

Variational Homotopy Perturbation Method for Solving Benjamin-Bona-Mahony Equation

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Abstract

In this article, the application of variational homotopy perturbation method is applied to solve Benjamin-Bona-Mahony equation. Then, we obtain the numerical solution of BBM equation using the initial condition. Comparison with Adomian's decomposition method, homotopy perturbation method, and with the exact solution shows that VHPM is more effective and accurate than ADM and HPM, and is reliable and manageable for this type of equation.

Keywords

Variational Homotopy Perturbation Method, Benjamin-Bona-Mahony Equation

1. Introduction

Most scientific problems arise in real-world physical problems such as plasma physics, fluid mechanic, solid state physics and in many branches of chemistry [1]. The Benjamin-Bona-Mahony (BBM) equation is inherently of nonlinearity. We know that except a limited number of these problems, most of them do not have analytical solution. Therefore, these nonlinear equations should be solved using numerical methods: variational iteration method (VIM) [2] and homotopy-perturbation method (HPM) [3]. These methods are the most effective and convenient ones for both weakly and strongly nonlinear equations. In this article, VHPM is used to solve nonlinear Benjamin-Bona-Mahony [4] and [5]. The Benjamin-Bona-Mahony equation (BBM equation) is the partial differential equation

$$u_t - u_{xxt} + u_x + uu_x = 0 \quad (1.1)$$

This equation was introduced in (Benjamin T. B., Bona J. L. & Mahony J. J. 1972) [5] as an improvement of

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the Korteweg-de Vries equation (KdV equation) for modeling long waves of small amplitude in 1 + 1 dimensions.

The main goal of this paper is to find the approximate solution of the BBM by the variational homotopy perturbation method that has already been successfully applied to several nonlinear problems.

2. Mathematical Model

The generalized BBM (Benjamin-Bona-Mahony) equation has a higher order nonlinearity of the form

$$u_t - u_{xxt} + u_x + au^n u_x = 0, \quad n \geq 1, \tag{2.1}$$

where a is constant.

The case $n = 1$ corresponds to the BBM equation

$$u_t - u_{xxt} + u_x + auu_x = 0, \tag{2.2}$$

which was first proposed in 1972 by Benjamin *et al.* [5]. This equation is an alternative to the Korteweg-de Vries (KdV) equation, and describes the unidirectional propagation of small-amplitude long waves on the surface of water in channel. The BBM equation is not only convenient for shallow water waves but also for hydro magnetic waves, acoustic waves, and therefore it has more advantages compared with the KdV equation. When $n = 2$, Equation (2.1) is called the modified BBM equation:

$$u_t - u_{xxt} + u_x + au^2 u_x = 0 \tag{2.3}$$

3. Materials and Methods

3.1. Variational Iteration Method

To clarify the basic ideas of VIM, we consider the following differential equation

$$Lu + Nu = g(x, t). \tag{3.1}$$

where L is a linear operator is defined by $L = \frac{\partial^m}{\partial t^m}$, $m \in \mathbb{N}$, N is a nonlinear operator and $g(x, t)$ is a known analytic function. According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(x, \tau) [Lu_n(x, \tau) + Nu_n(x, \tau) - g(x, \tau)] d\tau. \tag{3.2}$$

where λ is a general lagrangian multiplier [6] defined as:

$$\lambda(x, t) = \frac{(-1)^m}{(m-1)!} (x-t)^{m-1}, \quad m \geq 1. \tag{3.3}$$

The subscript n indicates the n th approximation and \tilde{u}_n is considered as a restricted variation [7].

3.2. Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \delta. \tag{3.2.1}$$

With the following boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \tau. \tag{3.2.2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ is a known analytical function and τ is the boundary of the domain δ . The operator A can be decomposed into two operators L and N , where L is a linear, and N a nonlinear operator.

