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Distribution of Points of Interpolation and of Zeros of Exactly Maximally Convergent Multipoint Padé Approximants

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Abstract

Given a regular compact set E in $\mathbb{C}$, a unit measure μ supported by ∂E, a triangular point set $\beta := \left\{\{\beta_{n,k}\}_{k=1}^{n}\right\}_{n=1}^{\infty}$, $\beta \subset \partial E$ and a function f, holomorphic on E, let $\pi_{n,m}^{\beta,f}$ be the associated multipoint β-Padé approximant of order (n,m). We show that if the sequence $\pi_{n,m}^{\beta,f}$, $n \in \Lambda$, $\Lambda \subseteq \mathbb{N}$, m-fixed, converges exactly μ-maximally to f with respect to the m-meromorphy, then the points $\beta_{n,k}$ are uniformly distributed on ∂E with respect to μ as $n \in \Lambda$. Furthermore, a result about the behavior of the zeros of the exact maximally convergent sequence Λ is provided, under the condition that Λ is "dense enough".

Keywords

Multipoint Padé Approximants, Maximal Convergence, Domain of *m*-Meromorphy

1. Introduction

We first introduce some needed notations.

Let Π_n, $n \in \mathbb{N}$ be the class of the polynomials of degree $\le n$ and $\mathcal{R}_{n,m} := \{r = p/q, p \in \Pi_n, q \in \Pi_m, q \not\equiv 0\}$.

Given a compact set E, we say that E is regular, if the unbounded component of the complement $E^c := \overline{\mathbb{C}} \setminus E$ is solvable with respect to Dirichlet problem. We will assume throughout the paper that E possesses a connected complement E^c. In what follows, we will be working with the max-norm $\|\cdots\|_E$ on E, that is $\|\cdots\|_E := \max_{z \in E} |\cdots|(z)$.

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Let $\mathcal{B}(E)$ be the class of the unit measures supported on E, that is $\operatorname{supp}(\cdots) \subseteq E$. We say that the infinite sequence of Borel measures $\{\mu_n\} \in \mathcal{B}(E)$ converges in the weak topology to a measure μ and write $\mu_n \to \mu$, if

$$\int g(t)\,\mathrm{d}\mu_n \to \int g(t)\,\mathrm{d}\mu$$

for every function g continuous on E. We associate with a measure $\mu \in \mathcal{B}(E)$, the logarithmic potential $U^{\mu}(z)$, that is,

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|}\,\mathrm{d}\mu .$$

Recall that U^{μ} ([1]) is a function superharmonic in $\mathbb{C}$, subharmonic in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\mu)$, harmonic in $\mathbb{C} \setminus \operatorname{supp}(\mu)$ and

$$U^{\mu}(z) = \ln \frac{1}{|z|} + o(1), \quad z \to \infty .$$

We also note the following basic fact ([2]):

Carleson's lemma: *Given the measures* μ_1, μ_2 *supported by* ∂E, *suppose that* $U^{\mu_1}(z) = U^{\mu_2}(z)$ *for every* $z \notin E$. *Then,* $\mu_1 = \mu_2$.

Finally, we associate with a polynomial $p \in \Pi_n$, the normalized counting measure μ_p of p, that is

$$\mu_p(F) := \frac{\text{number of zeros of } p \text{ on } F}{\deg p},$$

where F is a point set in $\mathbb{C}$.

Given a domain $B \subset \mathbb{C}$, a function g and a number $m \in \mathbb{N}$, we say that g is m-meromorphic in B $\left(g \in \mathcal{M}_m(B)\right)$ if g has no more than m poles in B (poles are counted with their multiplicities). We say that a function f is holomorphic on the compactum E and write $f \in \mathcal{A}(E)$, if it is holomorphic in some open neighborhood of E.

Let β be an infinite triangular table of points, $\beta := \left\{\{\beta_{n,k}\}_{k=1}^{n}\right\}_{n=1,2,\cdots}$, $\beta_{n,k} \in E$, with no limit points outside E (we write $\beta \in E$). Set

$$\omega_n(z) := \prod_{k=1}^{n}\left(z - \beta_{n,k}\right).$$

Let $f \in \mathcal{A}(E)$ and (n,m) be a fixed pair of nonnegative integers. The rational function $\pi_{n,m}^{\beta,f} := p/q$ where the polynomials $p \in \Pi_n$ and $q \in \Pi_m$ are such that

$$\frac{fq - p}{\omega_{n+m+1}} \in \mathcal{A}(E)$$

is called *a β-multipoint Padé approximant of f of order* (n,m). As is well known, the function $\pi_{n,m}^{\beta,f}$ always exists and is unique [3] [4]. In the particular case when $\beta \equiv 0$, the multipoint Padé approximant $\pi_{n,m}^{\beta,f}$ coincides with the classical Padé approximant $\pi_{n,m}^{f}$ of order (n,m) ([5]).

