} = \Omega \cup \partial\Omega$.

Our main aim here is to prove the above theorem. In order to prove this, let us first prove following results:

Lemma 1. *Let* (u,v) *be any nontrivial solution of* (4). *Then* $0 \le u(x) \le 1$ *and* $v(x) \ge 0$ *for* $x \in \bar{\Omega}$. *Furthermore, if* $(u,v) \ne S_1$, *then* $v(x) > 0$ *for* $x \in \bar{\Omega}$.

Proof. Let $\underline{u}_0 = u(\underline{x}_0) = \min_{x\in\bar{\Omega}} u(x)$. Then applying maximum principle at $\underline{x}_0$ we get $f(u(\underline{x}_0), v(\underline{x}_0)) \le 0$. That is, $1 - u(\underline{x}_0) - \alpha^2 u(\underline{x}_0) v^2(\underline{x}_0) \le 0$, which implies

$$u(\underline{x}_0) \ge \frac{1}{1+\alpha^2 v^2(\underline{x}_0)} > 0. \tag{5}$$

Therefore, $\min_{x\in\bar{\Omega}} u(x) > 0$. Let $\bar{u}_0 = u(\bar{x}_0) = \max_{x\in\bar{\Omega}} u(x)$. Again applying maximum principle at $\bar{x}_0$ we get $f(u(\bar{x}_0), v(\bar{x}_0)) \ge 0$. That is, $1 - u(\bar{x}_0) - \alpha^2 u(\bar{x}_0) v^2(\bar{x}_0) \ge 0$, which implies $u(\bar{x}_0) \le \frac{1}{1+\alpha^2 v^2(\bar{x}_0)} \le 1$. That is $\max_{x\in\bar{\Omega}} u(x) \le 1$. Since $0 \le u(x) \le 1$, from the second equation of (4) we have $d\Delta v - \lambda v = -\alpha^2 uv^2 \le 0$ in Ω. Applying strong maximum principle to the above equation we get $v(x) > 0$ in $\bar{\Omega}$, provided $v(x) \ne 0$. The proof is complete. □

Lemma 2. *Assume that* (u,v) *is any solution of* (4). *If* $\lambda > d$, *then* $u(x) + dv(x) \le 1$ *for* $x \in \bar{\Omega}$.

Proof. Let $p = u + dv - 1$. Then

$$\Delta p - p = \Delta u + d\Delta v - u - dv + 1 = -1 + u + \lambda v - u - dv + 1 = (\lambda - d)v \ge 0.$$

Also, $\nabla p \cdot \boldsymbol{n} = \nabla u \cdot \boldsymbol{n} + d\nabla v \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. Then applying maximum principle we have $\max_{x\in\bar{\Omega}} p(x) \le 0$, which implies the required inequality. □

Lemma 3. *Assume that* (u,v) *is any solution of* (4). *If* $\lambda < d$, *then* $u(x) + dv(x) \le d/\lambda$ *for* $x \in \bar{\Omega}$.

Proof. Put $q = u + dv - d/\lambda$, Then

$$\Delta q - \frac{\lambda}{d} q = \Delta u + d\Delta v - q = -1 + u + \lambda v - \frac{\lambda}{d}\left(u + dv - \frac{d}{\lambda}\right) = \left(1 - \frac{\lambda}{d}\right) u \ge 0.$$

Since $\nabla q \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, the maximum principle gives the required inequality. □

Lemma 4. *Let* (u,v) *be any solution for* (4). *Then there exist a constant* $C_1(d,\lambda,\alpha) > 0$, *such that* $u(x) \ge C_1(d,\lambda,\alpha)$ *for* $x \in \bar{\Omega}$.