Equation (3.2.1) can be written as follows:

$$L(u) + N(u) - f(r) = 0. \tag{3.2.3}$$

By using the homotopy technique, we construct a homotopy:

$$v(r, p) : \delta \times [0, 1] \rightarrow R, \tag{3.2.4}$$

Which are satisfies:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \delta. \tag{3.2.5}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{3.2.6}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation for the solution of Equation (3.2.1), which satisfies the boundary conditions. Obviously, from Equations (3.2.5) and (3.2.6) we have:

$$H(v, 0) = L(v) - L(u_0) = 0, \tag{3.2.7}$$

$$H(v, 1) = A(v) - f(r) = 0. \tag{3.2.8}$$

The changing process of p forms zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called homotopy. According to the (HPM), we can first use the embedding parameter p as a small parameter, And assume that the solution of Equations (3.2.5) and (3.2.6) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2v_2 + \dots \tag{3.2.9}$$

Setting $p = 1$, gives the solution of Equation (3.2.1)

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \tag{3.2.10}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques.

The series (3.2.10) is convergent for most cases. Some criteria are suggested for convergence of the series (3.2.10) [8].

3.3. Variational Homotopy Perturbation Method

To illustrate the concept of the variational homotopy perturbation method, we consider the general differential Equation (3.1). We construct the correction functional (3.2) and apply the homotopy perturbation method (3.2.9) to obtain:

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = u_0(x, t) + p \int_0^t \lambda(x, \tau) \left(\sum_{n=0}^{\infty} p^n L u_n(x, \tau) + N \sum_{n=0}^{\infty} p^n \tilde{u}_n(x, \tau) - g(x, \tau) \right) d\tau. \tag{3.3.1}$$

As we see, the procedure is formulated by the coupling of variational iteration method and homotopy perturbation method. A comparison of like powers of p gives solutions of various orders [9].

4. Numerical Example

Example:

Consider the nonlinear Benjamin-Bona-Mahony equation where $a = 1$ [10]:

$$u_t - u_{xxt} + u_x + uu_x = 0. \tag{4.1}$$

And initial condition

$$u(x, 0) = \operatorname{sech}^2\left(\frac{x}{4}\right). \tag{4.2}$$

And exact solution

$$u(x, t) = \operatorname{sech}^2\left(\frac{x-t}{4}\right). \tag{4.3}$$

The correct functional is given as

$$u_{n+1}(x, t) = u_0(x, t) + \int_0^t \lambda(x, \tau) \left((u_n)_\tau - (\tilde{u}_n)_{xx\tau} + (\tilde{u}_n)_x + \tilde{u}_n (\tilde{u}_n)_x \right) d\tau. \tag{4.4}$$

where \tilde{u}_n is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as Equation (3.3) is $\lambda(x, t) = -1$, which yields the following iteration formula:

$$u_{n+1}(x, t) = u_0(x, t) - \int_0^t \left((u_n)_\tau - (u_n)_{xx\tau} + (u_n)_x + u_n (u_n)_x \right) d\tau. \tag{4.5}$$

Applying the variational homotopy perturbation method, we have:

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = u_0(x, t) - p \int_0^t \left(\left(\sum_{n=0}^{\infty} p^n u_n \right)_\tau - \left(\sum_{n=0}^{\infty} p^n u_n \right)_{xx\tau} + \left(\sum_{n=0}^{\infty} p^n u_n \right)_x + \left(\sum_{n=0}^{\infty} p^n u_n \left(\sum_{n=0}^{\infty} p^n u_n \right)_x \right) \right) d\tau \tag{4.6}$$