Set

$$\pi_{n,m}^{\beta,f} := \frac{P_{n,m}^{\beta,f}}{Q_{n,m}^{\beta,f}}, \tag{1}$$

where the polynomials $P_{n,m}^{\beta,f}$ and $Q_{n,m}^{\beta,f}$ do not have common divisors. The zeros of $Q_{n,m}^{\beta,f}$ are called free zeros of $\pi_{n,m}^{\beta,f}$; $\deg Q_{n,m} \le m$.

We say that the points $\beta_{n,k}$ are *uniformly distributed relatively to the measure* μ, if

$$\mu_{\omega_n} \to \mu, \quad n \to \infty .$$

We recall the notion of m_1-Hausdorff measure (cf. [6]). For $\Omega \subset \mathbb{C}$, we set

$$m_1(\Omega) := \inf\left\{\sum_{\nu}|V_\nu|\right\}$$

where the infimum is taken over all coverings $\left\{\sum V_\nu\right\}$ of Ω by disks and $|V_\nu|$ is the radius of the disk V_ν.

Let D be a domain in $\mathbb{C}$ and φ a function defined in D with values in $\overline{\mathbb{C}}$. A sequence of functions $\{\varphi_n\}$, meromorphic in D, is said to converge to a function φ m_1-almost uniformly inside D if for any compact subset $K \subset D$ and every $\varepsilon > 0$ there exists a set $K_\varepsilon \subset K$ such that $m_1(K \setminus K_\varepsilon) < \varepsilon$ and the sequence $\{\varphi_n\}$ converges uniformly to φ on K_ε.

For $\mu \in \mathcal{B}(E)$, define

$$\rho_{\min} := \inf_{z\in E} \mathrm{e}^{-U^\mu(z)},$$

and

$$\varrho_{\max} := \max_{z\in E} \mathrm{e}^{-U^\mu(z)};$$

(U^μ is superharmonic on E; hence, it attains its minimum (on E)). As is known ([1] [7]),

$$\mathrm{e}^{-U^\mu(z)} \ge \rho_{\min}, \quad z \in E^c,$$

Set, for $r > \rho_{\min}$,

$$E_\mu(r) := \left\{z \in \mathbb{C}, \mathrm{e}^{-U^\mu(z)} < r\right\}.$$

Because of the upper semicontinuity of the function $\chi(z) := \mathrm{e}^{-U^\mu(z)}$, the set $E_\mu(r)$ is open; clearly $E_\mu(r_1) \subset E_\mu(r_2)$ if $r_1 \le r_2$ and $E_\mu(r) \supset E$ if $r > \varrho_{\max}$.

Let $f \in \mathcal{A}(E)$ and $m \in \mathbb{N}$ be fixed. Let $R_{m,\mu}(f) = R_{m,\mu}$ and $D_{m,\mu}(f) = D_{m,\mu} := E_\mu(R_{m,\mu})$ denote, respectively, *the radius and domain of m-meromorphy with respect to* μ, that is

$$R_{m,\mu} := \sup\left\{r, f \in \mathcal{M}_m(E_\mu(r))\right\}$$

Furthermore, we introduce the notion of a μ *-maximal convergence to f with respect to the m-meromorphy of a sequence of rational functions* $\{r_{n,\nu}\}$ (a μ-maximal convergence), that is, for any $\varepsilon > 0$ and each compact set $K \subset D_m$, there exists a set $K_\varepsilon \subset K$ such that $m_1(K \setminus K_\varepsilon) < \varepsilon$ and

$$\limsup_{n+\nu\to\infty} \left\|f - r_{n,\nu}\right\|_{K_\varepsilon}^{1/n} \le \frac{\left\|\mathrm{e}^{-U^\mu}\right\|_K}{R_{m,\mu}(f)}.$$

Hernandez and Calle Ysern proved the followings:

Theorem A [8]: *Let* E, μ, β *and* ω_n, $n = 1, 2, \cdots$ *be defined as above. Suppose that* $\mu_{\omega_n} \to \mu$ *as* $n \to \infty$ *and* $f \in \mathcal{A}(E)$. *Then, for each fixed* $m \in \mathbb{N}$, *the sequence* $\pi_{n,m}^{\beta,f}$ *converges to f* μ *-maximally with respect to the* m *-meromorphy.*

Theorem A generalizes Saff's theorem of Montessus de Ballore's type about multipoint Padé approximants (see [3]).

