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Zappa-Szép Products of Semigroups

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Abstract

The internal Zappa-Szép products emerge when a semigroup has the property that every element has a unique decomposition as a product of elements from two given subsemigroups. The external version constructed from actions of two semigroups on one another satisfying axiom derived by G. Zappa. We illustrate the correspondence between the two versions internal and the external of Zappa-Szép products of semigroups. We consider the structure of the internal Zappa-Szép product as an enlargement. We show how rectangular band can be described as the Zappa-Szép product of a left-zero semigroup and a right-zero semigroup. We find necessary and sufficient conditions for the Zappa-Szép product of regular semigroups to again be regular, and necessary conditions for the Zappa-Szép product of inverse semigroups to again be inverse. We generalize the Billhardt λ-semidirect product to the Zappa-Szép product of a semilattice E and a group G by constructing an inductive groupoid.

Keywords

Inverse Semigroups, Groups, Semilattice, Rectangular Band, Semidiret, Regular, Enlargement, Inductive Groupoid

1. Introduction

The Zappa-Szép product of semigroups has two versions internal and external. In the internal one we suppose that S is a semigroup with two subsemigroups A and B such that each $s \in S$ can be written uniquely as $s = ab$ with $a \in A$ and $b \in B$. Then since $ba \in S$, we have $ba = a'b'$ with $a' \in A$ and $b' \in B$ determined uniquely by a and b. We write $a' = b \cdot a$ and $b' = b^a$. Associativity in S implies that the functions $(a,b) \mapsto b \cdot a$ and $(a,b) \mapsto b^a$ satisfy axioms first formulated by Zappa [1]. In the external version we assume that we have semigroups A and B and assume that we have maps $A \times B \to A$, defined by $(a,b) \mapsto b \cdot a$ and a map $A \times B \to B$ defined by $(a,b) \mapsto b^a$ which satisfy Zappa axioms [1].

For groups, the two versions are equal, but as we show in this paper for semigroups this is true for only some

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special kinds of semigroups.

Zappa-Szép products of semigroups provide a rich class of examples of semigroups that include the self-similar group actions [2]. Recently, [3] uses Li's construction of semigroup C^*-algebras to associate a C^*-algebra to Zappa-Szép products and gives an explicit presentation of the algebra. They define a quotient C^*-algebra that generalises the Cuntz-Pimsner algebras for self-similar actions. They specifically discuss the Baumslag-Solitar groups, the binary adding machine, the semigroup $N \rtimes N^{\times}$, and the $ax+b$-semigroup $Z \rtimes Z^{\times}$.

In [4] they study semigroups possessing E-regular elements, where an element a of a semigroup S is E-regular if a has an inverse a° such that $aa^{\circ}, a^{\circ}a$ lie in $E \subseteq E(S)$. They also obtain results concerning the extension of (one-sided) congruences, which they apply to (one-sided) congruences on maximal subgroups of regular semigroups. They show that a reasonably wide class of $\tilde{D}_E$-simple monoids can be decomposed as Zappa-Szép products.

In [5] we look at Zappa-Szép products derived from group actions on classes of semigroups. A semidirect product of semigroups is an example of a Zappa-Szép product in which one of the actions is taken to be trivial, and semidirect products of semilattices and groups play an important role in the structure theory of inverse semigroups. Therefore Zappa-Szép products of semilattices and groups should be of particular interest. We show that they are always orthodox and $\mathfrak{I}$-unipotent, but are inverse if and only if the semilattic acts trivially on the group, that is when we have the semidirect product. In [5] we relate the construction (via automata theory) to the λ-semidirect product of inverse semigroups devised by Billhardt.

In this paper we give general definitions of the Zappa-Szép product and include results about the Zappa-Szép product of groups and a special Zappa-Szép product for a nilpotent group.

We illustrate the correspondence between the internal and external versions of the Zappa-Szép product. In addition, we give several examples of both kinds. We consider the structure of the internal Zappa-Szép product as an enlargement. We show how a rectangular band can be described as the Zappa-Szép product of a left-zero semigroup and a right-zero semigroup.

We characterize Green's relations ($\mathcal{L}$ and $\mathcal{R}$) of the Zappa-Szép product $M \bowtie G$ of a monoid M and a group G. We prove some results about regular and inverse Zappa-Szép product of semigroups.

We construct from the Zappa-Szép product of a semilattice E and a group G, an inverse semigroup by constructing an inductive groupoid.

We rely on basic notions from semigroup theory. Our references for this are [6] and [7].

2. Internal Zappa-Szép Products

Let S be a semigroup with subsemigroups A and B such that each element $s \in S$ is uniquely expressible in the form $s = ab$ with $a \in A$ and $b \in B$. We say that S is the "internal" Zappa-Szép product of A and B, and write $S = A \bowtie B$. Since $ba \in S$ with $b \in B$ and $a \in A$, we must have unique elements $a' \in A$ and $b' \in B$ so that $ba = a'b'$. This defines two functions $(a,b) \mapsto b \cdot a$ and $(a,b) \mapsto b^a$. Since $ba \in S$ with $b \in B$ and $a \in A$, we must have unique elements $a' \in A$ and $b' \in B$ so that $ba = a'b$. Write $a' = b \cdot a$ and $b' = b^a$. This defines two function $A \times B \to A$, $(a,b) \mapsto b \cdot a$ and $A \times B \to B$, $(a,b) \mapsto b^a$. Thus $ba = (b \cdot a) b^a$. Using these definitions, we have for all $a, a' \in A$ and $b, b' \in B$ that

$$(ab)(a'b') = a(b \cdot a') b^{a'} b'.$$

Thus the product in S can be described in terms of the two functions. Using the associativity of the semigroup S and the uniqueness property, we deduce the following axioms for the two functions. By the associativity of S, we have

$$b(aa') = (ba)a'.$$

Now

$$b(aa') = [b \cdot (aa')] b^{aa'}$$

and

$$(ba)a' = (b \cdot a) b^a a' = (b \cdot a)(b^a \cdot a')(b^a)^{a'}.$$

Thus, by uniqueness property, we have the following two properties

(ZS2) $b\cdot(aa')=(b\cdot a)(b^{a}\cdot a')$.

(ZS3) $(b^{a})^{a'}=b^{aa'}$.

Similarly by the associativity of S, we have

$$(bb')a=b(b'a).$$

Now

$$(bb')a=(bb'\cdot a)(bb')^{a}$$

and

$$b(b'a)=b((b'\cdot a)b'^{a})=b\cdot(b'\cdot a)b^{b'\cdot a}b'^{a}.$$

Thus, by uniqueness property, we have the following two properties

(ZS1) $bb'\cdot a=b\cdot(b'\cdot a)$.

(ZS4) $(bb')^{a}=b^{b'\cdot a}b'^{a}$.

In the following we illustrate which subsemigroups may be involved in the internal Zappa-Szép product.

Lemma 1. *If the semigroup S is the internal Zappa-Szép product of A and B then* $A\cap B=RI(A)\cap LI(B)$.

Proof. Consider $x\in A\cap B$. Then since $(ax)b=a(xb)$ we have $ax=a$ for all $a\in A, xb=b$ for all $b\in B$. Thus x is a right identity for A and left identity for B, whereupon $A\cap B\subseteq RI(A)\cap LI(B)$. Observe that $RI(A)\cap LI(B)\subseteq A\cap B$, thus $A\cap B=RI(A)\cap LI(B)$. □

Of course, if S is a monoid and A and B are submonoids then $RI(A)\cap LI(B)=\{1_S\}$.

Proposition 1. *If* $S=A\bowtie B$ *the internal Zappa-Szép product of A and B, then* $RI(A)\cap LI(B)\neq\varnothing$.

Proof. We use Brin's ideas in [8] Lemma 3.4. If $a\in A$ then $a=a'\beta(a)$ for unique $a'\in A$ and $\beta(a)\in B$, giving us a function $\beta:A\to B$, and likewise, if $b\in B$ then $b=\alpha(b)b'$ for some unique $b'\in B$ and some function $\alpha:B\to A$. But for all $b\in B$ we have $ab=a'\beta(a)b$, and therefore $a=a'$ and $\beta(a)b=b$: that is $\beta(a)$ is a left identity for B. Similarly, $b=b'$ and $\alpha(b)$ is a right identity for A. In particular, $\alpha(b)$ and $\beta(a)$ are idempotents. Now

$$a_1a_2=a_1a_2\beta(a_1a_2)=a_1a_2\beta(a_2)=a_1\beta(a_1)a_2=a_1(\beta(a_1)\cdot a_2)\beta(a_1)^{a_2}.$$

Therefore

$$\begin{gathered}a_1a_2=a_1(\beta(a_1)\cdot a_2)\\ \beta(a_1a_2)=\beta(a_2)=\beta(a_1)^{a_2}\end{gathered}\tag{1}$$

Similarly

$$b_1b_2=\alpha(b_1b_2)b_1b_2=b_1\alpha(b_2)b_2=(b_1\cdot\alpha(b_2))(b_1^{\alpha(b_2)}b_2)=\alpha(b_1)b_1b_2.$$

Therefore

$$\begin{gathered}\alpha(b_1b_2)=\alpha(b_1)=b_1\cdot\alpha(b_2)\\ b_1b_2=b_1^{\alpha(b_2)}b_2.\end{gathered}\tag{2}$$

Set $a_1=a$ and for any $b\in B, a_2=\alpha(b)$ in (1):

$$\beta(a)=\beta(a\alpha(b))=\beta(\alpha(b)).$$

Hence β is constant: $\beta(a)=f\in B$ for all $a\in A$. Similarly, setting $b_1=\beta(a)$ and $b_2=b$ in (2):

$$\alpha(b)=\alpha(\beta(a)b)=\alpha(\beta(a))$$

Hence α is constant: $\alpha(b)=e\in A$ for all $b\in B$. But now we have that for all $a\in A$ and $b\in B, ae=a=af$ and $fb=b=eb$. But then putting $a=e$ and $b=f$ we have $e^2=e=ef=f=f^2$ and in particular $e=f$. □

Lemma 2. *Let* $S = A \bowtie B$, *the internal Zappa-Szép product of A and B and* $e \in E(S)$ *be a right identity for A and a left identity for B. Then* $e^a = e = (b \cdot e)$ *and* $ea = e \cdot a, be = b^e$.

Proof. We have $e \in A \cap B, a \in A, b \in B$, then $ea = (e \cdot a)e^a$, but $ae = a$, thus $ea = e(ae) = (ea)e$ and by uniqueness we have $ea = e \cdot a$, $e = e^a$.

Similarly, since $eb = b$ we have $be = (b \cdot e)b^e$. Also $be = (eb)e = e(be)$. Thus $e = b \cdot e, be = b^e$. Hence $e^a = e = (b \cdot e)$ and $ea = e \cdot a, be = b^e$. □

In an internal Zappa-Szép product $S = A \bowtie B$, we find an idempotent $e \in RI(A) \cap LI(B)$. This shows (for example) that a free semigroup cannot be a Zappa-Szép product. But in a monoid Zappa-Szép product $M = A \bowtie B$ of submonoids *A* and *B* the special idempotent $e \in RI(A) \cap LI(B)$ must be 1_M, since we have $1_M = ab$ uniquely. Then for all $x \in A, x = xe = x1_M = x(ab) = (xa)b$ and thus $x = xa$ and $b = e$. Similarly $a = e$ and $1_M = ab = e^2 = e$.

In the following we give a definition of the enlargement of a semigroup introduced in [8] for regular semigroups, and in [9] this concept is generalized to non-regular semigroups by describing a condition (enlargement) under which a semigroup *T* is covered by a Rees matrix semigroup over a subsemigroup. We describe the enlargement concept for internal Zappa-Szép products.

