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Fixed Points of Two-Parameter Family of Function $\lambda\left(\frac{x}{b^x-1}\right)^n$

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Abstract

We establish sufficient conditions of the multiplicity of real fixed points of two-parameter family $\mathcal{T}=\left\{f_\lambda(x;b,n)=\lambda\left(\frac{x}{b^x-1}\right)^n \text{ if } x\neq 0, f_\lambda(0;b,n)=\frac{1}{(\ln b)^n}: n\in\mathbb{N}, \lambda>0, b>0, b\neq 1\right\}$. Moreover, the behaviors of these fixed points are studied.

Keywords

Fixed Points, Attracting Fixed Points, Rationally Indifferent Fixed Points, Repelling Fixed Points

1. Introduction

The introduction of chaos, fractal, and dynamical system could be found in many classical textbooks, such as Scheinerman [1]. A dynamical system has two parts, a state and a function. The second part of a dynamical system is a rule which tell us how the system changes over time. According to the time, we have the discrete and continuous system. The discrete dynamical system, in which we are interested, always does not have an analytical solution. Therefore, the behaviors of fixed points are very important. They play a vital role in the chaos, bifurcation, Julia sets problem in the dynamical system (see [2] [3]). Those problems have been studied for last thirty years. Using the dynamics of functions near the real fixed points, the dynamics of functions in complex plane were induced by the following researchers: The dynamics of families of entire functions $\lambda\frac{\sinh(z)}{z}$,

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$\lambda\frac{e^z-1}{z},\lambda\frac{\sinh^2(z)}{z^4},\lambda>0$ were studied by Prasad [4], Kapoor and Prasad [2], Sajid and Kapoor [5], respectively. The dynamics of λe^z is found in Devaney [6]. Recently, Sajid [7] [8] gave the results about the fixed points of one parameter family of function $\frac{x}{b^x-1}$. His work is motivated by the relationship of the function $\frac{zb^z}{b^z-1}\left(=\frac{-z}{b^{-z}-1}\right)$ with the well-known generating functions on base b by choosing $\mu=1$ and $t=1$ in the generalized Bernoulli generating function $\left(\frac{z}{e^z-1}\right)^{\mu}e^{tz}=\sum_{m=0}^{\infty}B_m^{\mu}(t)\frac{z^m}{m!}$. The proofs in [7] and [8] are too complicated. In this paper, we not only give a simple proof of the work of Sajid [7], but also generalize his work.

2. Main Results

We will determine the fixed points of $f_\lambda(x;b,n)=\lambda f(x;b,n)$, and $f(x;b,n)=[g(x;b)]^n$, where

$$g(x;b)=\begin{cases}\frac{x}{b^x-1}, & \text{if } x\neq 0\\ \frac{1}{\ln b}, & \text{if } x=0\end{cases},n\in\mathbb{N},\lambda>0,b>0,b\neq 1 \tag{2-1}$$

i.e., we will solve the equation $f_\lambda(x;b,n)=x$. Moreover, we also discuss the multiplicity and the behavior of the fixed points for two parameters b and n. For simplicity of notation, we denote $f_\lambda(x;b,n)$, $f(x;b,n)$, and $g(x;b)$ by $f_\lambda(x)$, $f(x)$, and $g(x)$, respectively.

For $n=1$, the following results in Theorem 1 were in Sajid [7], but we have a simpler proof.

Theorem 1. *Let* $g_\lambda(x)=\lambda g(x)$, *where* $g(x)$ *is given in* (2-1). *Then*

(1) *The function* $g_\lambda(x)$ *has a unique fixed point* x_λ *for any* $\lambda>0$.