We now utilize the *normalization of the polynomials* $Q_{n,m}(z)$ *with respect to a given open set* $D_{m,\mu}$, that is,

$$Q_{n,m}(z) = \prod\left(z - \alpha'_{n,k}\right)\prod\left(1 - z/\alpha''_{n,k}\right), \tag{2}$$

where $\alpha'_{n,k}$, $\alpha''_{n,k}$ are the zeros lying inside, resp. outside $D_{m,\mu}$. Under this normalization, for every compact set K and n large enough there holds

$$\left\|Q_{n,m}^{\beta,f}\right\|_K \le C_1,$$

where $C_1 = C_1(K)$ is a positive constant, depending on K. In the sequel, we denote by C_i positive constant, independent on n and different at different occurrences.

In [8], the set K_ε (look at the definition of a μ-maximal convergence) is explicitly written, namely

$K_\varepsilon := K \setminus \Omega(\varepsilon)$, where

$$\Omega(\varepsilon) := \bigcup_{n=1}^{\infty}\left(\bigcup_{\alpha'_{n,k}}\left\{z, \left|z-\alpha'_{n,k}\right| < \varepsilon/\left(2mn^2\right)\right\}\right).$$

For $\Omega(\varepsilon)$ we have

$$m_1\left(\Omega(\varepsilon)\right) \le \varepsilon .$$

For points $z \notin \Omega(\varepsilon)$, we have

$$\left|Q_{n,m}^{\beta,f}(z)\right| \ge C_2\left(\varepsilon/mn^2\right)^{k_n},$$

where k_n stands for the number of the zeros of $Q_{n,m}^{\beta,f}$ in $D_{m,\mu}$; $k_n \le m$.

Let Q be the monic polynomial, the zeros of which coincide with the poles of f in $D_{m,\mu}$; $\deg Q \le m$. It was proved in [8] (Proof of Lemma 2.3) that for every compact subset K of $D_{m,\mu}$

$$\limsup_{n\to\infty}\left\|fQQ_{n,m}^{\beta,f} - QP_{n,m}^{\beta,f}\right\|_K^{1/n} \le \frac{\left\|e^{-U^\mu}\right\|_K}{R_{m,\mu}}. \tag{3}$$

Hence, $-U^\mu(z) - \ln R_{m,\mu}$ is a harmonic majorant in $D_{m,\mu}$ of the family $\left\{\left|\left(fQQ_{n,m}^{\beta,f} - QP_{n,m}^{\beta,f}\right)(z)\right|^{1/n}\right\}_{n=1}^{\infty}$.

Theorem B [8]: *With* E, μ, m, ω_n *and f as in Theorem A, assume that K is a regular compact set for which* $\left\|e^{-U^\mu}\right\|_K$ *is not attained at a point on* E. *Suppose that the function f is defined on K and satisfies*

$$\limsup_{n\to\infty}\left\|f - \pi_{n,m}^{\beta,f}\right\|_K^{1/n} \le \left\|e^{-U^\mu}\right\|_K / R < 1 .$$

Then $R \le R_{m,\mu}(f)$.

Suppose that $\infty > R_{m,\mu} > \varrho_{\max}$ and $D_{m,\mu}$ is connected. Let V be a disk in $D_m \setminus E_\mu\left(\varrho_{\max}\right)$, centered at a point z_0 of radius $r > 0$ and such that f is analytic on V. Fix r_1, $0 < r_1 < r$ and set $A := \left\{z, r_1 \le |z - z_0| \le r\right\}$. Fix a number $\varepsilon < (r - r_1)/4$. Introduce, as before, the set $\Omega(\varepsilon)$. Recall that

$$m_1\left(\Omega(\varepsilon)\right) \le \varepsilon .$$

It is clear that the set $A \setminus \Omega(\varepsilon)$ contains a concentric circle Γ (otherwise we would obtain a contradiction with $m_1\left(\Omega(\varepsilon)\right) < (r - r_1)/4$.) We note that the function f and the rational functions $\pi_{n,m}^{\beta,f}$ are well defined on Γ. Viewing (3), we may write

$$\limsup_{n\to\infty}\left\|QQ_{n,m}^{\beta,f} f - QP_{n,m}^{\beta,f}\right\|_\Gamma^{1/N} \le \left\|e^{-U^\mu}\right\|_\Gamma / R_{m,\mu},$$