Definition 1. *A semigroup T is an enlargement of a subsemigroup S if* $STS = S$ *and* $TST = T$.

Example 1. [9] *Let S be any semigroup and let I be a set of idempotents in S such that* $S = SIS$. *Then S is an enlargement of ISI because* $S(ISI)S = (SIS)IS = SIS = S$, *and* $(ISI)S(ISI) = I(SIS)(ISI) = IS(ISI) = I(SIS)I = ISI$. *If* $I = \{e\}$ *and* $S = SeS$, *then S is an enlargement of the local submonoid* eSe.

Proposition 2. *Let S be the internal Zappa-Szép product of subsemigroups A and B. Then S is an enlargement of a local submonoid eSe for some* $e \in RI(A) \cap LI(B)$, *and eSe is the internal Zappa-Szép product of the submonoids* $\bar{A}$ *and* $\bar{B}$ *where* $\bar{A} = \{x : x = ea, a \in A\}, \bar{B} = \{y : y = be, b \in B\}$.

Proof. We have $e \in E(S)$ such that *e* is a right identity for *A* and a left identity for *B*. Then $S = AB = AeeB = AeB \subseteq SeS \subseteq S$ so $S = SeS$ for $e \in E(S)$. So *S* is an enlargement of the local submonoid eSe (eSe is a monoid with identity *e*). It is clear that $\bar{A}$ and $\bar{B}$ are submonoids of eSe. We must show that each element $z \in eSe$ is uniquely expressible as $z = xy$ with $x \in \bar{A}, y \in \bar{B}$. If $z \in eSe$ then $z = ese, s \in S$. But $s = ab$ for unique $a \in A, b \in B$, and so $z = e(ab)e = (ea)(be) = xy$, where $x = ea \in \bar{A}, y = be \in \bar{B}$. Since $\bar{A} \subseteq A$ and $\bar{B} \subseteq B$ this expression is unique, because $z \in S = A \bowtie B$. Therefore each element $z \in eSe$ is uniquely expressible as $z = xy$ with $x \in \bar{A}, y \in \bar{B}$. □

We note that if $T = A \bowtie B$ such that *T* is an enlargement of $S = eTe = \bar{A} \bowtie \bar{B}$, where $\bar{A} \subseteq A$ and $\bar{B} \subseteq B$, if $\bar{A}, \bar{B}$ are regular with the assumption that if $ea = ea'$ then $a = a'$ and if $be = b'e$ then $b = b'$. Then *A*, *B* are regular, since if $\bar{A}$ is regular monoid, then for each $x_1 \in \bar{A}$ there exists $x_2 \in \bar{A}$ such that $x_1x_2x_1 = x_1, x_2x_1x_2 = x_2$. Now $x_1x_2x_1 = x_1$ which implies $ea_1ea_2ea_1 = ea_1a_2a_1 = ea_1$. Since *A* has a right identity *e*, then $a_1a_2a_1 = a_1$. Similarly we get $a_2a_1a_2 = a_2$. Thus *A* is regular. Similarly, we get *B* is regular.

Following [9] we describe the Rees Matrix cover for the Zappa-Szép product $T = A \bowtie B$ such that $T = TeT$ and is an enlargement of $S = eTe$ for some idempotent $e \in T$ where $S^2 = S$ such that $Ae = A$ and $eB = B$ and *S* is the Zappa-Szép product of $\bar{A}$ and $\bar{B}$, $S = \bar{A} \bowtie \bar{B}$ with $\bar{A} = \{x : x = ea, a \in A\} \subseteq T$ and $\bar{B} = \{y : y = be, b \in B\} \subseteq T$. Then by Corollary 4 in [9] the Rees matrix semigroup is given by $M = M(S;U,V;P) = M(eTe;\bar{A},\bar{B};P)$ such that $T = \bigcup_{x \in A} xT = \bigcup_{y \in B} Ty$ since $T = TST$. For each $z \in \bar{A} \cup \bar{B}$, we can find $r_z \in TS$ and $r'_z \in ST$ such that $z = r_z r'_z$. So if $x \in \bar{A}$, then $r_x = e \in TS$ and $r'_x = a \in ST$ such that $x = ea = eeae \in TSST$. Similarly, for $y \in \bar{B}$, $y = be = r_b r'_b$, therefore $r_b = b$ and $r'_b = e$. Now for each $z \in \bar{A} \cup \bar{B}$, fix elements $r_z, r'_z \in T$, and define $\bar{B} \times \bar{A}$ matrix *P* by putting $p_{yx} = r'_y r_x \in (ST)(TS) \subseteq STS = S$. Thus $M = M(eTe;\bar{A},\bar{B};P)$ is the Rees matrix cover for $T = A \bowtie B$ where the map $\theta : M(S;\bar{A},\bar{B};P) \to T$ defined by $\theta(x,s,y) = r_x s r'_y = \theta(ea,s,be) = easbe$ is the covering map (θ is a strict local isomorphism from *M* to *T* along which idempotents can be lifted).

3. Green's Relations $\mathcal{L}$ and $\mathcal{R}$ on Zappa-Szép Products

In this Section we give some general properties of the Zappa-Szép product. We characterize Green's relations ($\mathcal{L}$ and $\mathcal{R}$) of the Zappa-Szép product $M \bowtie G$ of a monoid *M* and a group *G*.

Proposition 3. [10] *Let $A \rhd\lhd B$ be a Zappa-Szép product of semigroups A and B. Then*

(i) $(a_1,b_1)\mathcal{L}(a_2,b_2) \Rightarrow b_1\mathcal{L}b_2$ in *B*;

(ii) $(a_1,b_1)\mathcal{R}(a_2,b_2) \Rightarrow a_1\mathcal{R}a_2$ in *A*.

Proof. Suppose $(a_1,b_1)\mathcal{L}(a_2,b_2)$ in $A \rhd\lhd B$, then there exist $(x_1,y_1),(x_2,y_2) \in A \rhd\lhd B$ such that $(x_1,y_1)(a_1,b_1)=(a_2,b_2)$ and $(x_2,y_2)(a_2,b_2)=(a_1,b_1)$. Then

$$\left(x_1\left(y_1\cdot a_1\right), y_1^{a_1}b_1\right)=\left(a_2,b_2\right)$$

and

$$\left(x_2\left(y_2\cdot a_2\right), y_2^{a_2}b_2\right)=\left(a_1,b_1\right).$$

Hence

$$x_1\left(y_1\cdot a_1\right)=a_2,\ y_1^{a_1}b_1=b_2,\ x_2\left(y_2\cdot a_2\right)=a_1,\ y_2^{a_2}b_2=b_1.$$

It follows that $b_1\mathcal{L}b_2$ in *B*. Similar proof for (ii). □

Proposition 4. *In the Zappa-Szép product $M \rhd\lhd G$ of a monoid M and a group G. Then*

$$(m,g)\mathcal{R}(n,h) \Leftrightarrow m\mathcal{R}n \text{ in } M.$$

Proof. By Proposition 3 we have $(m,g)\mathcal{R}(n,h)$ implies that $m\mathcal{R}n$ in *M*. To prove the converse suppose that $m\mathcal{R}n$ in *M* then there exist z_1 and z_2 in *M* such that $mz_1=n$ and $nz_2=m$. To show that $(m,g)\mathcal{R}(n,h)$ we have to find (x_1,y_1) and (x_2,y_2) in $M \rhd\lhd G$ such that

$$(m,g)(x_1,y_1)=(n,h)$$

$$(n,h)(x_2,y_2)=(m,g).$$

Then $m(g\cdot x_1)=n, g^{x_1}y_1=h, n(h\cdot x_2)=m$ and $h^{x_2}y_2=g$. Hence

$$y_1=\left(g^{x_1}\right)^{-1}h \text{ and } y_2=\left(h^{x_2}\right)^{-1}g.$$

Therefore we set $x_1=g^{-1}\cdot z_1, x_2=h^{-1}\cdot z_2$. Hence

$$(m,g)(x_1,y_1)=(m,g)\left(g^{-1}\cdot z_1,\left(g^{g^{-1}\cdot z_1}\right)^{-1}h\right)=\left(m\left(g\cdot g^{-1}\cdot z_1\right), g^{g^{-1}\cdot z_1}\left(g^{g^{-1}\cdot z_1}\right)^{-1}h\right)$$

$$=\left(m\left(gg^{-1}\cdot z_1\right),h\right)=(mz_1,h)=(n,h)$$

Similarly $(n,h)(x_2,y_2)=(m,g)$. Hence $(m,g)\mathcal{R}(n,h)$. □

But from the following example we conclude that the action of the group *G* is a group action is a necessary condition.

Example 2. *Let $A=\{e,t,f,b\}$ be a Clifford semigroup with the following multiplication table. Note that $\{e,t\}\cong C_2$ and $\{f,b\}\cong C_2$,*

	e	*t*	*f*	*b*
e	*e*	*t*	*f*	*b*
t	*t*	*e*	*b*	*f*
f	*f*	*b*	*f*	*b*
b	*b*	*f*	*b*	*f*

Let $(\mathbb{Z},+)$, the group of integers. Suppose that the action of $\mathbb{Z}$ on *A* for each $m\in\mathbb{Z}$ is as follows:

$$\boxed{m\cdot e=f, m\cdot t=b, m\cdot f=f, m\cdot b=b.}$$

Observe that $m\cdot a=fa$ for all $a\in A$. The action of *A* on $\mathbb{Z}$ as follows:

$$\boxed{m^e = m, m^f = m, m^t = -m, m^b = -m.}$$

Thus Zappa-Szép axioms are satisfied, since define $\phi: A \to \{f, b\} \cong C_2$ by

$$e\phi = f\phi = f, t\phi = b\phi = b$$

is a morphism (this is easy to see from the fact that $\phi(a) = af$). Now define $\psi: \{f, b\} \to S_{\mathbb{Z}}$ (where $S_{\mathbb{Z}}$ is the group of permutations on $\mathbb{Z}$) by

$$f\psi = I_{\mathbb{Z}}, b\psi = \alpha$$

where $m\alpha = -m,$ for all $m \in \mathbb{Z}$. Clearly ψ is a morphism (of groups). Now for $a \in A$ and $m \in \mathbb{Z}$ we define the action arises from the composition $\phi\psi$ as follows

$$(a, m) \mapsto m^a \text{ by } m^a = m(a\phi\psi)$$

and

$$(a, m) \mapsto m \cdot a \text{ by } m \cdot a = fa.$$

We therefore have $(ZS1), (ZS3), (ZS2)$ and $(ZS4)$ as following.