Proof. From lemma (1), we have

$$u(\underline{x}_0) \ge \frac{1}{1+\alpha^2 \left(v(\underline{x}_0)\right)^2}. \tag{6}$$

From lemma (2) we get $v(x) \le \frac{1-u(x)}{d} \le \frac{1}{d}$ for all $x \in \bar{\Omega}$. From lemma (3) we get $v(x) \le \frac{1}{d}\left(\frac{d}{\lambda} - u(x)\right) \le \frac{1}{\lambda}$. Combining these two inequalities we have $v \le \max\left\{\frac{1}{d}, \frac{1}{\lambda}\right\} = C^*$ (say). Then from (5) we have

$$u(\underline{x}_0) \ge C_1 = \frac{1}{1+\alpha^2 C^{*2}}. \tag{7}$$

Therefore, $u(x) \ge u(\underline{x}_0) \ge C_1$ for all $x \in \bar{\Omega}$. □

Lemma 5. *Assume that* (u,v) *is any solution of* (4) *except* $S_1 \equiv (1,0)$. *Then there exist a positive constant* C_2 *such that* $v(x) \ge C_2$ *for all* $x \in \bar{\Omega}$.

Proof. The second equation of the system (4) can be written as $\Delta v + Av = 0$ in Ω, where

$A(x)=(\lambda-uv)/d$. From lemmas (1) and (3) we get $u(x)\le 1$ and $v(x)\le \max\left\{\frac{1}{d},\frac{1}{\lambda}\right\}$ for any $x\in\Omega$. Then $\|A(x)\|_\infty \le \frac{1}{d}\left\{\lambda+\max\left\{\frac{1}{d},\frac{1}{\lambda}\right\}\right\}$. Set $\mu=\frac{1}{d}\left\{\lambda+\max\left\{\frac{1}{d},\frac{1}{\lambda}\right\}\right\}$. According to Harnack inequality [15] there exists a parameter $C_2'(N,\Omega,\mu)>0$ such that

$$\min_{x\in\bar{\Omega}} v(x)\ge C_2'(N,\Omega,\mu)\max_{x\in\bar{\Omega}} v(x). \tag{8}$$

Denote $\max_{x\in\bar{\Omega}} v(x)=v(\bar{x}_0)=\hat{v}$ and $\max_{x\in\bar{\Omega}} u(\bar{x}_0)=\hat{u}$. Then applying maximum principle for the second equation of (4), we have $\hat{v}(\hat{u}\hat{v}-\lambda)\ge 0$. Since $\hat{v}>0$, we get

$$\hat{v}\ge\frac{\lambda}{\hat{u}}\ge\lambda \quad \left(\because u(x)\le 1 \text{ for all } x\in\bar{\Omega}\right). \tag{9}$$

From the inequalities (8) and (9) we get $v(x)\ge \min_{x\in\bar{\Omega}} v(x)\ge C_2'(N,\Omega,\mu)\max_{x\in\bar{\Omega}} v(x)\ge \lambda C_2'(N,\Omega,\mu)$ for all $x\in\bar{\Omega}$. That is $v(x)\ge C_2$ for all $x\in\bar{\Omega}$, where $C_2=\lambda C_2'$. □

Proof of Theorem (1): From lemma (3) we have, $u(x)\ge \underline{u}\ge C_1=\frac{1}{1+\alpha^2C^{*2}}$, $v(x)\le\max\left\{\frac{1}{d},\frac{1}{\lambda}\right\}$ and, from lemma (5) we have $v(x)\ge C_2$ for all $x\in\bar{\Omega}$. Set

$$C=\max\left\{\frac{1}{C_1},\frac{1}{C_2},1,\max\left\{\frac{1}{d},\frac{1}{\lambda}\right\}\right\}. \tag{10}$$

Then we have $\frac{1}{C}\le u(x),v(x)\le C.$ □

3. Existence of Non Constant Stationary Solutions

In this section we investigate the existence of non-constant solutions to (4). For this, the degree theory for compact operators in Banach spaces [15] [16] are used as the main mathematical tool. Define the spaces Θ and Y as follows:

$$\Theta=\left\{(u,v)\in C(\bar{\Omega})\times C(\bar{\Omega}):\frac{1}{C}<u,v<C\right\},$$