Suppose that

$$\limsup_{n\to\infty}\left\|QQ_{n,m}^{\beta,f} f - QP_{n,m}^{\beta,f}\right\|_\Gamma^{1/n} < \left\|e^{-U^\mu}\right\|_\Gamma / R_{m,\mu}.$$

or, what is the same,

$$\limsup_{n\to\infty}\left\|QQ_{n,m}^{\beta,f} f - QP_{n,m}^{\beta,f}\right\|_\Gamma^{1/n} \le \left\|e^{-U^\mu}\right\|_\Gamma / \left(R_{m,\mu} + \sigma\right) < 1 .$$

for an appropriate $\sigma > 0$. Then,

$$\left|\left(f - \pi_{n,m}^{\beta,f}\right)(z)\right|_\Gamma \le C_3\left(n^2 m/\varepsilon\right)^m \left(\left\|e^{-U^\mu}\right\|_\Gamma / \left(R_{m,\mu} + \sigma\right)\right)^n .$$

for all $z \in \Gamma$ and n large enough. This leads to

$$\limsup_{n\to\infty}\left\|f - \pi_{n,m}^{\beta,f}\right\|_\Gamma^{1/n} \le \left\|e^{-U^\mu}\right\|_\Gamma / \left(R_{m,\mu} + \sigma\right).$$

using Theorem B, we arrive at $R_{m,\mu}+\sigma<R_{m,\mu}$. The contradiction yields

$$\limsup_{n\to\infty}\left\|QQ_{n,m}^{\beta,f}f-QP_{n,m}^{\beta,f}\right\|_{\bar{V}_\Gamma}^{1/n}=\left\|e^{-U^\mu}\right\|_{\bar{V}_\Gamma}\Big/R_{m,\mu},$$

where V_Γ is the disk bounded by Γ.

Then the function $-U^\mu-\ln R_{m,\mu}$ is an exact harmonic majorant of the family $\left\{\left|fQQ_{n,m}^{\beta,f}-QP_{n,m}^{\beta,f}\right|^{1/n}\right\}$ in $D_{m,\mu}$ (see (3)). Therefore, there exists a subsequence Λ such that for every compact subset $K\subset D_{m,\mu}\setminus E$

$$\lim_{n\to\infty,n\in\Lambda}\left\|QfQ_{n,m}^{\beta,f}-P_{n,m}^{\beta,f}Q\right\|_K^{1/n}=\left\|e^{-U^\mu}\right\|_K\Big/R_{m,\mu}. \tag{4}$$

(see [9] [10]) for a discussion of exact harmonic majorant)). We will refer to this sequences as to an exact μ-*maximal convergent sequence* to *f* with respect to the *m*-meromorphy.

It is clear that for any $\varepsilon>o$ and each compactum $K\subset D_{m,\mu}$ there exists a set $K_\varepsilon\subset K$ such that $m_1(K\setminus K_\varepsilon)<\varepsilon$ and

$$\lim_{n\to\infty,n\in\Lambda}\left\|f-\pi_{n,m}^{\beta,f}\right\|_{K\setminus K_\varepsilon}^{1/n}=\left\|e^{-U^\mu}\right\|_\Gamma\Big/\left(R_{m,\mu}\right).$$

2. Main Results and Proofs

The main result of the present paper is

Theorem 1: *Under the same conditions on* E, *assume that* $\mu\in\mathcal{B}(\partial E)$ *and that* $\beta\subset\partial E$ *is a triangular set of points. Let* $m\in\mathbb{N}$ *be fixed,* $f\in\mathcal{A}(E)$ *and* $\varrho_{\max}<R_{m,\mu}<\infty$. *Suppose that* $D_{m,\mu}$ *is connected. If for a subsequence* Λ *of the multipoint Padé approximants* $\pi_{n,m}^{\beta,f}$ *condition* (4) *holds, then* $\mu_{\omega_n}\to\mu$ *as* $n\to\infty$, $n\in\Lambda$.