For $(ZS1)$: $m \cdot (m' \cdot a) = m \cdot (fa) = f^2 a = fa = (m + m') \cdot a,$ and for $(ZS3)$:

$$m^{aa'} = m((aa')\phi\psi) = m(a\phi\psi)(a'\phi\psi) = (m^a)(a'\phi\psi) = (m^a)^{a'},$$

For $(ZS2)$: $m \cdot (aa') = faa',$ and using $af = fa$ for all $a \in A$ we have

$$(m \cdot a)(m^a \cdot a') = fafa' = faa'$$

For $(ZS4)$:

$$(m + m')^a = (m + m')(a\phi\psi) = \begin{cases} m + m' & \text{if } a \in \{e, f\}, \\ -m - m' & \text{if } a \in \{t, b\}. \end{cases}$$

and

$$m^{m' \cdot a} + m'^a = m^{fa} + m'^a = \begin{cases} m + m' & \text{if } a \in \{e, f\}, \\ -m - m' & \text{if } a \in \{t, b\}. \end{cases}$$

Thus $M = A \bowtie \mathbb{Z} = \{(a, m) : a \in A, m \in B\}$ the Zappa-Szép product of A and B. The set $E(M) = \{(f, 0)\},$ since $(f, 0)(f, 0) = (f(0 \cdot f), 0^f 0) = (f, 0)$ and $(a, m) \in E(M)$ if and only if $a = a(m \cdot a)$ and $m = m^a m,$ since $m^a = m$ or $-m$ for all $a \in A$ so $m^a + m = 2m$ or $0 \neq m.$

Now, we note that $1_{\mathbb{Z}}$ acts non-trivially. We have $e\mathcal{R}t$ and $b\mathcal{R}f$ in A but (e, m) not $\mathcal{R}$-related to (t, n) where $m, n \in \mathbb{Z}$ since if we suppose $(e, m)\mathcal{R}(t, n)$ then there exist (x_1, y_1) and (x_2, y_2) in $A \bowtie \mathbb{Z}$ such that

$$(e, m)(x_1, y_1) = (t, n)$$

$$(e(m \cdot x_1), m^{x_1} y_1) = (t, n)$$

But $e(m \cdot x_1) \in \{f, b\}$ so (e, m) not $\mathcal{R}$-related to (t, n). To calculate the $\mathcal{R}$-class of $(e, m),$ if $(e, m)\mathcal{R}(a, n)$ then $e\mathcal{R}a$ and so $a = e$ or $a = t$ we prove $a = t$ is impossible so $a = e.$ If $(e, m)\mathcal{R}(e, n)$ for some $n \in \mathbb{Z}$ then we have

$$(e, m)(x_1, y_1) = (e, n)$$

$$(e(m \cdot x_1), m^{x_1} y_1) = (e, n)$$

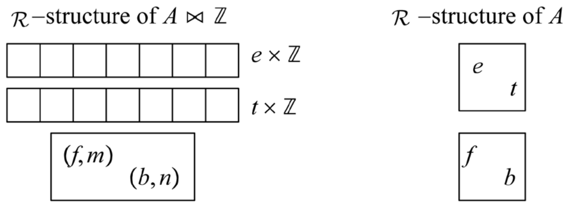
But $e(m \cdot x_1) \in \{f, b\}$ so (e, m) $\mathcal{R}$-related only to itself. Similarly (t, n) $\mathcal{R}$-related only to itself. To calculate the $\mathcal{R}$-class of (f, m) suppose $(f, m)\mathcal{R}(a, n)$ then $a = f$ or $a = b.$ Let $a = f$ so

$(f,m)\mathcal{R}(f,n)$ then

$$(f,m)(x_1,y_1)=(f,n)$$

$$(f,n)(x_2,y_2)=(f,m)$$

and so $(m\cdot x_1,,m^{x_1}+y_1)=(f,n)$ which implies $x_1=e$ or $x_1=f$ so $m^{x_1}=m$. Thus $(f,m)\mathcal{R}(f,n)$ for all $m,n\in\mathbb{Z}$. By similar calculation we have $(f,m)\mathcal{R}(b,n)$. So if $1_{\mathbb{Z}}$ acts non-trivially we have a different structure for the $\mathcal{R}$ -classes of $A\rhd\!\lhd\mathbb{Z}$ and A.



Proposition 5. *Let* $M\rtimes G$ *be the semidirect product of a monoid M and a group G. Then*

$$(m,g)\mathcal{L}(n,h)\Leftrightarrow\left(g^{-1}\cdot m\right)\mathcal{L}\left(h^{-1}\cdot n\right)\text{ in }M.$$

Proof. Suppose that $(m,g)\mathcal{L}(n,h)$ then there exist $(x_1,y_1),(x_2,y_2)$ in $M\rtimes G$ such that

$$(x_1,y_1)(m,g)=(n,h)$$

$$(x_2,y_2)(n,h)=(m,g)$$

Then $(x_1(y_1\cdot m),y_1g)=(n,h)$ and $(x_2(y_2\cdot n),y_2h)=(m,g)$. Hence $y_1=hg^{-1}$, $y_2=gh^{-1}=y_1^{-1}$ therefore

$$x_1(y_1\cdot m)=x_1\left(hg^{-1}\cdot m\right)=n$$

which implies

(1) $\left(h^{-1}\cdot x_1\right)\left(g^{-1}\cdot m\right)=h^{-1}\cdot n$

and

$$x_2(y_2\cdot n)=x_2\left(gh^{-1}\cdot n\right)=m$$

which implies

(2) $\left(g^{-1}\cdot x_2\right)\left(h^{-1}\cdot n\right)=g^{-1}\cdot m$

Thus by (1) and (2) we have $\left(g^{-1}\cdot m\right)\mathcal{L}\left(h^{-1}\cdot n\right)$ in *M*.

Now suppose $\left(g^{-1}\cdot m\right)\mathcal{L}\left(h^{-1}\cdot n\right)$ in *G* then there exist z_1 and z_2 in *G* such that $z_1\left(g^{-1}\cdot m\right)=h^{-1}\cdot n$ and $z_2\left(h^{-1}\cdot n\right)=g^{-1}\cdot m$. Therefore $(h\cdot z_1)\left(hg^{-1}\cdot m\right)=n.$ Hence

$$(n,h)=\left((h\cdot z_1)\left(hg^{-1}\cdot m\right),h\right)=\left(h\cdot z_1,hg^{-1}\right)(m,g)$$

Therefore we set $(x_1,y_1)=\left(h\cdot z_1,hg^{-1}\right)$ in $M\rtimes G$ other formula by symmetry $(x_2,y_2)=\left(g\cdot z_2,gh^{-1}\right)$ so $(m,g)\mathcal{L}(n,h)$. □

Proposition 6. *If* $\left(g^{-1}\cdot m\right)\mathcal{L}\left(h^{-1}\cdot n\right)$ *such that* $\left(g^{-1}\right)^m=g^{-1}$ *and* $\left(h^{-1}\right)^n=h^{-1}$ *in G, then* $(m,g)\mathcal{L}(n,h)$ *in* $M\rhd\!\lhd G$.

Proof. Suppose $\left(g^{-1}\cdot m\right)\mathcal{L}\left(h^{-1}\cdot n\right)$ then there exist z_1 and z_2 in *M* such that $z_1\left(g^{-1}\cdot m\right)=h^{-1}\cdot n$ which implies that

$$n=(h\cdot z_1)\left(h^{z_1}g^{-1}\cdot m\right)\text{ and }z_2\left(h^{-1}\cdot n\right)=g^{-1}\cdot m.$$

We set $(x_1, y_1) = (h \cdot z_1, h^{z_1} g^{-1})$ and $(x_2, y_2) = (g \cdot z_2, g^{z_2} h^{-1})$ then

$$(x_1, y_1)(m, g) = (x_1(y_1 \cdot m), y_1^m g) = \left((h \cdot z_1)(h^{z_1} g^{-1} \cdot m), (h^{z_1} g^{-1})^m g\right)$$
$$= \left(n, h^{z_1(g^{-1} \cdot m)} (g^{-1})^m g\right) = \left(n, h^{h^{-1} \cdot n} g^{-1} g\right) = \left(n, \left((h^{-1})^n\right)^{-1}\right) = (n, h).$$

Similarly $(x_2, y_2)(n, h) = (m, g)$. Hence $(m, g)\mathcal{L}(n, h)$ in $M \bowtie G$. □

4. Regular and Inverse Zappa-Szép Products

The main goal of this Section is to determine some of the algebraic properties of Zappa-Szép products of semigroups in terms of the algebraic properties of the semigroups themselves.

The (internal) Zappa-Szép product $M = A \bowtie B$, of the regular subsemigroups A and B need not to be regular in general. A special case of the Zappa-Szép product is the semidirect product for which one of the actions is trivial. We use Theorem 2.1 [11] to construct an example of regular submonoids such that their semidirect product is not regular.

Example 3. *Let* $T = \{1, e, f, 0\}$ *be a commutative monoid with* 0, *each of whose elements is idempotent and such that* $ef = f$. *Let* $S = \{1, a, b\}$ *be a monoid with two left zeros a and b. Then both S and T are regular semigroups. Let* 1 *acts trivially,* $e^a = e^b = e, f^a = 0^a = 0, f^b = 0^b = f, 1^a = 1^b = 1.$ *There is no* $e^2 = e \in S$ *such that* $Ss = Se, t \in t^e T$ *for all* $s \in S, t \in T$. *Thus the semidirect product* $R = S \ltimes T$ *is not regular. For example,* (a, f) *is not a regular element of R.*

Example 4. *Take* $A = \{1, e, f\}$ *such that* $e^2 = e, f^2 = f, ef = f = fe,$ *and* $B = \{1, b\}$ *such that* $b^2 = b$. *Let* $1 \in A$ *act trivially on B,* $1^e = 1^f = 1, b^e = b^f = 1,$ *and* $1 \in B$ *act trivially on A,* $b \cdot 1 = 1, b \cdot e = f, b \cdot f = f$. *Then A and B are regular monoids but their Zappa-Szép product* $M = A \bowtie B$ *is not regular.*

However, there are criteria we can prove that the internal Zappa-Szép product $M = A \bowtie B$ of regular A and B is regular as the following Propositions illustrated.

Proposition 7. *If A is a regular monoid, B is a group,* $1_B \cdot a = a, (1_B)^a = 1_B$ *for all* $a \in A$, *then* $M = A \bowtie B$ *is regular.* □

Proof. Let $(a, b) \in M = A \bowtie B$ where $a \in A$ and $b \in B$, we have to find $(c, d) \in M = A \bowtie B$ such that $(a, b)(c, d)(a, b) = (a, b)$. Now $(a, b)(c, d)(a, b) = \left(a(b \cdot c)((b^c d) \cdot a), (b^c d)^a b\right)$ and so we choose $(c, d) = \left(b^{-1} \cdot a', \left(b^{b^{-1} \cdot a'}\right)^{-1}\right)$, where $a' \in V(a)$. Since we must have $(b^c d)^a b = b$ but B is a group, so $(b^c d)^a = 1_B$. Suppose we are given c, and choose $d = (b^c)^{-1}$ since B is a group, so $(b^c d)^a = (1_B)^a = 1_B$, and then $a(b \cdot c)((b^c d) \cdot a) = a(b \cdot c)(1_B \cdot a) = a(b \cdot c)a$. Since A is regular, we choose any $a' \in V(a)$ and set $c = b^{-1} \cdot a'$, thus $b \cdot c = b \cdot (b^{-1} \cdot a') = (bb^{-1}) \cdot a' = 1_B \cdot a' = a'$. Thus $(a, b)\left(b^{-1} \cdot a', \left(b^{b^{-1} \cdot a'}\right)^{-1}\right)(a, b) = (a, b)$, whereupon $M = A \bowtie B$ is regular. □

Proposition 8. *Let A be a left zero semigroup and B be a regular semigroup. Suppose that for all* $b \in B$, *there exists some* $a \in A$ *such that* $b^a = b$, *and for all* $x \in A$ *there exists some* $b' \in V(b)$ *such that* $(b')^x = b'$. *Then* $M = A \bowtie B$ *is regular.*

Proof. Let $(a, b) \in M = A \bowtie B$ where $a \in A$ and $b \in B$. We have to find $(c, d) \in M = A \bowtie B$ such that $(a, b)(c, d)(a, b) = (a, b)$. Now $(a, b)(c, d)(a, b) = \left(a(b \cdot [c(d \cdot a)]), b^{c(d \cdot a)} d^a b\right)$ and we choose $(c, d) = (c, b')$, where $b' \in V(b)$. Now since A is a left zero semigroup $\left(a(b \cdot [c(d \cdot a)]), b^{c(d \cdot a)} d^a b\right) = (a, b^c d^a b)$, and by our assumptions we can choose $c \in A$ such that $b^c = b$, and $b' = d$ that is fixed by a. Then $(a, b)(c, d)(a, b) = (a, b^c b'^a b) = (a, bb'b) = (a, b)$. Thus $M = A \bowtie B$ is regular. □

Theorem 1. [12] *For any arbitrary semigroup S,* $Reg(S)$ *is a subsemigroup of S if and only if the product* *of any pair of idempotents in S is regular*.