$Y=\left\{(u,v)\in C^2(\bar{\Omega})\times C^2(\bar{\Omega}):\nabla u\cdot\boldsymbol{n}=\nabla v\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega\right\}$ and $Y^+=\left\{(u,v)\in Y:u,v>0\right\}$. Here C is the constant defined in Equation (10) and (u,v) is any solution of the system (4). Set an auxiliary parameter $d_t=td+(1-t)M$ for $t\in[0,1]$, where M is a large constant to be determined. Let $S=\boldsymbol{w}_*=(u_*,v_*)$ denote any constant solution of the system (4). Linearizing the system (4) when $d=d_t$ at S takes the form:

$$\left.\begin{aligned}&\Delta u+f_u(u_*,v_*)u+f_v(u_*,v_*)v=0, && x\in\Omega\\ &\Delta v+\frac{g_u(u_*,v_*)}{d_t}u+\frac{g_v(u_*,v_*)}{d_t}v=0, && x\in\Omega\\ &\nabla u\cdot\boldsymbol{n}=\nabla v\cdot\boldsymbol{n}=0 && x\in\partial\Omega.\end{aligned}\right\} \tag{11}$$

Denote

$$\boldsymbol{G}_t(\boldsymbol{w})=\begin{pmatrix} f(u,v)\\ \dfrac{g(u,v)}{d_t}\end{pmatrix},$$

and

$$A=\begin{pmatrix} f_u(u_*,v_*) & f_v(u_*,v_*) \\ \dfrac{g_u(u_*,v_*)}{d_t} & \dfrac{g_v(u_*,v_*)}{d_t} \end{pmatrix}.$$

Thus, $D_w\boldsymbol{G}_t(\boldsymbol{w}_*)=A$. Then (4) and (11) can be written as

$$\begin{aligned} &-\Delta\boldsymbol{w}=\boldsymbol{G}_t(\boldsymbol{w}) \text{ in } \Omega,\ \nabla\boldsymbol{w}=0 \text{ on } \partial\Omega, \\ &\text{and} -\Delta\boldsymbol{w}=A\boldsymbol{w}=D_w\boldsymbol{G}_t(\boldsymbol{w}_*) \text{ in } \Omega,\ \nabla\boldsymbol{w}=0 \text{ on } \partial\Omega, \end{aligned} \tag{12}$$

respectively. Define $\boldsymbol{T}_t(\boldsymbol{w})=(-\Delta+I)^{-1}(\boldsymbol{G}_t(\boldsymbol{w})+\boldsymbol{w})$, and $\boldsymbol{F}_t(\boldsymbol{w})=\boldsymbol{w}-\boldsymbol{T}_t(\boldsymbol{w})$. That is $\boldsymbol{F}_t(.)$ is a compact perturbation of the identity operator. According to the definition of Θ there is no fixed point of T on the boundary $\partial\Theta$. Thus, $\boldsymbol{w}$ is a positive solution of (12) if and only if $\boldsymbol{F}_t(\boldsymbol{w})=0$ in Y^+. So, the Leray-Schauder degree $\deg(\boldsymbol{F}_t(.),\Theta,0)$ is well defined. Furthermore, we have $D_w\boldsymbol{F}_t(\boldsymbol{w}_*)=I-(-\Delta+I)^{-1}(A+I)$.

The index of $\boldsymbol{F}_t$ at $\boldsymbol{w}_*$ is defined as

$$\text{Index}(\boldsymbol{F}_t(.),\boldsymbol{w}_*)=(-1)^{\sigma_*(t)},$$

where $\sigma_*(t)$ is the number of negative eigenvalues of $D_w\boldsymbol{F}_t(\boldsymbol{w}_*)$.

Lemma 6. *The eigenvalues,* μ *of* $D_w\boldsymbol{F}_t(\boldsymbol{w}_*)$ *are given by the equation*

$$(1+\mu_m)^2\mu^2+P\mu+Q=0, \tag{13}$$

where $P=(1+\mu_m)(p-2\mu_m)$ and $Q=\mu_m^2-p\mu_m+q$. Here p and q are the trace and determinant of the matrix A respectively and μ_m $(m=1,2,\cdots)$ are the positive eigenvalues of the eigenvalue problem

$$\left.\begin{aligned} -\Delta u=\mu u \quad & \text{in } \Omega \\ \frac{\partial u}{\partial \boldsymbol{n}}=0 \quad & \text{on } \partial\Omega \end{aligned}\right\}, \tag{14}$$

such that $\mu_1<\mu_2<\mu_3<\cdots$. Also the discriminant D of (13) is given by