The problem of the distribution of the points of interpolation of multipoint Padé approximants has been investigated, so far, only for the case when the measure μ coincides with the equilibrium measure μ_E of the compact set *E*. It was first raised by Walsh ([11], Chp. 3) while considering maximally convergent polynomials with respect to the equilibrium measure. He showed that the sequence μ_{ω_n} converged weakly to μ_E through the entire set $\mathbb{N}$ (respectively their associated balayage measures onto the boundary of *E*) iff the interpolating polynomials at the points of β of every function $f_t(z)$ of the form $f_t(z):=1/(t-z)$, t-fixed, $t\notin E$, converged μ_E-maximally to f_t. Walsh's result was extended to multipoint Padé approximants with a fixed number of the free poles by Ikonomov in [12], as well as to generalized Padé approximants, associated with a regular condenser [13]. The case of polynomial interpolation of an arbitrary function $f\in\mathcal{A}(E)$ was considered by Grothmann [14]; he established the existence of an appropriate sequence Λ such that $\mu_{\omega_n}\to\mu_E$, $n\to\infty$, $n\in\Lambda$, respectively the balayage measures onto ∂E. Grothmann's result was extended to multipoint Padé approximants $\pi_{n,m}^{\beta,f}$ with a fixed number of the free poles (see [15]). Finally, in [16] the case was considered, when the degrees of the denominators tended slowly to infinity, namely, $m_n=o(n/\ln n)$, $n\to\infty$.

As a consequence of Theorem 1, we derive

Theorem 2: *Under the conditions of Theorem* 1, *suppose that the* μ*-exact maximally convergent sequence* $\Lambda:=\{n_k\}_{k=1}^\infty$ *satisfies the condition to be* "*dense enough*", *that is*

$$\limsup_{n_k\to\infty,n_k\in\Lambda}\frac{n_{k+1}}{n_k}<\infty.$$

Then, there is at least one point $z_0\in\partial D_{m,\mu}(f)$ such that

$$\limsup_{n\to\infty,n\in\Lambda}\mu_{P_{n,m}^{\beta,f}}\left(V_{z_0}(r)\right)>0.$$

Proof of Theorem 1: Set $Q_{n,m}^{\beta,f}:=Q_n$, $P_{n,m}^{\beta,f}:=P_n$ and $F:=fQ$. Fix numbers R,τ,r such that $\varrho_{\max}<R<\tau<r<R_{m,\mu}$ and $E_\mu(R)$ is connected. Then, by the conditions of the theorem, for every compactum $K\subset D_{m,\mu}$ (comp. (4))

$$\lim_{n\in\Lambda}\left\|FQ_n-QP_n\right\|_K^{1/n}=\left\|e^{-U^\mu}\right\|_K\Big/R_{m,\mu},\quad n\in\Lambda. \tag{5}$$

Select a positive number η such that $R+\eta<\tau<\tau+\eta<r<R_{m,\mu}$. Let Γ be an analytic curve in

$E_\mu(r)\setminus E_\mu(\tau+\eta)$ such that Γ winds around every point in $E_\mu(\tau)$ exactly once. In an analogous way, we select a curve $\gamma\subset E_\mu(R+\eta)\setminus E_\mu(R)$. Additionally, we require that U^μ is constant on Γ and γ. Set

$$F_n(z):=\frac{1}{n}\ln\left|FQ_n-P_nQ\right|(z)+U^\mu(z)+\ln R_{m,\mu},\quad n\in\Lambda. \tag{6}$$

Let $\sigma>0$ be arbitrary. The functions F_n are subharmonic in $E_\mu(r)\setminus E_\mu(R)$. By (5) and the choice of Γ,

$$\max_{t\in\Gamma}F_n(t)\le-\min_{t\in\Gamma}+\max_{t\in\Gamma}+\sigma\le\sigma,\quad N\in\Lambda,\quad n\ge n_1 0 n_1(\sigma),$$

and, analogously,

$$\max_{t\in\gamma}F_n(t)\le-\min_{t\in\Gamma}+\max_{t\in\Gamma}\le\sigma,\quad N\in\Lambda,\quad n>n_1.$$

Then, by the max-principle of subharmonic functions,

$$\max_{z\in A_{\gamma,\Gamma}}F_n(z)\le\sigma,\quad n\in\Lambda,\quad n\ge n_1,\quad N\in\Lambda, \tag{7}$$

where $A_{\gamma,\Gamma}$ is the "annulus", bounded by Γ and γ.