We now give a general necessary and sufficient condition for Zappa-Szép products of regular semigroups to be regular. Consider the internal Zappa-Szép product $M = A \bowtie B$ of regular semigroups *A* and *B*. Then each $m \in M$ is uniquely a product of regular elements: $m = ab$ where $a, b \in Reg(M)$. Hence *M* is regular if and only if $Reg(M)$ is a subsemigroup of *M* and so by Hall's Theorem 1, *M* is regular if and only if the product of any two idempotents is regular. But in fact we need only consider products of idempotents $e \in A$ and $f \in B$, as our next theorem shows.

Theorem 2. *Let A and B be regular subsemigroups and* $e \in E(A)$ *and* $f \in E(B)$. *Then* $ef \in Reg(M)$ *if and only if* $M = A \bowtie B$ *is regular*.

Proof. Given $m \in M$ with $m = ab$ (uniquely) where $a \in A, b \in B$, since *A* and *B* are regular subsemigroups, then there exist $a' \in V(a)$ and $b' \in V(b)$. Then $a'a \in E(A)$ and $bb' \in E(B)$. Set $e = a'a$ and $f = bb'$. Then by the assumption $(a'a)(bb') \in Reg(M)$ and the set

$$M(a'a, bb') = M(e, f) = \{g \in V(ef) \cap E(M) : ge = fg = g\}.$$

is not empty, see [14]. Because $ef \in Reg(M)$ let $x \in V(ef)$, and let $g = fxe$. Then

$$(ef)g(ef) = (ef)fxe(ef) = ef^2xe^2f = efxef = ef$$

and

$$g(ef)g = fxe(ef)fxe = fxe^2f^2xe = f(xefx)e = fxe = g$$

and so $g \in V(ef)$. Also

$$g^2 = f(xefx)e = fxe = g,$$

and so $g \in E(M)$. Also

$$ge = fxee = g,$$

$$fg = ffxe = g.$$

Thus $g \in M(e, f)$. Also $b'ga' \in V(m)$ because

$$(ab)(b'ga')(ab) = afgeb = ageb = agb = aa'agbb'b = a(egf)b$$

we have

$$egf = ef,$$

$$egf = e(fge)f = (ef)g(ef) = ef.$$

Then

$$(ab)(b'ga')(ab) = a(ef)b = (aa'a)(bb'b) = ab,$$

and

$$(b'ga')ab(b'ga') = b'gefga' = b'g^2a' = b'ga'.$$

and so $b'ga' \in V(m)$. Then $m = ab$ is a regular element which implies that $M = A \bowtie B$ is regular.

Conversely, if $M = A \bowtie B$ is the regular internal Zappa-Szép product of the regular subsemigroups *A* and *B*, each element *m* of *M* is uniquely written in the form $m = ab$ where $a \in A$ and $b \in B$. Thus if $e \in A$ and $f \in B$ this implies that $ef \in M$, then $ef \in Reg(M)$. □

Corollary 1. *If A and B are regular and* $E(A), E(B)$ *act trivially, then* $M = A \bowtie B$ *is regular*.

Proof. If we take $e \in E(A)$ and $f \in E(B)$, then ef is an idempotent in *M*. Because $(ef)(ef) = e(f \cdot e)(f^e)f = eeff = ef$, since $E(A), E(B)$ act trivially. Therefore $ef \in Reg(M)$. Hence $M = A \bowtie B$ is regular. □

In this case: if $m = ab \in M$, we can find $m' \in V(m)$. First find $g \in M(a'a, bb') = M(e, f), e = a'a$ and $f = bb'$, $g = fefe = fe$ since idempotents of *A* commute with those of *B*. Then $m' = b'(ef)a' = xy$ for some $x \in A, y \in B$. Thus $m' = (b' \cdot e)b'^e(f \cdot a')f^{a'} = (b' \cdot e)(b'^e \cdot (f \cdot a'))(b'^e)^{(f \cdot a')} f^{a'}$ where $x = b' \cdot a'$ and

$y=(b')^{a'}$. Then $m'=xy=(b'\cdot a')(b')^{a'}$.

Now we discuss inverse Zappa-Szép products. Let S and T be inverse semigroups/monoids with $\theta:S\to End(T)$ and let $P=S\ltimes T$ be the semidirect product of S and T. We can see from the following example that P need not be inverse.

Example 5. [11] *Let* $S=\{1,a\}$ *be the commutative monoid with one non zero-identity idempotent* a. *Let* $T=\{1,e,0\}$ *be the commutative monoid with zero an with* $e=e^2$. *Then S and T are both inverse monoids, and there is a homomorphism* $\theta:S\to End(T)$ *given by* $1^a=1$ *and* $e^a=0^a=e$. *Then* $P=S\ltimes T$ *is regular. However, the element* $(a,e)\in P$ *has two inverses, namely* (a,e) *and* $(a,0)$, *and hence P cannot be an inverse monoid*.

A complete characterization of semidirect products of monoids which are inverse monoids is given in Nico [11].

Theorem 3. [11] *A semidirect product* $P=S\ltimes T$ *of two inverse semigroups S and T will be inverse if and only if* $E(S)$ *acts trivially*.

In the general case of the Zappa-Szép product of inverse semigroups $P=S\bowtie T$ we have also P need not be inverse semigroup as we can see from the following example.

Example 6. *Let* $A=E=\{e,f,ef\}$ *where* $e^2=e, f^2=f, ef=fe$ *and* $B=G=\{1,a,b,ab\}$*—Klein* 4*-group where* $a^2=1, b^2=1, ab=ba$. *Suppose that the action of G on E is defined by*:

$$\boxed{a\cdot e=f, b\cdot e=f, a\cdot ef=ef, a\cdot f=e, b\cdot f=e,}$$

and

$$\boxed{b\cdot ef=ef, ab\cdot e=e, ab\cdot f=f, ab\cdot ef=ef.}$$

and $1\in G$ acts trivially (that is each of a,b permutes $\{e,f\}$ non-trivially). The action of E on G is defined by:

$$\boxed{a^e=b, b^e=b, a^f=b, b^f=b, a^{ef}=b,}$$

and

$$\boxed{b^{ef}=b, (ab)^e=1, (ab)^f=1, (ab)^{ef}=1.}$$

We check Zappa-Szép axioms by the following: define $\phi:G\to C_2=\{1,t\}$ by

$$\phi(a)=t, \phi(b)=t, \phi(ab)=1.$$

This is a homomorphism of groups since C_2 is the automorphism group of E. We have an action of G on E using ϕ: if $g\in G$ and $x\in E$ define

$$g\cdot x=\phi(g)(x).$$

We have a homomorphism $\psi:G\to G$ given by $a\psi=b$ and $b\psi=b$. The action of E on G is given by:

$$g^e=g^f=g^{ef}=g\psi$$

We note that $\phi(g\psi)=\phi(g)$, for all $g\in G$. For $(ZS1)$ we have for $g,h\in G$

$$g\cdot(h\cdot x)=g(\phi(h)(x))=\phi(g)\phi(h)(x)=\phi(gh)(x)=gh\cdot x.$$

Since $\psi^2=\psi$ it is clear that $(ZS3)$ holds. For $(ZS2)$ we have for $x,y\in E$

$$g\cdot(xy)=\phi(g)(xy)=\phi(g)(x)\phi(g)(y)$$

and

$$(g\cdot x)(g^x\cdot y)=\phi(g)(x)\phi(g\psi)(y)=\phi(g)(x)\phi(g)(y)$$

So $(ZS2)$ holds and for $(ZS4)$

$$(gh)^x=(gh)\psi=(g\psi)(h\psi)$$

and

$$g^{h \cdot x} h^{x} = (g\psi)(h\psi).$$

Thus $(ZS4)$ holds. Then $M = E \bowtie G$. Since every element of *M* is regular, then *M* is regular. $E(M)$ is a closed subsemigroup of *M*, so $M = E \bowtie G$ is orthodox, but since $E(M)$ is not commutative for example $(f,a)(f,1) = (ef,b)$ while $(f,1)(f,a) = (f,a)$, then *M* is not inverse.

The achievement of necessary and sufficient conditions was difficult; so we try to find an inverse subset of the Zappa-Szép product of inverse semigroups. This achieved and described in Section 9. We have given the necessary conditions for Zappa-Szép products of inverse semigroups to be inverse in the following theorem.

Theorem 4. $P = S \bowtie T$ *is an inverse semigroup if*

(i) *S* and *T* are inverse semigroups;

(ii) $E(S)$ and $E(T)$ act trivially;

(iii) For each $p = (s,t) \in E(P)$ where $s \in S$ and $t \in T$, then *s* and *t* act trivially on each other.

Proof. We know that $P = S \bowtie T$ is regular. Since a regular semigroup is inverse if and only if its idempotents commutes, it suffices to show that idempotents of $P = S \bowtie T$ commute. If $(a,t),(b,u)$ are idempotents of $P = S \bowtie T$, then

$$(a,t)(a,t) = (a,t) = (a(t \cdot a), t^{a} t)$$

Thus

$$a = a(t \cdot a) \text{ and } t = t^{a} t,$$

and

$$b = b(u \cdot b) \text{ and } u = u^{b} u,$$

By (iii) *a* and *t* act trivially on each other, *b* and *u* act trivially on each other, then

$$a = a^2, b = b^2, t = t^2, u = u^2.$$

But since *S* and *T* are inverse semigroup, then idempotents commutes that is

$$ab = ba \in S, tu = ut \in T,$$

Then $(a,t)(b,u) = (a(t \cdot b), t^{b} u)$, but *t* and *c* are idempotents they are act trivially then

$$(a,t)(b,u) = (ab,tu) = (ba,ut) = (b(u \cdot a), u^{a} t) = (b,u)(a,t).$$

Thus $P = S \bowtie T$ is inverse. □

5. External Zappa-Szép Products

Let A and B be semigroups, and suppose that we are given functions $A \times B \to A$, $(a,b) \mapsto b \cdot a \in A$ and $A \times B \to B$, $(a,b) \mapsto b^{a} \in B$ where $a \in A$ and $b \in B$. satisfying the Zappa-Szép rules $(ZS1),(ZS2),(ZS3)$ and $(ZS4)$. Then the set $A \times B$ with the product defined by: $(a,b)(c,d) = (a(b \cdot c), b^{c} d)$ is a semigroup, the external Zappa-Szép product of *A* and *B*, which is written as $A \bowtie B$.

If *A* and *B* are semigroups that both have zero elements (0_A and 0_B respectively), and we have in addition to $(ZS1),(ZS2),(ZS3)$ and $(ZS4)$ for all $a \in A$ and $b \in B$ the following rule:

($ZS5$) $b \cdot 0_A = 0_A, (b)^{0_A} = 0_B, 0_B \cdot a = 0_A, (0_B)^{a} = 0_B.$

Then by Proposition [10] we have that *S* is a semigroup with zero $(0_A, 0_B)$. But from the following example we deduce that $(ZS9)$ is not a necessary condition:

Example 7. *If A and B are semigroups with* 0_A *and* 0_B *respectively, acting trivially on each other. Then* ($ZS5$) *is not satisfied. However, in the Zappa-Szép product* $S = A \times B$ *we have*

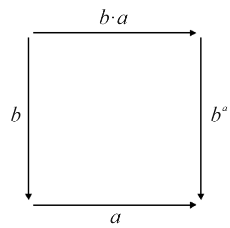
$(0_A, 0_B)(a,b) = (0_A(0_B \cdot a), 0_B^{a} b) = (0_A a, 0_B b) = (0_A, 0_B)$, *and*

$(a,b)(0_A, 0_B) = (a(b \cdot 0_A), b^{0_A} 0_B) = (a0_A, b0_B) = (0_A, 0_B)$. *Thus* $(0_A, 0_B)$ *is zero for* $A \bowtie B$.