Comparing the coefficient of like powers of p , we have

$$p^{(0)} : u_0(x, t) = \operatorname{sech}^2\left(\frac{x}{4}\right). \tag{4.7}$$

$$p^{(1)} : u_1(x, t) = -\int_0^t \left((u_0)_\tau - (u_0)_{xx\tau} + (u_0)_x + u_0 (u_0)_x \right) d\tau. \tag{4.8}$$

$$p^{(2)} : u_2(x, t) = -\int_0^t \left((u_1)_\tau - (u_1)_{xx\tau} + (u_1)_x + u_0 (u_1)_x + u_1 (u_0)_x \right) d\tau. \tag{4.9}$$

$$p^{(3)} : u_3(x, t) = -\int_0^t \left((u_2)_\tau - (u_2)_{xx\tau} + (u_2)_x + u_0 (u_2)_x + u_2 (u_0)_x + u_1 (u_1)_x \right) d\tau. \tag{4.10}$$

⋮

Then

$$u_0(x, t) = \operatorname{sech}^2\left(\frac{x}{4}\right).$$

$$u_1(x, t) = t \left(\frac{1}{2} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) + \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right). \tag{4.11}$$

$$u_2(x, t) = \frac{1}{256} \operatorname{sech}^8\left(\frac{x}{4}\right) \left(-104t^2 + 23t^2 \cosh\left(\frac{x}{2}\right) + 16t^2 \cosh(x) + t^2 \cosh\left(\frac{3x}{2}\right) - 159t \sinh\left(\frac{x}{2}\right) - 24t \sinh(x) - 3t \sinh\left(\frac{3x}{2}\right) \right). \tag{4.12}$$

⋮

The other components of the VHPM can be determined in a similar way. Finally, the approximate solution of Equation (4.1) in a series form is

$$u(x, t) \simeq u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots \tag{4.13}$$

5. Figures and Tables

In this section, the results obtained by VHPM are tabulated in the following tables, followed by their figures (Figures 1-12) and comparisons.

6. Conclusions

In this paper, we have successfully used variational homotopy perturbation method for solving the Benjamin-Bona-Mahony equation, it is apparently seen that VHPM is very powerful and efficient technique in finding

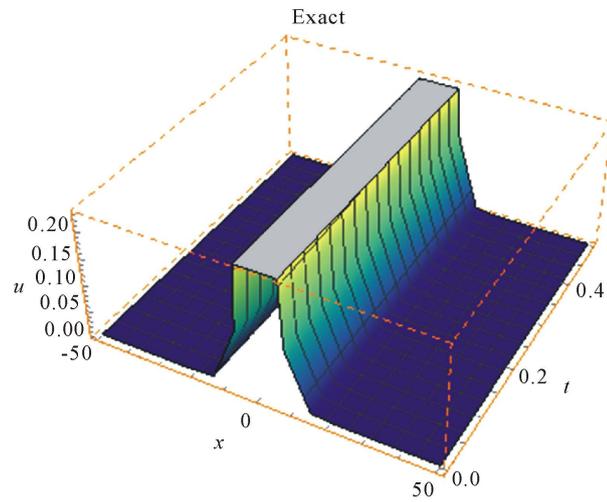


Figure 1. Exact solution at $-50 \leq x \leq 50, 0 \leq t \leq 0.5$.

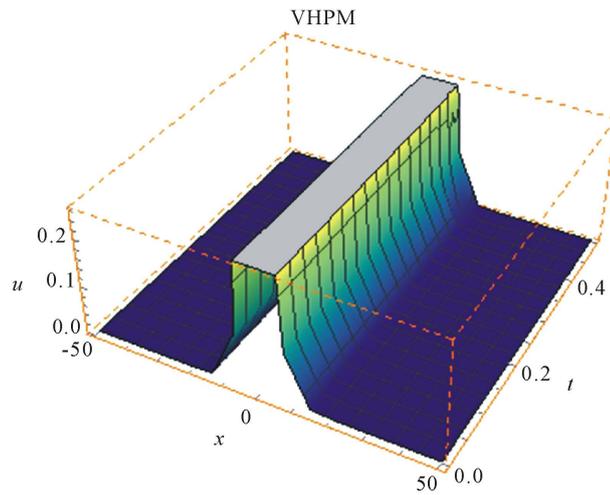


Figure 2. VHPM at $-50 \leq x \leq 50, 0 \leq t \leq 0.5$.