On the other hand, by (5), there exists, for every compact set $K\subset E_r\setminus E_R$ and n large enough, a point $z_{n,K}\in K$ such that

$$-U^\mu(z_{n,K})-\ln R_{m,\mu}-\sigma\le\frac{1}{n}\ln\left|FQ_n(z_{n,K})-QP_n(z_{n,K})\right|,\quad n\ge n_3(K),\quad n\in\Lambda.$$

Therefore,

$$-\sigma\le F_n(z_{n,K}),\quad n\ge n_2(K,\sigma). \tag{8}$$

Further, by the formula of Hermite-Lagrange, for $z\in\gamma$ we have

$$FQ_n(z)-QP_n(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\omega_{n+m+1}(z)}{\omega_{n+m+1}(t)}\frac{FQ_n(t)-QP_n(t)}{t-z}\mathrm{d}t.$$

Hence, by (5),

$$\begin{aligned}\frac{1}{n}\ln\left|FQ_n(z)-QP_n(z)\right|&\le\max_{t\in\Gamma}U^{\omega_{n+m+1}}(t)-U^{\omega_{n+m+1}}(z)+\frac{1}{n}\ln\left\|FQ_n-QP_n\right\|_\Gamma+\frac{1}{n}\text{const}\\&\le\max_{t\in\Gamma}U^{\omega_{n+m+1}}(t)-U^{\omega_{n+m+1}}(z)-\min_{t\in\Gamma}U^\mu(t)-\ln R_{m,\mu}+\sigma,\\&\qquad n\in\Lambda,\quad n\ge n_3=n_3(\sigma)>n_1,\end{aligned}$$

where $U^{\omega_{n+m+1}}:=U^{\mu_{\omega_{n+m+1}}}$. To simplify the notations, we set $U^{\omega_{n+m+1}}:=U^{\omega_n}$ (the correctness will be not lost, since $m\in\mathbb{N}$ is fixed). Involving into consideration the functions F_n (see (6)), we get for $z\in\gamma$

$$\begin{aligned}F_n(z)\le\max_{t\in\Gamma}\left(U^{\omega_n}(t)-U^\mu(t)\right)+\max_{t\in\Gamma}U^\mu(t)+\left(U^\mu(z)-U^{\omega_n}(z)\right)\\-\min U^\mu(t)+\sigma,\quad n\in\Lambda,\quad n\ge n_2\ge n_1\end{aligned}.$$

By Helly's selection theorem [1], there exists a subsequence of Λ which we denote again by Λ such that $\mu_{\omega_{n+m+1}}:=\mu_{\omega_n}\to\omega$, $n\in\Lambda$. Passing to the limit, we obtain

$$\limsup_{\Lambda}\left|F_n(z)\right|\le\max_{t\in\Gamma}\left(U^\omega(t)-U^\mu(t)\right)+\left(U^\mu(z)-U^\omega(z)\right),\quad z\in\gamma. \tag{9}$$

Consider the function ϕ, harmonic in $A_{\Gamma,\gamma}$ and

$$\phi:=\begin{cases}0, & \Gamma,\\ \min\left(0,-\min_{t\in\gamma}\left(U^\mu(t)-U^\omega(t)\right)+\left(U^\mu(z)-U^\omega(z)\right)\right), & \gamma.\end{cases}$$

From (7) and (9), we arrive at

$$\limsup F_n(z) \le \phi,$$

for z in $A_{\Gamma,\gamma}$. Being harmonic, ϕ obeys the maximum and the minimum principles in this region. The definition yields

$$\phi(z) \le 0, \quad z \in A_{\Gamma,\gamma},$$

We will show that

$$\phi(z) \equiv 0, \tag{10}$$

Suppose that (10) is not true. Let Υ be a closed curve in the set $E_{R+\eta} - \gamma^o$, where γ^o stands for the interior of γ. Then there exists a number $\theta > 0$ such that $\phi \le -\Theta$ for every $z \in \Upsilon$. This inequality contradicts (8), for σ close enough to the zero and $n \in \Lambda$ sufficiently large.