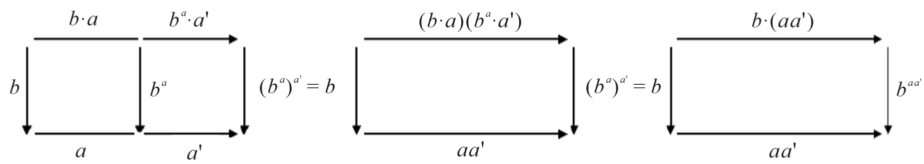
From the following example we deduce that the zeros 0_A and 0_B of *A* and *B* respectively do not necessarily give a zero for the external Zappa-Szép product $A \bowtie B$.

Example 8. *If A is a monoid with identity* 1_A *and zero* 0_A *and B is a semigroup with zero* 0_B, *such that the action of A on B is trivial action* $(a,b) \mapsto b^a = b$ *and the action of A on B is* $(a,b) \mapsto b \cdot a = 1_A$ *for all* $a \in A, b \in B$. *Then Zappa-Szép rules* $(ZS1)$-$(ZS4)$ *are satisfied. But* (*ZS*5) *is not satisfied since* $b \cdot 0_A = 1_A \neq 0_A$, *and* $(0_A, 0_B)$ *is not a zero for* $A \bowtie B$.

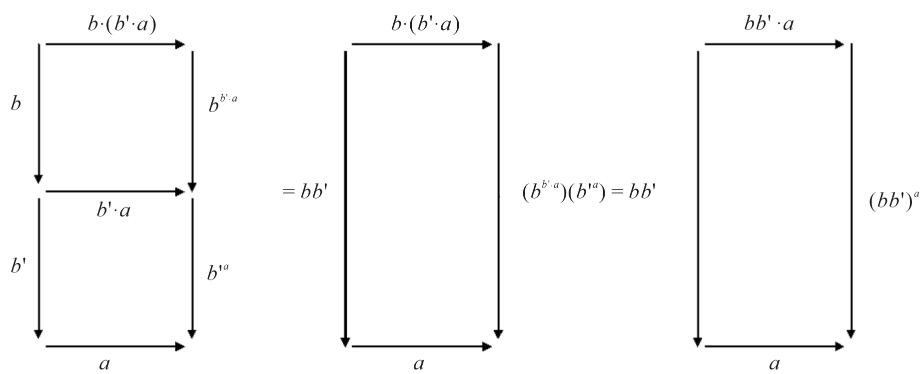
The Zappa-Szép rules can be demonstrated using a geometric picture: draw elements from *A* as horizontal arrows and elements from *B* as vertical arrows. The rule $ba = (b \cdot a)(b^a)$ completes the square



From the horizontal composition we get $(ZS2)$ and $(ZS3)$ as follows:



From the vertical composition we get (*ZS*1) and (*ZS*4) as follows:



These pictures show that a Zappa-Szép product can be interpreted as a special kind of double category. This viewpoint on Zappa-Szép products underlies the work of Fiedorowicz and Loday [13]. In the theory of quantum groups Zappa-Szép product known as the bicrossed (bismash) product see [14].

6. Internal and External Zappa-Szép Products

In general, there is not a perfect correspondence between the internal and external Zappa-Szép product of semigroups. For one thing, embedding of the factors might not be possible in an external product as the following example demonstrates.

Example 9. *Consider the external Zappa-Szép product* $P = \mathbb{Z} \bowtie \mathbb{N}$, *where for all* $m \in \mathbb{Z}$ *and* $n \in \mathbb{N}$ *we have* $n \cdot m = n + m$ *and* $n^m = 0$, *so that the multiplication in P is* $(m,n)(m',n') = (m(n \cdot m'), n^{m'}n') = (m + n + m', n')$. *Then for each* $k \in \mathbb{N}$ *the subset* $\mathbb{Z}_k = \{(m,k) : m \in \mathbb{Z}\}$ *is a subgroup of P isomorphic to* $\mathbb{Z}$ *(with identity* $(-k,k)$*). However P cannot be an internal Zappa-Szép product of subsemigroups Z, N isomorphic to* $\mathbb{Z}$ *and* $\mathbb{N}$ *respectively: If* (p,q) *generates N, then the second* *coordinate of every non-identity element of N is q, and so the second coordinate of any product* xy *with* $x \in Z$ *and* $y \in N$ *is equal to q.*

However, under some extra hypotheses, the external product can be made to correspond to an internal product for example:

- if we assume the two factors *A* and *B* involving in the external Zappa-Szép product have an identities elements 1_A and 1_B respectively such that the following is satisfied

  $(ZS6)$ $b \cdot 1_A = 1_A, b^{1_A} = b, 1_B \cdot a = a, 1_B^a = 1_B,$ for all $a \in A$ and $b \in B.$

So if $M = A \bowtie B$, the external Zappa-Szép product of *A* and *B*, then each *A* and *B* are embedded in $M = A \bowtie B$ Define $\tau_M : A \to A \bowtie B$ by $\tau_M(a) = (a, 1_B)$. We claim τ_M is an injective homomorphism since $\tau_M(a_1a_2) = (a_1a_2, 1_B)$ and $\tau_M(a_1)\tau_M(a_2) = (a_1, 1_B)(a_2, 1_B) = (a_1(1_B \cdot a_2), 1_B^{a_2}1_B) = (a_1a_2, 1_B)$. Thus τ_M is a homomorphism, also τ_M is injective since $\tau_M(a_1) = \tau_M(a_2) \Leftrightarrow (a_1, 1_B) = (a_2, 1_B) \Leftrightarrow a_1 = a_2$. Denote its image by $\bar{A}$. Define $\psi_M : B \to A \to A \bowtie B$ by $\psi_M(a) = (1_A, b)$, then ψ_M is also an injective homomorphism. Denote its image by $\bar{B}$. Observe that $(a, b) = (a, 1_B)(1_A, b)$. Thus $M = A \bowtie B = \bar{A}\bar{B}$. This decomposition is evidently unique. Thus $M = \bar{A} \bowtie \bar{B}$ is the internal Zappa-Szép product of $\bar{A}$ and $\bar{B}$.

- If *A* is a left zero semigroup and *B* is a right zero semigroup, then the external Zappa-Szép product of *A* and *B* is a rectangular band and it is the internal Zappa-Szép product of $\bar{A} = \{(a, b_\circ) : a \in A\}$ and $\bar{B} = \{(a_\circ, b) : b \in B\}$ where $a_\circ, b_\circ$ are fixed elements of *A* and *B* respectively. Note that in a left-zero semigroup *A*, $RI(A) = A$ and in a right-zero semigroup *B*, $LI(B) = B,$ and we have the following Theorem:

Theorem 5. *M is the internal Zappa-Szép product of a left-zero semigroup A and a right-zero semigroup B if and only if M is a rectangular band.*

Proof. Let *A* be a left-zero semigroup and *B* a right-zero semigroup. In the rectangular band $M = A \times B$, let $\bar{A} = \{(a, b_\circ) : a \in A\}$ and $\bar{B} = \{(a_\circ, b) : b \in B\}$, where $a_\circ, b_\circ$ are fixed elements. Then $(a, b) = (a, b_\circ)(a_\circ, b) \in \bar{A} \times \bar{B}$ uniquely and $(a, b_\circ)(a', b_\circ) = (a, b_\circ) \in \bar{A},$ and $(a_\circ, b)(a_\circ, b') = (a_\circ, b) \in \bar{B}.$ So $M = A \times B$ is the internal Zappa-Szép product of $\bar{A}$ and $\bar{B}$, $M = \bar{A} \bowtie \bar{B}$, where $\bar{A} \cong A$ (as left-zero semigroup) and $\bar{B} \cong B$ (as right-zero semigroup).

Conversely, Let $M = A \bowtie B$ where *A* is left-zero semigroup and *B* is right-zero semigroup. Then $a_1a_2 = a_1$ for all $a_1, a_2 \in A$ and $b_1b_2 = b_2$ for all $b_1, b_2 \in B.$ M is a rectangular band if for all $x, y \in M$ then $xyx = x,$ where $x = a_1b_1$ and $y = a_2b_2$ for unique $a_1, a_2 \in A$ and $b_1, b_2 \in B.$ Now

$$xyx = a_1b_1a_2b_2a_1b_1 = a_1(b_1 \cdot a_2)b_1^{a_2}(b_2 \cdot a_1)b_2^{a_1}b_1 = a_1(b_1 \cdot a_2)b_1^{a_2}(b_2 \cdot a_1)b_1$$

$$= a_1(b_1 \cdot a_2)\left(b_1^{a_2} \cdot (b_2 \cdot a_1)\right)\left(b_1^{a_2}\right)^{(b_2 \cdot a_1)} b_1 = a_1\left(b_1^{a_2}\right)^{(b_2 \cdot a_1)} b_1 = a_1b_1 = x$$

Thus *M* is a rectangular band. □

7. Examples

1) Let $A = \{e, t, f, b\}$ be a Clifford semigroup. Note that $\{e, t\} \cong C_2$ and $\{f, b\} \cong C_2$. Let $B = (\mathbb{Z}, +)$, the group of integers. Suppose that the action of $\mathbb{Z}$ on *A* for each $m \in \mathbb{Z}$ is as follows: $m \cdot e = f, m \cdot t = b, m \cdot f = f, m \cdot b = b.$ The action of *A* on $\mathbb{Z}$ as follows: $m^e = m, m^f = m, m^t = -m, m^b = -m.$ Then the Zappa-Szép multiplication is associative. Thus $M = A \bowtie B = \{(x, m) : x \in A, m \in B\}$ the Zappa-Szép product of *A* and *B*. The set $E(M)$ of idempotents of *M* is the empty set, since $(x, m) \in E(M)$ if and only if $x = x(m \cdot x)$ and $m = m^x m,$ since $m^x = m$ or $-m$ for all $x \in A$ so $m^x m = 2m$ or $0 \neq m.$

2) Suppose that *A* is a band. Then the left and right regular actions of *A* on itself allows us to form the Zappa-Szép product $M = A \bowtie A,$ since if we define $b \cdot a = ba$ and $b^a = ba$ with $a, b \in A,$ we obtain the multiplication $(a_1, b_1)(a_2, b_2) = (a_1(b_1 \cdot a_2), b_1^{a_2}b_2) = (a_1b_1a_2, b_1a_2b_2).$ Then $M = A \bowtie A$ is the external Zappa-Szép product of *A* and *A*. Moreover *M* is a band if and only if *A* is a rectangular band, in which case *M* is a rectangular band.