(2) *The unique fixed point* x_λ *is negative if* $0<b<1$, *and positive if* $b>1$. *Moreover, we have* x_λ *is decreasing if* $0<b<1$, *increasing if* $b>1$ *as* λ *is increasing and*

$$\lim_{\lambda\to 0^+}x_\lambda=0,\ \lim_{\lambda\to\infty}x_\lambda=\begin{cases}\infty, & \text{if } b>1\\ -\infty, & \text{if } 0<b<1\end{cases}. \tag{2-2}$$

(3) *There exists* $\lambda^*>0$ *such that the fixed point* x_λ *of the function* $g_\lambda(x)$ *is* (i) *attracting, i.e.,* $|g'_\lambda(x_\lambda)|<1$, *for* $0<\lambda<\lambda^*$, (ii) *rationally indifferent, i.e.,* $|g'_\lambda(x_\lambda)|=1$, *at* $\lambda=\lambda^*$, *and* (iii) *repelling, i.e.,* $|g'_\lambda(x_\lambda)|>1$, *for* $\lambda>\lambda^*$.

Proof. Suppose that $\lambda>0,b>0$, *and* $b\neq 1$. Statements (1) and (2) can be proved directly as follows. The expression $g_\lambda(x)=x$ if and only if $\lambda\frac{x}{b^x-1}=x,x\neq 0$, because $g_\lambda(0)=\frac{\lambda}{\ln b}\neq 0$. Let x_λ be the fixed point of the function $g_\lambda(x)$ *for* $\lambda>0$. Then the fixed point

$$x_\lambda=\frac{\ln(\lambda+1)}{\ln b} \tag{2-3}$$

is unique. Moreover, (2-3) easily implies statement (2).

Next, we proved statement (3). It is easy that

$$g'(x)=\begin{cases}\frac{b^x-1-xb^x\ln b}{(b^x-1)^2}, & \text{if } x\neq 0\\ \frac{-1}{2}, & \text{if } x=0\end{cases}, \tag{2-4}$$

and the function g' is contionus on $\mathbb{R}$. Therefore, substituting (2-3) into (2-4), we have

$$g'_\lambda(x_\lambda) = \lambda g'(x_\lambda) = \frac{\lambda - (1+\lambda)\ln(1+\lambda)}{\lambda}. \tag{2-5}$$

Hence,

$$\lim_{\lambda\to 0^+} g'(x_\lambda) = 0, \tag{2-6}$$

$$\lim_{\lambda\to\infty} g'(x_\lambda) = -\infty, \tag{2-7}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} g'(x_\lambda) = \frac{\ln(1+\lambda)-\lambda}{\lambda^2} < 0 \quad \text{for } \lambda > 0. \tag{2-8}$$

Therefore, statement (3) are true by (2-6), (2-7), and (2-8).

The results about $f_\lambda(x;b,n) = x$ depend on the parameters n and b. If n is odd, then the behavior of the fixed points is similar to the case $n=1$. If n is even, the behavior of the fixed points depends on the parameter b. We have the simple facts about the functions f and g. It is easy that

$$g(x)\begin{cases} >0, & \text{if } b>1 \\ <0, & \text{if } 0<b<1 \end{cases}, \text{ and } g'(x)<0 \text{ if } b>0, b\neq 1, \text{for all } x\in\mathbb{R} \tag{2-9}$$

Hence, if the integer n is even, then $f(x) > 0, \forall x \in \mathbb{R}$. If the integer n is odd, then

$$f(x)\begin{cases} >0, & \text{if } b>1, \\ <0, & \text{if } 0<b<1, \end{cases} \quad \text{for all } x\in\mathbb{R}. \tag{2-10}$$

Suppose that the fixed point of $f_\lambda(x)$ exists. If n is odd and $0<b<1$, then the fixed point is negative. Otherwise, it is positive. Moreover,

$$f'(x) = \begin{cases} n\left(\dfrac{x}{b^x-1}\right)^{n-1} \dfrac{b^x - 1 - xb^x \ln b}{\left(b^x-1\right)^2}, & \text{if } x\neq 0, \\ \dfrac{-n}{2(\ln b)^{n-1}}, & \text{if } x = 0, \end{cases} \tag{2-11}$$

and $f'(x)$ is continuous in $\mathbb{R}$. Hence,

$$f'(0)\begin{cases} >0, & \text{if } 0<b<1 \text{ and } n \text{ is even,} \\ <0, & \text{otherwise} \end{cases} \tag{2-12}$$