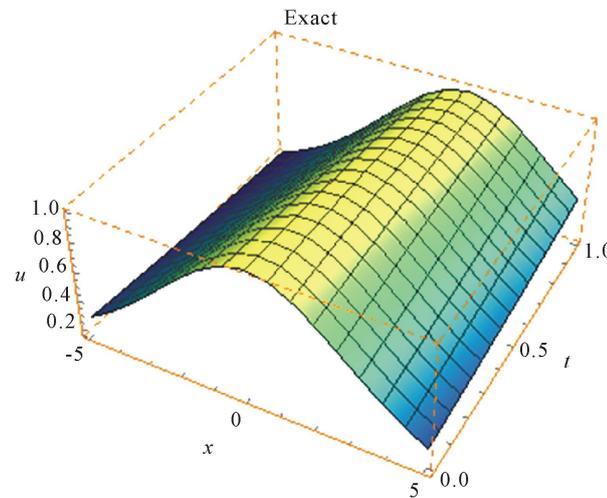


Figure 3. Exact solution at $-5 \leq x \leq 5, 0 \leq t \leq 1$.

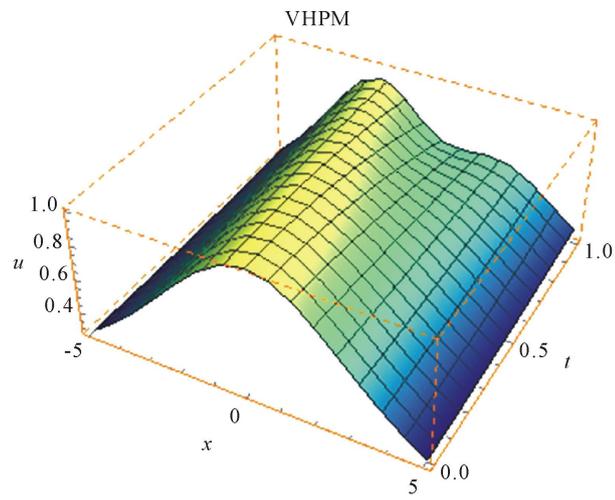


Figure 4. VHPM at $-5 \leq x \leq 5, 0 \leq t \leq 1$.

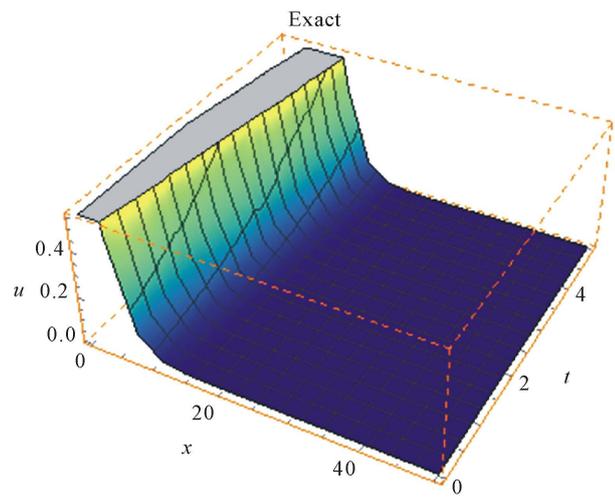


Figure 5. Exact solution at $0 \leq x \leq 50, 0 \leq t \leq 5$.