3) Let $A = E = \{e, f, ef\}$ where $e^2 = e, f^2 = f, ef = fe$ and $B = G = \{1, a, b, ab\}$—Klein 4-group, where $a^2 = 1, b^2 = 1, ab = ba.$ Suppose that the action of *G* on *E* is defined by: $a \cdot e = f, b \cdot e = f, a \cdot ef = ef, a \cdot f = e, b \cdot f = e,$ and $b \cdot ef = ef, ab \cdot e = e, ab \cdot f = f, ab \cdot ef = ef.$ and $1 \in G$ acts trivially. The action of E on G is defined by: $a^e = b, b^e = b, a^f = b, b^f = b, a^{ef} = b,$ and

$b^{ef} = b, (ab)^e = 1, (ab)^f = 1, (ab)^{ef} = 1$. Then $M = E \bowtie G$ is the Zappa-Szép product of E and G. Since every element of M is regular, then M is regular. $E(M)$ is a closed subsemigroup of M, so $M = E \bowtie G$ is orthodox, but since $E(M)$ is not commutative for example $(f,a)(f,1) = (ef,b)$ while $(f,1)(f,a) = (f,a)$, then M is not inverse.

4) For groups, $M = A \bowtie B$ is the Zappa-Szép product of subgroups A and B if and only if, $M = B \bowtie A$ since for any $m \in M$ we have $m^{-1} = ab$ for unique $a \in A$ and $b \in B$. This implies that $m = b^{-1}a^{-1}$ for unique $b^{-1} \in B$ and $a^{-1} \in A$. Thus $M = B \bowtie A$. But this is not true in general for semigroups or monoids. Let $A = \{1, a\}$ be a commutative monoid with one non-identity idempotent a. Let $B = \{1, e, f\}$ be a commutative monoid with two idempotents e and f and $ef = fe = e$. Let B act trivially on A and $1 \in A$ act trivially on B and $1^a = 1, e^a = f^a = e$. Then $\overline{A} = \{(1,1),(a,1)\} \cong A$ and $\overline{B} = \{(1,1),(1,e),(1,f)\} \cong B$. Then $M = \overline{A} \bowtie \overline{B}$ is the internal Zappa-Szép product of $\overline{A}$ and $\overline{B}$. But $M \neq \overline{B} \bowtie \overline{A}$, since $(1,b)(a,1) = (1(b \cdot a), b^a 1) = (a, b^a)$, so (a, f) can not be written as $b'a'$. Moreover, $(a,1)(1,f) = (a,f)$ so $\overline{BA}$ is not a submonoid of M.

8. Zappa-Szép Products and Nilpotent Groups

In this section we consider a particular Zappa-Szép product for nilpotent groups. Note that G being nilpotent of class at most 2 is equivalent to the commutator subgroup G' being contained in the center $Z(G)$ of G. Now, let G be a group and let G act on itself by left and right conjugation as follows:

$$b \cdot a = bab^{-1} \text{ and } b^a = a^{-1}ba.$$

In the following we show that these actions let us form a Zappa-Szép product $P = G \bowtie G$ if and only if G is nilpotent group of class at most 2.

Proposition 9. *Let G be nilpotent group of class at most 2. Then the left and right conjugation actions of G on it self can be used to form the Zappa-Szép product $P = G \bowtie G$.*

Proof. Let G act on itself by left and right conjugation as follows:

$$b \cdot a = bab^{-1} \text{ and } b^a = a^{-1}ba$$

where $a, b \in G$. Thus the multiplication is given by:

$$(a_1, b_1)(a_2, b_2) = (a_1 b_1 a_2 b_1^{-1}, a_2^{-1} b_1 a_2 b_2).$$

We prove that the Zappa-Szép rules are satisfied if G is a nilpotent group of class less than or equal 2, which implies that $[[a,b],c] = 1$ for all $a,b,c \in G$. For $(ZS1)$ and $(ZS3)$ clear they are hold.

$(ZS2)$ $b \cdot (a_1 a_2) = (b \cdot a_1)(b^{a_1} \cdot a_2)$.

$L.H.S.:$ $b \cdot (a_1 a_2) = b a_1 a_2 b^{-1}$,

$R.H.S.:$

$$(b \cdot a_1)(b^{a_1} \cdot a_2) = (b a_1 b^{-1})(a_1^{-1} b a_1 \cdot a_2) = b a_1 b^{-1} a_1^{-1} b a_1 a_2 a_1^{-1} b^{-1} a_1$$

since G is nilpotent of class ≤ 2,then

$$(b \cdot a_1)(b^{a_1} \cdot a_2) = b a_1 b^{-1} a_1^{-1} b a_1 a_2 a_1^{-1} b^{-1} a_1 = b a_1 a_2 a_1^{-1} a_1 b^{-1} a_1^{-1} b b^{-1} a_1 = b a_1 a_2 b^{-1} = R.H.S.$$

Thus $(ZS2)$ holds.

$(ZS4)$ $(b_1 b_2)^a = b_1^{b_2 \cdot a} b_2^a$.

$R.H.S.:$ $(b_1 b_2)^a = a^{-1} b_1 b_2 a$,

$L.H.S.:$

$$b_1^{b_2 \cdot a} b_2^a = b_1^{b_2 a b_2^{-1}} b_2^a$$

since G is nilpotent of class 2, then

$$b_1^{b_2 \cdot a} b_2^a = b_2 a^{-1} b_2^{-1} b_1 b_2 a b_2^{-1} a^{-1} b_2 a = b_2 a^{-1} b_2^{-1} b_2 a b_2^{-1} a^{-1} b_1 b_2 a = a^{-1} b_1 b_2 a = R.H.S.$$

Thus $(ZS4)$ holds. Hence $P = G \bowtie G$ is the Zappa-Szép product. □

Proposition 10. *If the left and right conjugation actions of G on itself satisfy the Zappa-Szép rules, then G is nilpotent of class at most* 2.

Proof. Suppose the Zappa-Szép rules satisfied, we prove that *G* is nilpotent of class ≤ 2. If $(ZS2)$ holds, then for all $a_1, a_2, b \in G$ we have $b \cdot (a_1 a_2) = (b \cdot a_1)(b^{a_1} \cdot a_2)$. Thus

$$ba_1a_2b^{-1} = ba_1b^{-1}a_1^{-1}ba_1a_2a_1^{-1}b^{-1}a_1$$
$$a_2b^{-1} = b^{-1}a_1^{-1}ba_1a_2a_1^{-1}b^{-1}a_1$$

Therefore $a_1^{-1}b^{-1}a_1ba_2 = a_2a_1^{-1}b^{-1}a_1b.$ Hence $a_1^{-1}b^{-1}a_1b$ is central in *G*. Similarly if $(ZS4)$ holds. □

Combining Propositions 9 and 10 we prove the following:

Proposition 11. *P is the Zappa-Szép product of the group G and G with left and right conjugation actions of G on itself if and only if G is nilpotent of class at most* 2.

Next we prove the following:

Lemma 3. *The center of* $P = G \bowtie G$ *is* $Z(G) \times Z(G)$.

Proof. Suppose $(a,b) \in Z(P)$. Since for all $(c,d) \in P$ we have $(a,b)(c,d) = (abcb^{-1}, c^{-1}bcd)$. and $(c,d)(a,b) = (cdad^{-1}, a^{-1}dab)$. Then

(1) $abcb^{-1} = cdad^{-1}$.

and

(2) $c^{-1}bcd = a^{-1}dab$.

Put in (1) $c = 1$: then $a = dad^{-1}$ for all $d \in G$. Therefore $ad = da$. So $a \in Z(G)$. Put in (2) $d = 1$: then $b = c^{-1}bc$ for all $c \in G$. Therefore $cb = bc$. So $b \in Z(G)$. So $Z(P) = \{(a,b) : a,b \in Z(G)\} \cong Z(G) \times Z(G)$, since if $a,a',b,b' \in Z(G)$. Then $(a,b)(a',b') = (a',b')(a,b) = (aa',bb') \in P$. □

Lemma 4. *If* $P = G \bowtie G$ *then P is abelian if and only if G is abelian.*

Proof. If *G* is abelian then *G* is nilpotent of class 1 if and only if $Z(G) = G$. This implies $Z(P) = G \times G = P$ and so *P* is abelian.

If *P* is abelian then $Z(P) = P$, but $Z(P) = Z(G) \times Z(G)$. Thus $P = G \bowtie G = Z(G) \times Z(G)$ if and only if $G = Z(G)$. Hence *G* is abelian group. In which case $P = G \times G$. □

Proposition 12. *If P is the non-abelian Zappa-Szép product* $P = G \bowtie G$ *and G is nilpotent group of class at most* 2, *then P is nilpotent of class* 2.

Proof. We have *G* is nilpotent group of class ≤ 2 if and only if for all $a,b,c \in G$ we have $a^{-1}b^{-1}abc = ca^{-1}b^{-1}ab$, that is $[a,b]c = c[a,b]$ the commutator elements are central. Let $[(a,b),(c,d)]$ be a commutator in *P*. We prove it is central in *P*. We have

$$[(a,b),(c,d)] = (a,b)^{-1}(c,d)^{-1}(a,b)(c,d) = ((c,d)(a,b))^{-1}(a,b)(c,d).$$

Now

$$(a,b)(c,d) = (abcb^{-1}, c^{-1}bcd) = (acc^{-1}bcb^{-1}, c^{-1}bcb^{-1}bd) = (ac[c,b^{-1}], [c,b^{-1}]bd)$$
$$= (ac[c,b^{-1}], bd[c,b^{-1}]) = (ac,bd)([c,b^{-1}],[c,b^{-1}]).$$

and

$$(c,d)(a,b) = (cdad^{-1}, a^{-1}dab) = (ca[a,d^{-1}], [a,d^{-1}]db)$$
$$= (acc^{-1}a^{-1}ca[a,d^{-1}], [a,d^{-1}]bdd^{-1}b^{-1}db)$$
$$= (ac[c,a][a,d^{-1}], [a,d^{-1}]bd[d,b])$$
$$= (ac,bd)([c,a],[d,b])([a,d^{-1}],[a,d^{-1}]).$$

Write $x = [a,d^{-1}], y = [c,b^{-1}], u = [c,a], v = [d,b]$. Then

$$[(a,b),(c,d)] = ((c,d)(a,b))^{-1}(a,b)(c,d) = (x,x)^{-1}(u,v)^{-1}(ac,bd)^{-1}(ac,bd)(y,y)$$

Since $[c,b^{-1}],[a,d^{-1}],[c,a],[d,b]$ are commutators, then

$$[(a,b),(c,d)] = ([a,d^{-1}],[a,d^{-1}])^{-1}([c,a],[d,b])^{-1}([c,b^{-1}],[c,b^{-1}]).$$

This implies that $[(a,b),(c,d)] \in G' \times G'$. Thus $P' \subseteq G' \times G' \subseteq Z(G) \times Z(G) = Z(P)$. So commutators in P are in the center (and P is not abelian) so P is nilpotent of class 2. □

Combining Propositions 11, 12 and Lemma 4 we have the following.

Theorem 6. *Let G be a group that is nilpotent of class at most* 2, *and let* $P = G \bowtie G$ *with left and right conjugation action of G on it self. Then*:

1) P is abelian if and only if G is abelian, in which case $P = G \times G$;

2) If G is non-abelian and hence nilpotent of class 2, then P is also nilpotent of class 2.

9. Zappa-Szép Products of Semilattices and Groups

The Zappa-Szép product of inverse semigroups need not in general be an inverse semigroup. This is even the case for the semidirect product as we see (Nico [11] for example) However, Bernd Billhardt [15] showed how to get around this difficulty in the semidirect product of two inverse semigroups by modifying the definition of semidirect products in the inverse case to obtain what he termed λ-semidirect products. The λ-semidirect product of inverse semigroups is again inverse. In this Section, we construct from the Zappa-Szép product P of a semilattice E and a group G, an inverse semigroup by constructing an inductive groupoid. We assume the additional axiom for the identity element $1 \in G$ we have $1 \cdot e = e$.