Lemma 2. *Let* $f(x) = [g(x)]^n$*, where* $g(x) = \dfrac{x}{b^x-1}$*, and* $n\in\mathbb{N}, n>1$*. Then* (1) $f(x)$ *is concave downward in* $\mathbb{R}$ *if* n *is odd and* $0<b<1$*.* (2) $f(x)$ *is concave upward in* $\mathbb{R}$ *otherwise.*

Proof. Suppose that $f(x) = [g(x)]^n$, and $g(x) = \dfrac{x}{b^x-1}, x>0$.

$$f'(x) = n[g(x)]^{n-1} g'(x)$$

$$\begin{aligned} f''(x) &= n(n-1)[g(x)]^{n-2}[g'(x)]^2 + n[g(x)]^{n-1} g''(x) \\ &= n[g(x)]^{n-2}\left\{(n-1)[g'(x)]^2 + g(x)g''(x)\right\} \end{aligned}$$

Therefore, (2-9) implies that $f''(x)$ and $(n-1)[g'(x)]^2 + g(x)g''(x)$ have the opposite sign if and only if

n is odd and $0<b<1$. Moreover, $\frac{\mathrm{d}}{\mathrm{d}x}\left[g(x)g'(x)\right]=\left[g'(x)\right]^2+g(x)g''(x)>0$ $\Rightarrow(n-1)\left[g'(x)\right]^2+g(x)g''(x)>0$ since $n>1$. Thus, it suffices to prove that $\frac{\mathrm{d}}{\mathrm{d}x}\left[g(x)g'(x)\right]>0$ in $(0,\infty)$, where $g(x)=\frac{x}{b^x-1}$.

(2-4) implies that $g(x)g'(x)=\frac{x\left(b^x-1-xb^x\ln b\right)}{\left(b^x-1\right)^3}$. We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(b^x-1-xb^x\ln b\right)=-xb^x\ln^2 b,$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[g(x)g'(x)\right]&=\frac{\left(b^x-1\right)^3\left(b^x-1-xb^x\ln b-x^2b^x\ln^2 b\right)-x\left(b^x-1-xb^x\ln b\right)3\left(b^x-1\right)^2b^x\ln b}{\left(b^x-1\right)^6}\\&=\frac{h(x)}{\left(b^x-1\right)^4},\end{aligned}$$

where

$$\begin{aligned}h(x)&=\left(b^x-1\right)\left[b^x\left(1-x\ln b-x^2\ln^2 b\right)-1\right]-x\left[b^x\left(1-x\ln b\right)-1\right]b^x3\ln b\\&=b^{2x}\left(1-4x\ln b+2x^2\ln^2 b\right)+b^x\left(-2+4x\ln b+x^2\ln^2 b\right)+1.\end{aligned}\tag{2-13}$$

Let $y=b^x>0$. Equation (2-13) is transformed to

$$h\left[x(y)\right]=y^2\left[1-4\ln y+2(\ln y)^2\right]+y\left[-2+4\ln y+(\ln y)^2\right]+1,\ y>0\tag{2-14}$$

Moreover, let $y=\mathrm{e}^t,t\in\mathbb{R}$. (2-14) is transformed to

$$H(t)=h\left\{x\left[y(t)\right]\right\}=\mathrm{e}^{2t}\left(1-4t+2t^2\right)+\mathrm{e}^t\left(-2+4t+t^2\right)+1,\ t\in\mathbb{R}\tag{2-15}$$

In fact, the graph of $H(t)$ has 2 critical points, including $t=0$, and $H(0)=0$ is the global minimum.

$$H(t)=\mathrm{e}^{2t}\left(1-4t+2t^2\right)+\mathrm{e}^t\left(-2+4t+t^2\right)+1,\ H(0)=0.$$

$$\begin{aligned}H'(t)&=\mathrm{e}^{2t}\left(2-8t+4t^2-4+4t\right)+\mathrm{e}^t\left(-2+4t+t^2+4+2t\right)\\&=\mathrm{e}^{2t}\left(4t^2-4t-2\right)+\mathrm{e}^t\left(t^2+6t+2\right)\\&=\mathrm{e}^t\left[\mathrm{e}^t\left(4t^2-4t-2\right)+\left(t^2+6t+2\right)\right]\end{aligned}$$

By the algorithm of bisection, $H'(t)=0\Rightarrow t=0$, or $t\approx-5.5373$, and $H(0)=0$, $H(-5.5373)\approx1.027$. It is easy to check that $\lim_{t\to\infty}H(t)=\infty$, and $\lim_{t\to-\infty}H(t)=1$. Therefore, $H(0)=0$ is the minimum of H on $\mathbb{R}$. This completes the proof of Lemma 2.