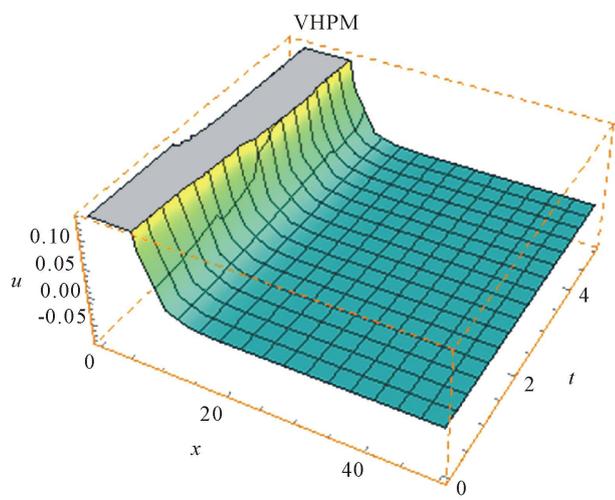


Figure 6. VHPM at $0 \leq x \leq 50, 0 \leq t \leq 5$.

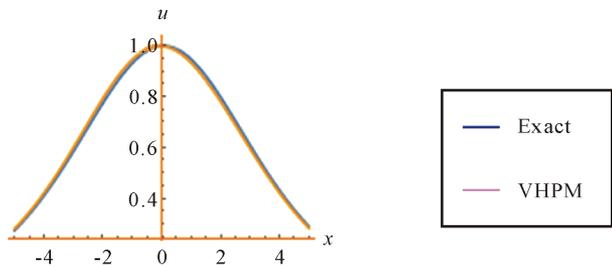


Figure 7. Exact solution and VHPM at $-5 \leq x \leq 5, t = 0.1$.

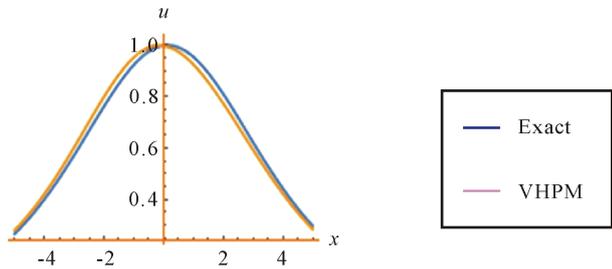


Figure 8. Exact solution and VHPM at $-5 \leq x \leq 5, t = 0.5$.

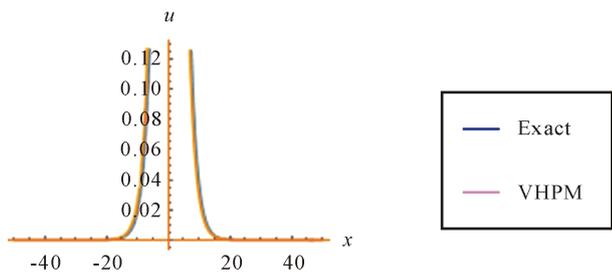


Figure 9. Exact solution and VHPM at $-50 \leq x \leq 50, t = 0.5$.

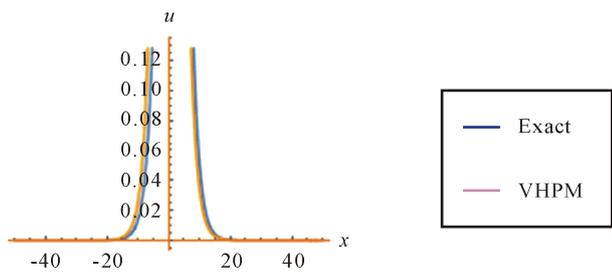


Figure 10. Exact solution and VHPM at $-50 \leq x \leq 50, t = 1$.

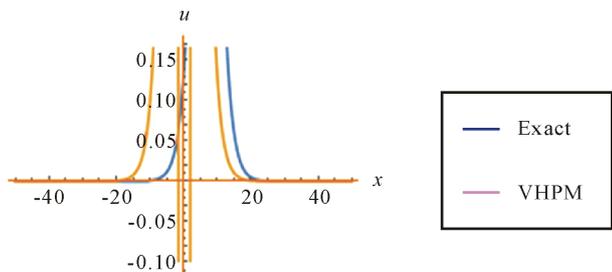


Figure 11. Exact solution and VHPM at $-50 \leq x \leq 50, t = 5$.