$$D=P^2-4(1+\mu_m)^2Q=(1+\mu_m)^2(p^2-4q).$$

Proof. The eigenvalues μ of $D_w\boldsymbol{F}_t(\boldsymbol{w}_*)$ satisfies

$$\begin{aligned} &D_w\boldsymbol{F}_t(\boldsymbol{w}_*)=\mu\boldsymbol{w} \\ &(I-D_w\boldsymbol{T}_t(\boldsymbol{w}_*))\boldsymbol{w}=\mu\boldsymbol{w} \\ &\left(I-(-\Delta+I)^{-1}(A+I)\right)\boldsymbol{w}=\mu\boldsymbol{w} \\ &(-(\Delta+A))\boldsymbol{w}=\mu(-\Delta+I)\boldsymbol{w} \\ &((\mu-1)\Delta I-(\mu I+A))\boldsymbol{w}=0. \end{aligned}$$

This implies

$$\begin{vmatrix} (1-\mu)\mu_m-\mu-f_u(u_*,v_*) & f_v(u_*,v_*) \\ -d_t^{-1}g_u(u_*,v_*) & \mu_m(1-\mu)-\mu-d_t^{-1}g_v(u_*,v_*) \end{vmatrix}=0. \tag{15}$$

By simplifying we get

$$\begin{aligned} &(1+\mu_m)^2\mu^2+(1+\mu_m)\left(f_u(u_*,v_*)+d_t^{-1}g_v(u_*,v_*)-2\mu_m\right)\mu+\mu_m^2-d_t^{-1}g_v(u_*,v_*) \\ &+f_u(u_*,v_*)+d_t^{-1}\left(f_u(u_*,v_*)g_v(u_*,v_*)-g_u(u_*,v_*)f_v(u_*,v_*)\right)=0. \end{aligned}$$

This implies

$$(1+\mu_m)^2\mu^2+P\mu+Q=0,$$

where $P=(1+\mu_m)(p-2\mu_m)$ and $Q=\mu_m^2-p\mu_m+q$. The discriminant of (13) is

$$P^2-4(\mu_m+1)^2 Q=(\mu_m+1)^2\left[(p-2\mu_m)^2-4(\mu_m^2-p\mu_m+q)\right]$$
$$=(\mu_m+1)^2(p^2-4q).$$
□

Now we consider the cases $\alpha>2\lambda$ and $\alpha=2\lambda$ separately.

3.1. The Case $\alpha > 2\lambda$

In this case there are two constant fixed points of $\boldsymbol{T}_t$ in Θ which are $\boldsymbol{w}_*\equiv\boldsymbol{w}_2\equiv S_2\equiv(u_2,v_2)$ and $\boldsymbol{w}_3\equiv S_3\equiv(u_3,v_3)$. Now we deal with the case $\boldsymbol{w}_*\equiv(u_2,v_2)$. Let P_2, Q_2 and D_2 be corresponding P value, Q value and the discriminant of (13) respectively. Also let p_2 and q_2 be the corresponding p and q values.

3.1.1. The Case $\boldsymbol{w}_*\equiv(u_2,v_2)$

The solutions for μ of the Equation (13) can be written as

$$\mu^*=\frac{-(p_2-2\mu_m)+\sqrt{p_2^2-4q_2}}{2(1+\mu_m)}\quad\text{and}\quad\mu_*=\frac{-(p_2-2\mu_m)-\sqrt{p_2^2-4q_2}}{2(1+\mu_m)}.$$

If $p_2^2-4q_2>(p_2-2\mu_m)^2$ then $\mu^*>0$ and $\mu_*<0$. It can be shown that $Q_2=(p_2-2\mu_m)^2-(p_2^2-4q_2)$. That is, if $Q_2<0$ then only one negative solution exists for (13). It follows that if Q_2 is negative we can find m_1, m_2 $(0<m_1<m_2)$ such that $\mu_{m_1}<\mu_m<\mu_{m_2}$. Therefore, $\text{Index}(T_t,\boldsymbol{w}_2)=(-1)^{\sigma_2(t)}=(-1)^{(m_2-m_1-2)}$.