To study the behavior of the fixed points in Theorem 5, we need Lemma 3 and Lemma 4 as follows.

Lemma 3. *Suppose that*

$$h(x)=\begin{cases}1-\frac{x\cdot\ln(x)}{x-1}, & \text{if } x>0, x\neq1\\0, & \text{if } x=1\end{cases}.\tag{2-16}$$

Then (1) $\lim_{x\to 0^+} h(x)=1$, $\lim_{x\to 1} h(x)=0$, *and* $\lim_{x\to\infty} h(x)=-\infty$. (2) *The function h is decreasing, and concave upward in* $(0,\infty)$.

Proof. The statement (1) is easy. (2-16) implies $h'(x)=\dfrac{\ln x+1-x}{(x-1)^2}<0, x>0.$ Specially, we have $h'(1)=\dfrac{-1}{2}$. Therefore, the function *h* is decreasing in $(0,\infty)$. Moreover,

$$h''(x)=\frac{x-\frac{1}{x}-2\ln x}{(x-1)^3},\quad x>0.$$

Let $k(x)=x-\frac{1}{x}-2\ln x, x>0.$ Then $k(1)=0, k'(x)=1+\frac{1}{x^2}-\frac{2}{x}=\frac{x^2-2x+1}{x^2}>0.$ We have $k(x)<0$ for $0<x<1$, and $k(x)>0$ for $x>1$. Therefore, $h''(x)>0$, for $x>0$, and the function h is concave upward in $(0,\infty)$. The proof of Lemma 3 is completed.

Lemma 4. *Suppose that*

$$h\in\mathcal{C}^2[0,\infty),\ h''(x)>0 \text{ for } x\in[0,\infty), \tag{2-17}$$

$h(0)>0$ and $\exists\tilde{x}\in(0,\infty)$, *such that*

$$h'(\tilde{x})-h(\tilde{x})>0, \tag{2-18}$$

and

$$\lim_{x\to\infty}\frac{h(x)}{x}=\infty. \tag{2-19}$$

Then there is a unique $x^*\in(0,\infty)$, *such that*

$$h'(x^*)=\frac{h(x^*)}{x^*}. \tag{2-20}$$

Moreover, if $m=h'(x^*)$, *then* $y=mx$ *intersects* $y=h(x)$ *at exactly one point. If* $m<h'(x^*)$, *then* $y=mx$ *does not intersect* $y=h(x)$. *If* $m>h'(x^*)$, *then* $y=mx$ *intersects* $y=h(x)$ *at exactly two points.*

Proof. Let $H(x)=xh'(x)-h(x), x\in[0,\infty).$ Then $H(0)=-h(0)<0,$ (2-18) implies $H(\tilde{x})>0.$ The Intermediate Value Theorem of the continuous function implies $\exists x^*\in(0,\tilde{x}),$ such that $H(x^*)=0,$ *i.e.*, Equation (2-20) holds.

Suppose to the contrary that x^* is not unique.

There exist $x_1^*<x_2^*,$ such that

$$h'(x_i^*)=\frac{h(x_i^*)}{x_i^*},\quad i=1,2. \tag{2-21}$$

Then $h'(x_1^*)<h'(x_2^*)$ by (2-17).