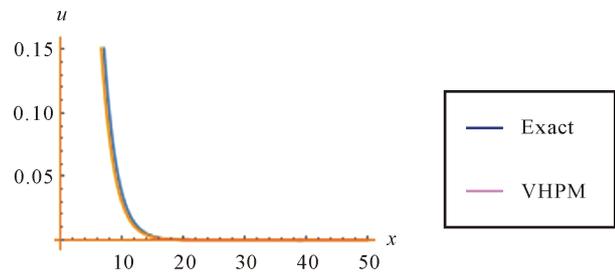


Figure 12. Exact solution and VHPM at $0 \leq x \leq 50, t = 0.5$.

Table 1. Comparing absolute error of ADM, HPM obtained [10] and present method with exact solution.

$x \rightarrow$	0.03		0.04		0.05	
t	ADM & HPM	VHPM	ADM & HPM	VHPM	ADM & HPM	VHPM
0.01	2.26646e-004	1.1543e-004	2.77073e-004	1.4926e-004	3.27453e-004	1.8307e-004
0.02	6.03525e-004	2.5862e-004	7.04304e-004	3.2626e-004	8.04969e-004	3.9387e-004
0.03	1.13061e-003	4.2956e-004	1.28165e-003	5.3101e-004	1.43250e-003	6.3239e-004
0.04	1.80786e-003	6.2827e-004	2.00908e-003	7.6350e-004	2.20999e-003	8.9864e-004
0.05	2.63524e-003	8.5474e-004	2.88653e-003	1.0237e-003	3.13739e-003	1.1926e-003

Table 2. Absolute error of VHPM with exact solution.

	$t = 0.1$	$t = 0.3$	$t = 0.5$
$x = 50$	5.631561237089383e-011	5.825992451205052e-011	6.075975440786746e-011
$x = 30$	1.240433164768256e-006	1.283259519582440e-006	1.338321997161011e-006
$x = 10$	2.696255591595309e-002	2.791563560904538e-002	2.915192190432746e-002
$x = -10$	2.629268246421708e-002	2.590601471553702e-002	2.580255186686319e-002
$x = -30$	1.209842932718460e-006	1.191488823433051e-006	1.185370836912030e-006
$x = -50$	5.492681798439742e-011	5.409354135256130e-011	5.381578247538546e-011

Table 3. Absolute error between VHPM and exact solution.

	$t = 0$	$t = 0.5$	$t = 1$
$x = -5$	5.551115123125783e-017	7.314427490900022e-002	1.560686720003175e-001
$x = 0$	0	3.522864401908321e-002	1.466295596049143e-001
$x = 5$	5.551115123125783e-017	6.534308332294098e-002	1.230071227985541e-001

Table 4. Absolute error between VHPM and exact solution.

	$t = 0$	$t = 2.5$	$t = 5$
$x = 0$	0	1.097018849581008e+000	5.383034896646911e+000
$x = 25$	0	4.770884185166814e-005	3.469615990472011e-004
$x = 50$	0	1.778067925425482e-010	1.293335597251119e-009

analytical solutions for wide classes of nonlinear problems. They also do not require large computer memory. This method is reliable and manageable. The results show that:

- As shown in (Table 1) VHPM is more accurate than ADM and HPM obtained [10].
- As shown in (Table 2) and (Figure 1, Figure 2, Figure 9, Figure 10 and Figure 12), when the time small in a wide space the solution by VHPM approaches to exact solution.
- As shown in (Table 3) and (Figure 3, Figure 4, Figure 7 and Figure 8), when the time small in a small space the error decreases.
- As shown in (Table 4) and (Figure 5, Figure 6 and Figure 11), when time increases in a wide space the error increases.
- In general, whenever a space gets extended the error decreases and closer gets to zero.

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