3.1.2. The Case $\boldsymbol{w}_*\equiv(u_3,v_3)$

Next we deal with the case $\boldsymbol{w}_*=(u_3,v_3)$. Let P_3, Q_3 and D_3 be corresponding P value, Q value and the corresponding discriminant of (13). Also let p_3 and q_3 be the corresponding p and q values. In this case we can find m_3, $(1<m_3)$ such that Q_3 is negative when $0<\mu_m<\mu_{m_3}$. Therefore there are exactly one negative solutions for the corresponding Equation (13) when $0<\mu_m<\mu_{m_3}$. Therefore $\text{Index}(T_t,\boldsymbol{w}_3)=(-1)^{m_3}$. Also,

$$\deg(I-T_t,\Theta,0)=\text{Index}(T_t,\boldsymbol{w}_2)+\text{Index}(T_t,\boldsymbol{w}_3)=(-1)^{(m_2-m_1-2)}+(-1)^{m_3}. \tag{16}$$

Theorem 2. *Assume that* $\alpha>2\lambda$, $Q_2<0$ *and* $Q_3<0$ *are satisfied. If* $m_3+(m_2-m_1)$ *is even, then* (4) *has at least one positive nontrivial solution.*

Proof. Homotopy invariance property show that

$$\deg(I-T_0,\Theta,0)=\deg(I-T_1,\Theta,0).$$

By setting $d_0=M$ as sufficiently large constant we get $\text{Index}(T_0,\boldsymbol{w}_2)=-1$, $\text{Index}(T_0,\boldsymbol{w}_3)=1$. Therefore,

$$\deg(I-T_0,\Theta,0)=\text{Index}(T_0,\boldsymbol{w}_2)+\text{Index}(T_0,\boldsymbol{w}_3)=0. \tag{17}$$

Also, we have

$$\begin{aligned}\deg(I-T_1,\Theta,0)&=\text{Index}(T_1,\boldsymbol{w}_2)+\text{Index}(T_1,\boldsymbol{w}_3)\\&=(-1)^{m_2-m_1-2}+(-1)^{m_3}=\pm2\end{aligned} \tag{18}$$

The relations (17) and (18) contradict the homotopy invariance property for $\deg(I-T_t,\Theta,0)$, $(0\le t\le1)$. Thus the proof is complete. □

3.2. The Case $\alpha = 2\lambda$

In this case the constant fixed point of $\boldsymbol{T}_t$ in Θ is uniquely determined by $\boldsymbol{w}_0=\left(\frac{1}{2},\frac{1}{2\lambda}\right)$. The Leray-Schauder index at this point is:

$$\text{Index}(T,w_0)=(-1)^{\sigma_0},$$

where σ_0 is the number of real negative eigenvalues (counting algebraic multiplicity) of $I-D_wT(w_0)$.

In this case $p = \frac{\lambda - 2d_t}{d_t}$ and $q = 0$. Then,

$$P = (1 + \mu_m)\left(\frac{\lambda - 2d_t}{d_t} - 2\mu_m\right)$$

and

$$Q = \mu_m^2 - p\mu_m = \mu_m\left(\frac{(\mu_m + 2)d_t - \lambda}{d_t}\right).$$

If $\mu_m = 0$:

Then $P = p = \frac{\lambda - 2d_t}{d_t}$ and $Q = 0$. Therefore, if $d_t < \lambda/2$, then $P > 0$. That is if $d_t < \lambda/2$, there is exactly one negative solution for (13). No negative solutions for (13) if $d_t \ge \lambda/2$.

If $\mu_m > 0$:

In this case, Q is negative if $d_t < \frac{\lambda}{2 + \mu_m}$. Then there is exactly one negative solution for (13).

Let m^* be the number of μ_m, satisfying $Q < 0$. Then, $\text{Index}(T_1, \boldsymbol{w}_1) = (-1)^{\sigma_1(1)} = (-1)^{m^*}$.