For $x\in[x_1^*,x_2^*],$ the Mean Value Theorem and (2-17) imply $\exists\tilde{x}\in(x_1^*,x_2^*)$ such that $h(x_1^*)-h(x_2^*)=h'(\tilde{x})(x_1^*-x_2^*)>h'(x_2^*)(x_1^*-x_2^*).$ Therefore, the Equation (2-21) implies that $h'(x_2^*)(x_1^*-x_2^*)<x_1^*h'(x_1^*)-x_2^*h'(x_2^*)<x_1^*h'(x_2^*)-x_2^*h'(x_2^*)=h'(x_2^*)(x_1^*-x_2^*)$. It is impossible.

Suppose that $m=h'(x^*),$ and $y=mx$ intersects $y=h(x)$ at x_0. Then we have $h'(x^*)=m=\dfrac{h(x_0)}{x_0}.$

(2-20) and the uniqueness of x^* imply $x^* = x_0$.

Suppose to the contrary that $m < h'(x^*)$, but $y = mx$ intersects $y = h(x)$ at the smallest x_1 on $[0,\infty)$. Let $H_1(x) = h(x) - mx$. Then $H_1(0) > 0$, $H_1(x_1) = 0$, $H_1(x^*) > 0$, and $H_1^*(x^*) > 0$. This implies $x^* > x_1$ and there exists the minimum of H_1 on $[0, x^*]$.

Let the minimum occurs at $x = x_1^*$. Then $H_1'(x_1^*) = 0$ and $h'(x_1^*) = \frac{h(x_1^*)}{x_1^*}$. This contradicts to the uniqueness of x^* in (2-20).

Finally, suppose that $m > h'(x^*)$, we prove that $y = mx$ intersects $y = h(x)$ at exactly two points. Let $H_1(x) = h(x) - mx, x \in [0, x^*]$. Then $H_1(0) > 0$ and $H_1(x^*) = h(x^*) - mx^* = x^* h'(x^*) - mx^* < 0$. Intermediate Value Theorem and (2-19) imply that there exists $\hat{x}_0 \in [0, x^*], \tilde{x}_0 \in [x^*, \infty)$ at which $y = mx$ intersects $y = h(x)$.

Suppose to the contrary that $y = mx$ does not intersect $y = h(x)$ at exactly two points. By the previous proofs, the line $y = mx$ will intersect $y = h(x)$ at more than two points. Let $y = mx$ intersect $y = h(x)$ at the three consecutive points $x_1 < x_2 < x_3$, then $x_i \neq x^*$ for $i = 1,2,3$. Without loss of generality, we may suppose that $h'(x_1) < m$. Then $h'(x_2) > m$ and $h'(x_3) < m$. It is impossible that $m > h'(x_3) > h'(x_2) > m$ by (2-17). The proof of Lemma 4 is completed.

Theorem 5. *Let* n *be even and* $0 < b < 1$. *Then*

(1) *There exists a unique* $\lambda^* > 0$, *such that* $f_{\lambda^*}(x)$ *has a unique fixed point at* $x^*_{\lambda^*,n}$, *and* $f_\lambda(x)$ *has two fixed points, say* $x^*_{l,\lambda,n}$, $x^*_{u,\lambda,n}$, *and* $x^*_{l,\lambda,n} < x^*_{\lambda^*,n} < x^*_{u,\lambda,n}$, *for* $0 < \lambda < \lambda^*$, *and no fixed points for* $\lambda > \lambda^*$.

(2) *Let n be fixed. If* $0 < \lambda < \lambda^*$, *then* $x^*_{l,\lambda,n}$ *is decreasing, and* $x^*_{u,\lambda,n}$ *is increasing as* λ *decreases.*

(3) *Let b be fixed. Then* $x^*_{\lambda^*,n}$ *is decreasing to* 0 *as n increases.*

(4) *Suppose* $x^*_{l,\lambda,n} < x^*_{u,\lambda,n}$ *exist for fixed* λ. *If* $\frac{1}{e} < b < 1$, *then* $x^*_{l,\lambda,n}$ *is increasing, and* $x^*_{u,\lambda,n}$ *is decreasing as n increases. If* $0 < b < \frac{1}{e}$, *then there exists a unique* x_0, *such* $g(x_0) = -1$, and $f(x_0) = 1$ *for any even n. Moreover,* $x^*_{l,\lambda,n}$ *is decreasing if it is less than* x_0, *and* $x^*_{u,\lambda,n}$ *is increasing if it is less than* x_0, $x^*_{l,\lambda,n}$ *is increasing if it is greater than* x_0, *and* $x^*_{u,\lambda,n}$ *is decreasing if it is greater than* x_0, *as n increases.*

(5) *The fixed points* $x^*_{l,\lambda,n}, x^*_{\lambda^*,n}, x^*_{u,\lambda,n}$ *are attracting, rationally indifferent, and repelling, respectively.*

Proof. Let n be even and $0 < b < 1$.