Hence, $\phi \equiv 0$. Then the definition of ϕ yields

$$U^{\mu}(z) - U^{\omega}(z) \equiv \min_{t\in\gamma}\left(U^{\mu}(t) - U^{\omega}(t)\right), \quad z \in \gamma.$$

The function $U^{\mu}(z) - U^{\omega}(z)$ is harmonic in the unbounded complement G of γ, and by the maximum principle,

$$U^{\mu}(z) - U^{\omega}(z) \equiv \text{Constant}, \quad z \in G,$$

consequently,

$$U^{\mu}(z) - U^{\omega}(z) \equiv \text{Constant}, \quad z \in E^c.$$

On the other hand, $\left(U^{\mu} - U^{\omega}\right)(\infty) = 0$, which yields $U^{\mu} \equiv U^{\omega}$ in E^c. By Carleson's Lemma, $\mu = \omega$. On this, Theorem 1 is proved. Q.E.D.

The proof of Theorem 2 will be preceded by an auxiliary lemma

Lemma 1 [17]: *Given a domain* U, *a regular compact subset* S *and a sequence* $\vartheta := \{n_k\}$ *of positive integers*, $n_k < n_{k+1}$, $k = 1, 2, \cdots$, *such that*

$$\limsup_{n_k\to\infty, n_k\in\Lambda} \frac{n_{k+1}}{n_k} < \infty,$$

Suppose that $\{\phi_{n_k}\}$ is a sequence of rational functions, $\phi_{n_k} \in R_{n_k,n_k}$, $k - 1, 2, \cdots$, $\phi_{n_k} = \phi'_{n_k}/\phi''_{n_k}$ having no more that m poles in U and converging uniformly of ∂S to a function $\phi \not\equiv 0$ such that

$$\limsup_{n_k\to\infty, n_k\in\Lambda} \left\|\phi_{n_k} - \phi\right\|_{\partial S}^{1/n} < 1.$$

Assume, in addition, that on each compact subset of U

$$\lim_{n_k\to\infty, n_k\in\Lambda} \mu_{\phi'_{n_k}}(K) = 0. \tag{11}$$

Then the function ϕ admits a continuation into U as a meromorphic function with no more than m poles.

Proof of Theorem 2: We preserve the notations from the proof of Theorem 1.

The proof of Theorem 2 follows from Lemma 1 and Theorem 1. Indeed, under the conditions of the theorem the sequence $\{\pi_n\}_{n\in\Lambda}$ converges maximally to f with respect to the measure μ and the domain $D_{m,\mu}$. Hence, inside $D_{m,\mu}$ (on compact subsets) condition (11) if fulfilled. From the proof of Theorem 1, we see that there is a regular compact subset S of $D_{m,\mu}$ such that $\limsup_{n\in\Delta}\left\|f - \pi_n\right\|_S^{1/n} < 1$.

Suppose now that the statement of Theorem 2 is not true. Then there is, for every $z \in \partial D_{m,\mu}$ a disk $V_z(r_z) := V_z$, $r_z > 0$ with $\lim_{n_k} \mu_{P_n}(V_z) = 0$. We select a finite covering of disks V_{z_j} such that $W := \bigcup V_{z_j} \supset \partial D_{m,\mu}$. Condition (11) holds inside W. Applying Lemma 1 with respect to the sequence π_n and to the domain $D_{m,\mu} \bigcup W$, we conclude that $f \in \mathcal{M}_m\left(\overline{D_{m,\mu}}\right)$. This contradicts the definition of $D_{m,\mu}$.

On this, the proof of Theorem 2 is completed. Q.E.D.

Using again Lemma 1 and applying Theorem A, we obtain a more general result about the zero distribution of the sequence $\left\{\pi_{n,m}^{\beta,f}\right\}$.

Theorem 3: *Let E be a regular compactum in $\mathbb{C}$ with a connected complement, let $\mu \in \mathcal{B}(E)$ and $\beta \in E$ be a triangular point set. Let the polynomials ω_n, $n = 1, 2, \cdots$, be defined as above. Suppose that $\mu_{\omega_n} \to \mu$ as $n \to \infty$ and $f \in \mathcal{A}(E)$. Let $m \in \mathbb{N}$ be fixed, and suppose that $R_{m,\mu} < \infty$. Then there is at least one point $z_0 \in \partial D_{m,\mu}$ such that $\limsup_{n\to\infty} \mu_{\pi_{n,m}^{\beta,f}}\left(\bar{V}_{z_0}(r)\right) > 0$ for every positive r.*

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