Note that if $1 \cdot e = e$ for all $e \in E$, then $(ZS8)\, 1^e = 1$ holds, since by cancellation in the group G.

$$g^e = (g1)^e = g^{1\cdot e}1^e = g^e 1^e$$
$$1 = 1^e.$$

We consider the following where E a semilattice and G a group, and subset $B_{\bowtie}(P)$ of the Zappa-Szép product $P = E \bowtie G$:

$$B_{\bowtie}(P) = \left\{(e,g) \in P : (g^{-1})^e = g^{-1}\right\}.$$

We form a groupoid from the action of the group G on the set E which has the following features:

- vertex set: $E(B_{\bowtie}(P)) = \{(e,1) : e \in E\} \cong E$;
- arrow set: $B_{\bowtie}(E \bowtie G)$;
- an arrow (e,g) starts at $(e,g)(e,g)^{-1} = (e,1)$, finishes at $(e,g)^{-1}(e,g) = (g^{-1} \cdot e, 1)$;
- the inverse of the arrow $(e,g) \in B_{\bowtie}(P)$ is $(g^{-1} \cdot e, g^{-1})$;
- the identity arrow at e is $(e,1)$; and
- an arrows (e,g) and (f,h) are composable if and only if $g^{-1} \cdot e = f$, in which case the composite arrow is $(e(g \cdot f), g^f h)$.

Lemma 5. *If* $(e,g) \in B_{\bowtie}(P = E \bowtie G)$, *then* $g = g^{g^{-1}\cdot e}$.

Proof. we have $1 = 1^e$ for all $e \in E$, then $1 = 1^e = (gg^{-1})^e = g^{g^{-1}\cdot e}(g^{-1})^e = g^{g^{-1}\cdot e}g^{-1}$. Then $g = g^{g^{-1}\cdot e}$. □

Lemma 6. *Suppose that* $(e,g) \in B_{\bowtie}$, *then* (e,g) *is a regular element and* $(g^{-1} \cdot e, g^{-1}) \in V(e,g)$.

Proof. We have

$$(e,g)(g^{-1} \cdot e, g^{-1})(e,g) = (e(gg^{-1} \cdot e), g^{g^{-1}\cdot e}g^{-1})(e,g) = (e,1)(e,g) = (e(1 \cdot e), 1^e g) = (e,g)$$

and

$$(g^{-1} \cdot e, g^{-1})(e,g)(g^{-1} \cdot e, g^{-1}) = (g^{-1} \cdot e, g^{-e}g)(g^{-1} \cdot e, g^{-1}) = (g^{-1} \cdot e, 1^{g^{-1}\cdot e}g^{-1}) = (g^{-1} \cdot e, g^{-1})$$

Thus (e,g) is a regular. □

Proposition 13. *If* $P = E \bowtie G$, *where E is a semilattice and G is a group then*

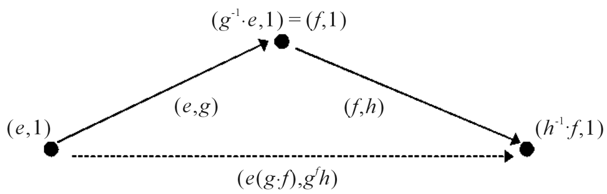
$$B_{\bowtie}(P) = \left\{(e,g) \in P : \left(g^{-1}\right)^{e} = g^{-1}\right\}$$

with composition defined by

$$(e,g)(f,h) = \left(e(g \cdot f), g^{f}h\right)$$

if $\left(g^{-1} \cdot e, 1\right) = (f,1)$ and $1 \in G$ acts trivially on E is a groupoid.

Proof.



We have to prove

◄ $\left(e(g \cdot f), g^{f}h\right) \in B_{\bowtie}(P)$

$$\left(e(g \cdot f), g^{f}h\right) \in B_{\bowtie}(P) \Leftrightarrow \left[\left(g^{f}h\right)^{-1}\right]^{e(g \cdot f)} = \left(g^{f}h\right)^{-1}$$

Now

$$\left[\left(g^{f}h\right)^{-1}\right]^{e(g \cdot f)} = \left[h^{-1}\left(g^{f}\right)^{-1}\right]^{e(g \cdot f)} = \left[h^{-1}\right]^{\left(g^{f}\right)^{-1} \cdot e(g \cdot f)}\left[\left(g^{f}\right)^{-1}\right]^{e(g \cdot f)}$$

Since $g^{-1} \cdot e = f$ implies that $g \cdot g^{-1} \cdot e = g \cdot f$, this implies that $e = g \cdot f$, therefore

$$\left(g^{f}\right)^{-1} = \left(g^{-1}\right)^{g \cdot f}\left(g^{-1}\right)^{g \cdot f} = \left(g^{-1}\right)^{e} = g^{-1}.$$

Then

$$\left[\left(g^{f}h\right)^{-1}\right]^{e(g \cdot f)} = \left[h^{-1}\right]^{\left(g^{f}\right)^{-1} \cdot e(e)}\left[\left(g^{f}\right)^{-1}\right]^{e(e)} = \left[h^{-1}\right]^{\left(g^{f}\right)^{-1} \cdot e}\left[\left(g^{f}\right)^{-1}\right]^{e}$$

$$= \left[h^{-1}\right]^{g^{-1} \cdot e}\left[g^{-1}\right]^{e} = h^{-1}\left(g^{f}\right)^{-1} = \left(g^{f}h\right)^{-1}.$$

◄ $\left(e(g \cdot f), g^{f}h\right)$ starts at $(e,1)$

But $\left(e(g \cdot f), g^{f}h\right)$ starts at $\left(e(g \cdot f), 1\right)$ and $g \cdot f = e$, so $e(g \cdot f) = e$,

◄ $\left(e(g \cdot f), g^{f}h\right)$ ends at $\left(h^{-1} \cdot f, 1\right)$

But $\left(e(g \cdot f), g^{f}h\right)$ ends at $\left(h^{-1}\left(g^{f}\right)^{-1} \cdot e(g \cdot f), 1\right)$

$$h^{-1}\left(g^{f}\right)^{-1} \cdot e(g \cdot f) = h^{-1}g^{-1} \cdot e^{2} = h^{-1} \cdot g^{-1} \cdot e = h^{-1} \cdot f.$$

Thus $B_{\bowtie}(P)$ is a groupoid. □

Now we introduce an ordering on $B_{\bowtie}(P)$. The ordering is giving as follows:

$$\boxed{(e,g) \le (f,h) \Leftrightarrow e \le f \text{ and } g = h^{h^{-1} \cdot e}.}$$

Lemma 7. *The ordering on* $B_{\bowtie}(P)$ *defined by*

$$(e,g) \le (f,h) \Leftrightarrow e \le f \text{ and } g = h^{h^{-1} \cdot e}.$$

is transitive.

Proof. We have to prove that if $(e,g)\le(f,h)\le(l,k)$, then $(e,g)\le(l,k)$. We have $(e,g)\le(f,h)\Leftrightarrow e\le f$ and $g=h^{h^{-1}\cdot e}$ and $(f,h)\le(l,k)\Leftrightarrow f\le l$ and $h=k^{k^{-1}\cdot f}$. Since $1=1^e=(g^{-1}g)^e=(g-1)^{g\cdot e}g^e$. Thus $(g^e)^{-1}=(g^{-1})^{g\cdot e}$. We conclude $g=h^{h^{-1}\cdot e}=(h^{-e})^{-1}$ and $h=k^{k^{-1}\cdot f}=(k^{-f})^{-1}$. Now, $e\le f\le l$ implies that $e\le l$ and $g=(h^{-e})^{-1}=\left((k^{-f})^e\right)^{-1}=(k^{-fe})^{-1}=(k^{-e})^{-1}=k^{k^{-1}\cdot e}$. Thus $\le$ is transitive. □

Lemma 8. *The ordering on* $B_{\bowtie}(P)$ *defined by*

$$B_{\bowtie}(P)(e,g)\le(f,h)\Leftrightarrow e\le f \text{ and } g=h^{h^{-1}\cdot e}.$$

is antisymmetric.

Proof. We have to prove if $(e,g)\le(f,h)$ and $(f,h)\le(e,g)$, then $(e,g)=(f,h)$. Now $(e,g)\le(f,h)\Leftrightarrow e\le f$ and $g=h^{h^{-1}\cdot e}$ and $(f,h)\le(e,g)\Leftrightarrow f\le e$ and $h=g^{g^{-1}\cdot f}$. Thus $e=f$ and $g=(h^{-e})^{-1}=\left((g^{-f})^e\right)^{-1}=(g^{-f})^{-1}=g^{g^{-1}\cdot f}=h$. Thus $\le$ is antisymmetric. □

Proposition 14. $B_{\bowtie}(P)=\left\{(e,g)\in P:(g^{-1})^e=g^{-1}\right\}$ *with ordering defined by*

$$(e,g)\le(f,h)\Leftrightarrow e\le f \text{ and } g=h^{h^{-1}\cdot e}.$$

is a partial order set.

Proof. Clear from the definition of the ordering that $\le$ is reflexive. By Lemma 7 and Lemma 8 $\le$ is transitive and antisymmetric. Thus $(B_{\bowtie}(P),\le)$ is a partial order set. □

Next we prove that $(B_{\bowtie}(P),\le)$ is an ordered groupoid.

Lemma 9. *If* $(e,g)\le(f,h)$, *then* $(e,g)^{-1}\le(f,h)^{-1}$ *for all* $(e,g),(f,h)\in B_{\bowtie}(P)$.

Proof. Suppose that $(e,g)\le(f,h)$, so that $e\le f$ and $g=h^{h^{-1}\cdot e}$. Now, we have

$$g^{-1}\cdot e=\left(h^{h^{-1}\cdot e}\right)^{-1}\cdot e=h^{-e}\cdot e=h^{-1}\cdot e.$$

Thus

$$\begin{aligned}(g^{-1}\cdot e)(h^{-1}\cdot f)&=(h^{-1}\cdot e)(h^{-1}\cdot f)=(h^{-1}\cdot f)(h^{-1}\cdot e)\\&=(h^{-1}\cdot f)\left((h^{-1})^f\cdot e\right)=h^{-1}\cdot fe=h^{-1}\cdot e=g^{-1}\cdot e\end{aligned}$$

and

$$(h^{-1}\cdot f)(g^{-1}\cdot e)=(h^{-1}\cdot f)(h^{-1}\cdot e)=(h^{-1}\cdot f)\left((h^{-1})^f\cdot e\right)=h^{-1}\cdot fe=h^{-1}\cdot e=g^{-1}\cdot e.$$

Therefore $g^{-1}\cdot e\le h^{-1}\cdot f$. Also

$$(h^{-1})^{h\cdot g^{-1}\cdot e}=(h^{-1})^{h\cdot h^{-1}\cdot e}=(h^{-1})^{hh^{-1}\cdot e}=(h^{-1})^{1.e}=(h^{-1})^e=g^{-1}.$$

and hence $(e,g)^{-1}\le(f,h)^{-1}$ as required. □

Lemma 10. *If* $(p,s)\le(e,g)$ *and* $(q,t)\le(f,h)$ *such that the composition* $(p,s)(q,t)$ *and* $(e,g)(f,h)$ *are defined, then*

$$(p,s)(q,t)\le(e,g)(f,h)$$

for all $(p,s),(e,g),(q,t),(f,h)\in B_{\bowtie}(P)$.