(1) We want to solve the equation $f_\lambda(x) = x$, where $f_\lambda(x) = \lambda f(x)$, and $f(x) = [g(x)]^n$, and $g(x)$ is given as in (2-1). Note that the equation $f_\lambda(x) = x$ is equivalent to $f(x) = \frac{x}{\lambda}$.

Because of $f(x) > 0$ and $\lambda > 0$, we only consider $x > 0$.

By $f(0) > 0$, $f \in \mathcal{C}^2[0,\infty)$, $f''(x) > 0$ for $x \in [0,\infty)$, Lemma 4 implies that $\exists \lambda^* > 0, \ni$ the number of intersections of $y = f(x)$ and $y = \frac{x}{\lambda}$ are exactly two for $0 < \lambda < \lambda^*$, unique at $\lambda = \lambda^*$, and none for $\lambda > \lambda^*$. i.e., $f(x) - \frac{x}{\lambda} = 0$ has two roots say $x^*_{l,\lambda,n}$, $x^*_{u,\lambda,n}$, and $x^*_{l,\lambda,n} < x^*_{u,\lambda,n}$, for $0 < \lambda < \lambda^*$, one root, say $x^*_{\lambda^*,n}$ at $\lambda = \lambda^*$, and no root for $\lambda > \lambda^*$. Therefore,

$$f'\left(x^*_{\lambda^*,n}\right)=\frac{f\left(x^*_{\lambda^*,n}\right)}{x^*_{\lambda^*,n}}, \text{ and } f'\left(x^*_{\lambda^*,n}\right)=\frac{1}{\lambda^*}, \textit{ i.e.}, \lambda^* f'\left(x^*_{\lambda^*,n}\right)=1. \tag{2-22}$$

(2) The statement (2) is easy by Lemma 2 and Part (1).

(3) In fact, $x^*_{\lambda^*,n}$ can be determined by solving the equation $f'(x)=\frac{f(x)}{x}$. (2-4) implies to solve

$$g(x)=nxg'(x) \tag{2-23}$$

(2-1) and (2-4) imply to solve

$$b^x-1=n\left(b^x-1-xb^x\ln b\right). \tag{2-24}$$

Let $u=b^x$. (2-24) is equivalent to

$$\frac{1}{n}=1-\frac{u\ln u}{u-1},\ u\in(0,1). \tag{2-25}$$

Lemma 3 and (2-25) imply that $x^*_{\lambda^*,n}$ is decreasing to 0 as n is increasing for any $b\in(0,1)$.

(4) Suppose that $\frac{1}{e}<b<1$. We have $|g(0)|\ge 1$. Then $f(x)$ is increasing as even number increases by (2-9). Suppose that $0<b<\frac{1}{e}$. Then $|g(0)|<1$. Let x_0 be the solution of $f(x)=\left(\frac{x}{b^x-1}\right)^n=1$. Therefore, $\frac{x_0}{b^{x_0}-1}=-1$, *i.e.*, $b^{x_0}=1-x_0$. Since $f(x)$ is increasing, and concave upward, Statement (4) holds.