Theorem 3. *Assume that* $\alpha = 2\lambda$. *If* $\sigma_1(1) = m^*$ *is odd, then* (4) *admits at least one positive non-constant solution.*

Proof. From the Homotopy invariance property we have

$$\deg(I - T_0, \Theta, 0) = \deg(I - T_1, \Theta, 0).$$

Suppose that (4) has no non-constant solutions if $d_t = d$. Also

$$\deg(I - T_0, \Theta, 0) = \text{Index}(T_0, \boldsymbol{w}_1) = 1,$$

provided $d_0 = M$ is sufficiently large. On the other hand

$$\deg(I - T_1, \Theta, 0) = \text{Index}(T_1, \boldsymbol{w}_1) = (-1)^{\sigma_1(1)} = -1.$$

These two relations contradict the homotopy invariance property for $\deg(I - T_t, \Theta, 0)$, $(0 \le t \le 1)$. Thus the proof is complete. □

4. Discussion

Stationary problem corresponding to a model mathematical model for formation of coral patterns is considered. We have proved the existence of non-constant positive solutions of the stationary problem (4). Existence of non-constant solutions to the stationary problem gives a guarantee for the existence of spatially variant time invariant solutions to the proposed reaction-diffusion system. In other words, the solution of the system reaches a steady state with spatial patterns. This is a physically important feature which gurantees the the existence of stable coral patterns of the system.

References

[1] Merks, R.M.H. (2003) Models of Coral Growth: Spontaneous Branching, Compactification and Laplacian Growth Assumption. *Journal of Theoretical Biology*, **224**, 153-166. http://dx.doi.org/10.1016/S0022-5193(03)00140-1

[2] Castro, P. and Huber, M.E. (1997) Marine Biology. WCB/McGraw-Hill, New York.

[3] Kaandorp, J.A., *et al*. (1996) Effect of Nutrient Diffusion and Flow on Coral Morphology. *Physical Review Letters*, **77**, 2328-2331. http://dx.doi.org/10.1103/PhysRevLett.77.2328

[4] Kaandorp, J.A., *et al*. (2005) Morphogenesis of the Branching Reef Coral Madracis Mirabilis. *Proceedings of the Royal Society B*, **77**, 127-133. http://dx.doi.org/10.1098/rspb.2004.2934

[5] Kaandorp, J.A., *et al*. (2008) Modelling Genetic Regulation of Growth and Form in a Branching Sponge. *Proceedings of the Royal Society B*, **275**, 2569-2575. http://dx.doi.org/10.1098/rspb.2008.0746

[6] Merks, R.M.H. (2003) Branching Growth in Stony Corals: A Modelling Approach. Ph.D. Thesis, Advanced School of Computing and Imaging, University of Amsterdam, Amsterdam.

[7] Merks, R.M.H. (2003) Diffusion-Limited Aggregation in Laminar Flows. *International Journal of Modern Physics C*, **14**, 1171-1182. http://dx.doi.org/10.1142/S0129183103005297

[8] Mistr, S. and Bercovici, D. (2003) A Theoretical Model of Pattern Formation in Coral Reefs. *Ecosystems*, **6**, 61-74. http://dx.doi.org/10.1007/s10021-002-0199-0

[9] Maxim, V.F., *et al*. (2010) A Comparison between Coral Colonies of the Genus Madracis and Simulated Forms. *Proceedings of the Royal Society B*, **277**, 3555-3561. http://dx.doi.org/10.1098/rspb.2010.0957

[10] Merks, R.M.H. (2010) Problem Solving Environment for Modelling Stony Coral Morphogenesis.

[11] Somathilake, L.W. and Wedagedera, J.R. (2012) On the Stability of a Mathematical Model for Coral Growth in a Tank. *British Journal of Mathematics and Computer Science*, **2**, 255-280. http://dx.doi.org/10.9734/BJMCS/2012/1387

[12] Somathilake, L.W. and Wedagedera, J.R. (2014) A Reaction-Diffusion Type Mathematical Model for Formation of Coral Patterns. *Journal of National Science Foundation*, **42**, 341-349.

[13] Kien, B.T., *et al*. (2007) On the Degree Theory for General Mappings of Monotone Type. *Journal of Mathematical Analysis and Applications*, **340**, 707-720. http://dx.doi.org/10.1016/j.jmaa.2007.07.058

[14] Dhruba, R.A. (2007) Applications of Degree Theories to Nonlinear Operator Equations in Banach Spaces. Ph.D. Thesis, University of South Florida, Tampa.

[15] Norihiro, S. (2007) A Study on the Set of Stationary Solutions for the Gray-Scott Model. Ph.D. Thesis, Waseda University, Japan.

[16] Mawhin, J. (1999) Leray-Schauder Degree: A Half Century of Extension and Applications. *Topological Methods in Nonlinear Analysis*, **14**, 195-228.