Proof. Suppose that $(p,s)(q,t)=(p,u)$ and $(e,g)(f,h)=(e,k)$ are defined

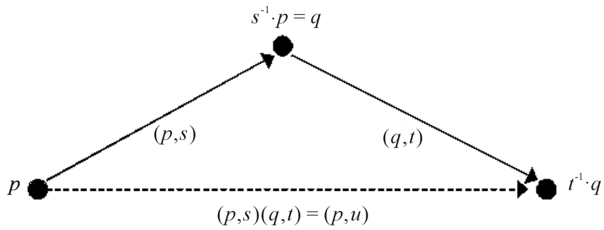
Then we have

$$p \le e \text{ and } s = g^{g^{-1}\cdot p}.$$

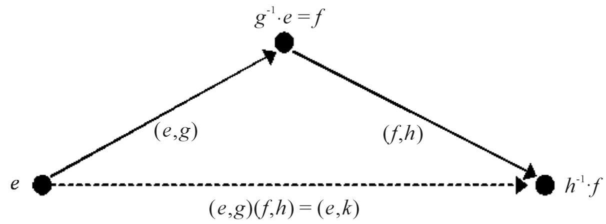
and

$$q \le f \text{ and } t = h^{h^{-1}\cdot q}.$$

and we have the following



where $u = s^q t = s^{s^{-1}\cdot p} t = st$



where $k = g^f h = g^{g^{-1}\cdot e} h = gh.$ Now

$$\begin{aligned} u = st &= \left(g^{g^{-1}\cdot p}\right)\left(h^{h^{-1}\cdot q}\right) = \left(g^{g^{-1}\cdot p}\right)\left(h^{h^{-1}\cdot s^{-1}\cdot p}\right) \\ &= \left(g^{g^{-1}\cdot p}\right)\left(h^{h^{-1}\cdot g^{-1}\cdot p}\right) = \left(g^{g^{-1}\cdot p}\right)\left(h^{h^{-1}g^{-1}\cdot p}\right) \end{aligned}$$

and

$$k^{k^{-1}\cdot p} = gh^{(gh)^{-1}\cdot p} = gh^{h^{-1}g^{-1}\cdot p} = g^{h\cdot\left(h^{-1}g^{-1}\cdot p\right)} h^{h^{-1}g^{-1}\cdot p} = \left(g^{g^{-1}\cdot p}\right)\left(h^{h^{-1}g^{-1}\cdot p}\right).$$

Then $u = k^{k^{-1}\cdot p}$. Moreover, $p \le e$. Thus $(p,u) \le (e,k)$ as required. □

Lemma 11. *If* $(e,g) \in B_{\bowtie}(P)$ *and* $(f,1)$ *is an identity such that* $(f,1) \le (e,1)$, *then* $\left(f, g^{g^{-1}\cdot f}\right)$ *is the restriction of* (e,g) *to* $(e,1)$.

Proof. Suppose $(e,g) \in B_{\bowtie}(P)$ and $(f,1)$ is an identity such that $(f,1) \le (e,1)$, since $\left(g^{g^{-1}\cdot f}\right)^{-1} = g^{-f} = \left(g^{-f}\right)^f$, then $\left(f, g^{g^{-1}\cdot f}\right) \in B_{\bowtie}(P)$. Also

$$\begin{aligned} \left(f, g^{g^{-1}\cdot f}\right)\left(f, g^{g^{-1}\cdot f}\right)^{-1} &= \left(f, g^{g^{-1}\cdot f}\right)\left(g^{-f}\cdot f, g^{-f}\right) \\ &= \left(f\left[\left(g^{-f}\right)^{-1}\cdot g^{-f}\cdot f\right], \left[\left(g^{-f}\right)^{-1}\right]^{g^{-f}\cdot f} g^{-f}\right) = (f,1). \end{aligned}$$

Moreover, $\left(f, g^{g^{-1}\cdot f}\right) \le (e,g)$ and unique by definition.

Thus $\left(f, g^{g^{-1}\cdot f}\right)$ is the restriction of (e,g) to $(e,1)$. □

Proposition 15. $\left(B_{\bowtie}(P), \le\right)$ *is an inductive groupoid.*

Proof. We prove that $(OG1), (OG2)$ and $(OG3)$ hold. By Lemma 9 we have $(OG1)$, by Lemma 10

$(OG2)$ holds and by Lemma 11 $(OG3)$ holds. Since the partially ordered set of identities forms a meet semilattice $E\left(B_{\bowtie}(P)\right) \cong E$. Thus $\left(B_{\bowtie}(P), \leq\right)$ is an inductive groupoid. □

Theorem 7. *If* $P = E \bowtie G$, *where E is a semilattice and G is a group, then*

$$B_{\bowtie}(P) = \left\{(e,g) \in P : \left(g^{-1}\right)^{e} = g^{-1}\right\}$$

is an inverse semigroup with multiplication defined by

$$(e,g)(f,h) = \left(e(g \cdot f), g^{f} h^{h^{-1}g^{-1}\cdot e}\right).$$

Proof. Let $(e,g),(f,h) \in B_{\bowtie}(P)$ since $(e,g)^{-1}(e,g) = \left(g^{-1}\cdot e, 1\right)$ and $(f,h)(f,h)^{-1} = (f,1)$, we form the pseudoproduct $(e,g) \otimes (f,h)$ using the greatest lower pound.

$$\ell = (e,g)^{-1}(e,g) \wedge (f,h)(f,h)^{-1} = \left(g^{-1}\cdot e, 1\right) \wedge (f,1) = \left(g^{-1}\cdot e, 1\right)(f,1) = \left(\left(g^{-1}\cdot e\right) f, 1\right)$$

and

$$\left(\ell \mid (f,h)\right) = \left(\left(\left(g^{-1}\cdot e\right) f, 1\right) \middle| (f,h)\right) = \left(\left(g^{-1}\cdot e\right) f, h^{\left(h^{-1}g^{-1}\cdot e\right)\left(\left(h^{-1}\right)^{g^{-1}\cdot e}\cdot f\right)}\right)$$

and

$$\left((e,g) \mid \ell\right) = \left(\ell \middle| \left(g^{-1}\cdot e, g^{-1}\right)\right)^{-1} = \left(\left(g^{-1}\cdot e\right) f, \left(g^{-1}\right)^{g\cdot\left(g^{-1}\cdot e\right) f}\right)^{-1}$$

$$= \left(\left(g^{-1}\cdot e\right) f, \left(g^{-1}\right)^{\left(g\cdot g^{-1}\cdot e\right)\left(g^{\left(g^{-1}\cdot e\right)}\cdot f\right)}\right)^{-1} = \left(\left(g^{-1}\cdot e\right) f, \left(g^{-1}\right)^{e(g\cdot f)}\right)^{-1}$$

$$= \left(\left[\left(g^{-1}\right)^{e(g\cdot f)}\right]^{-1} \cdot \left(g^{-1}\cdot e\right) f, \left[\left(g^{-1}\right)^{e(g\cdot f)}\right]^{-1}\right).$$

Now, since $(e,g) \in B_{\bowtie}(P)$ then $g^{-e} = g^{-1}$ and so $\left[\left(g^{-1}\right)^{e(g\cdot f)}\right]^{-1} = \left[\left(g^{-1}\right)^{(g\cdot f)}\right]^{-1}$. We have $\left[\left(g^{-1}\right)^{(g\cdot f)}\right]^{-1} = g^{f}$. Then

$$\left((e,g) \mid \ell\right) = \left(g^{f} \cdot \left(g^{-1}\cdot e\right) f, g^{f}\right),$$

Therefore

$$(e,g) \otimes (f,h) = \left((e,g) \mid \ell\right)\left(\ell \mid (f,h)\right)$$

$$= \left(g^{f} \cdot \left(g^{-1}\cdot e\right) f, g^{f}\right)\left(\left(g^{-1}\cdot e\right) f, h^{\left(h^{-1}g^{-1}\cdot e\right)\left(\left(h^{-1}\right)^{g^{-1}\cdot e}\cdot f\right)}\right)$$

$$= \left(\left(g^{f} \cdot \left(g^{-1}\cdot e\right) f\right), \left(g^{f}\right)^{\left(g^{-1}\cdot e\right) f} h^{\left(h^{-1}g^{-1}\cdot e\right)\left(\left(h^{-1}\right)^{g^{-1}\cdot e}\cdot f\right)}\right)$$

$$= \left(g^{f} \cdot \left(g^{-1}\cdot e\right) f, g^{f} h^{h^{-1}\left[\left(g^{-1}\cdot e\right) f\right]}\right).$$

Now, we have in the ordering defined on $B_{\bowtie}(P)$, $(g^{-1}\cdot e)f \le g^{-1}\cdot e$ and $g^{-1}\cdot e$ acts trivially on g this implies that $g^{(g^{-1}\cdot e)f} = g^f$. Then we have

$$\begin{aligned} g^f\cdot(g^{-1}\cdot e)f &= g^{(g^{-1}\cdot e)f}\cdot\left[(g^{-1}\cdot e)f\right] = g\cdot\left[(g^{-1}\cdot e)f\right] \\ &= (g\cdot g^{-1}\cdot e)\left(\left(g^{g^{-1}\cdot e}\cdot f\right)\right) = (gg^{-1}\cdot e)(g\cdot f) = e(g\cdot f) \end{aligned}$$

and

$$\begin{aligned} g^f h^{h^{-1}\left[(g^{-1}\cdot e)f\right]} &= g^f h^{h^{-1}\left[f(g^{-1}\cdot e)\right]} = g^f h^{(h^{-1}\cdot f)\left((h^{-1})^f\cdot(g^{-1}\cdot e)\right)} \\ &= g^f\left(h^{h^{-1}\cdot f}\right)^{h^{-1}(g^{-1}\cdot e)} = g^f h^{h^{-1}g^{-1}\cdot e}. \end{aligned}$$

Therefore

$$(e,g)(f,h) = \left(e(g\cdot f), g^f h^{h^{-1}g^{-1}\cdot e}\right).$$

Thus $B_{\bowtie}(P)$ is an inverse semigroup. □

We summarize the main results of this paper in the following:

1) We characterize Green's relations ($\mathcal{L}$ and $\mathcal{R}$) of the Zappa-Szép product $M \bowtie G$ of a monoid M and a group G we prove that $(m,g)\mathcal{R}(n,h) \Leftrightarrow m\mathcal{R}n$ in M. And If $(g^{-1}\cdot m)\mathcal{L}(h^{-1}\cdot n)$ such that $(g^{-1})^m = g^{-1}$ and $(h^{-1})^n = h^{-1}$ in G, then $(m,g)\mathcal{L}(n,h)$ in $M \bowtie G$.

2) We prove that the internal Zappa-Szép product S of subsemigroups A and B is an enlargement of a local submonoid eSe for some $e \in RI(A)\cap LI(B)$, and eSe is the internal Zappa-Szép product of the submonoids $\overline{A}$ and $\overline{B}$ where $\overline{A} = \{x : x = ea, a \in A\}, \overline{B} = \{y : y = be, b \in B\}$. And M is the internal Zappa-Szép product of a left-zero semigroup A and a right-zero semigroup B if and only if M is a rectangular band.

3) We give the necessary and sufficient conditions for the internal Zappa-Szép product $M = A \bowtie B$ of regular subsemigroups A and B to again be regular. We prove that $M = A \bowtie B$ is regular if and only if $ef \in Reg(M)$ where $e \in E(A)$ and $f \in E(B)$.

4) The Zappa-szep products $P = G \bowtie G$ of the nilpotent group G with left and right conjugation action of G on it self is abelian if and only if G is abelian, in which case $P = G \times G$; and if G is non-abelian and hence nilpotent of class 2, then P is also nilpotent of class 2.

5) The Zappa-Szép product of inverse semigroups need not in general be an inverse semigroup. In this paper we give the necessary conditions for their existence and we modified the definition of semidirect products in the inverse case to obtain what we termed λ-semidirect products. The λ-semidirect product of inverse semigroups is again inverse. We construct from the Zappa-Szép product P of a semilattice E and a group G, an inverse semigroup by constructing an inductive groupoid.

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