(5) Let x_λ be the fixed point of $f_\lambda(x)$. Then

$$f'_\lambda(x_\lambda)=\lambda f'(x_\lambda)=\lambda n\left[g(x_\lambda)\right]^{n-1}g'(x_\lambda)=\lambda n\left[g(x_\lambda)\right]^n\frac{g'(x_\lambda)}{g(x_\lambda)}=nf_\lambda(x_\lambda)\frac{g'(x_\lambda)}{g(x_\lambda)}$$

$$=nx_\lambda\frac{g'(x_\lambda)}{g(x_\lambda)}=nx_\lambda\frac{\dfrac{b^{x_\lambda}-1-x_\lambda b^{x_\lambda}\ln b}{\left[b^{x_\lambda}-1\right]^2}}{\dfrac{x_\lambda}{b^{x_\lambda}-1}}=n\cdot\frac{b^{x_\lambda}-1-x_\lambda b^{x_\lambda}\ln b}{b^{x_\lambda}-1}.$$

Let $y_\lambda=b^{x_\lambda}, x_\lambda>0$. Then

$$f'_\lambda(x_\lambda)=n\cdot\left(1-\frac{y_\lambda\cdot\ln(y_\lambda)}{y_\lambda-1}\right), \text{ where } 0<y_\lambda<1.$$

$$x_\lambda\to 0 \text{ implies } y_\lambda\to 1, \text{and}$$

$$x_\lambda\to\infty \text{ implies } y_\lambda\to 0$$

Lemma 3, (2-21) and $b^{x^*_{l,\lambda,n}}>b^{x^*_{\lambda^*,n}}>b^{x^*_{u,\lambda,n}}$ imply $0<f'_\lambda\left(x^*_{l,\lambda,n}\right)<1,\ f'_{\lambda^*}\left(x^*_{\lambda^*,n}\right)=1,\ f'_\lambda\left(x^*_{u,\lambda,n}\right)>1,$ *i.e.*, the statement (5) holds.

Theorem 6. *Let* n *be even,* $b>1$. *Then*

(1) *The function* $f_\lambda(x)$ *has a unique fixed point* $x^*_\lambda=x^*_\lambda(b,n)$ *which is increasing as* λ *increases.*

$$\lim_{\lambda\to 0^+}x_\lambda=0,\ \lim_{\lambda\to\infty}x_\lambda=\infty. \tag{2-27}$$

(2) *Let* λ and b *be fixed. If* $b\ge e$, *then* x^*_λ *is increasing as n increases. If* $1<b<e$, *then there exists*

a unique number x_0, *such* $f(x_0)=1$ *for any even n, and* x_λ^* *is increasing if* $x_\lambda^* < x_0$, *and decreasing if* $x_\lambda^* \ge x_0$ *as n increases.*

(3) *There exists* $\lambda^* > 0$ *such that the fixed point* x_λ *of the function* $f_\lambda(x)$ *is* (i) *attracting, i.e.,* $\left|f_\lambda'(x_\lambda^*)\right| < 1,$ *for* $0<\lambda<\lambda^*$, (ii) *rationally indifferent, i.e.,* $\left|f_\lambda'(x_\lambda^*)\right| = 1,$ *at* $\lambda=\lambda^*$, *and* (iii) *repelling, i.e.,* $\left|f_\lambda'(x_\lambda^*)\right| > 1,$ *for* $\lambda>\lambda^*$.

Proof. The proof of Theorem 6 is similar to that of Theorem 5. We just mention some key points. The function *f* is positive, decreasing, and concave upward. Let x_λ^* be the fixed point of $f_\lambda(x)$. Then

$$f_\lambda'(x_\lambda^*) = n\cdot\frac{b^{x_\lambda^*}-1-x_\lambda^* b^{x_\lambda^*}\ln b}{b^{x_\lambda^*}-1} \tag{2-26}$$

Let $y_\lambda = b^{x_\lambda^*}$. Then $y_\lambda \in (1,\infty)$ for $b>1,$ and $f_\lambda'(x_\lambda^*) = n\cdot\left(1-\frac{y_\lambda\cdot\ln(y_\lambda)}{y_\lambda-1}\right)<0.$

Lemma 3 and (2-27) imply that there exists a unique λ^* such that $f_\lambda'(x_{\lambda^*}^*)=-1$, and $0<f_\lambda'(x_\lambda^*)<-1$ if $0<\lambda<\lambda^*$ and $f_\lambda'(x_\lambda^*)>-1$ if $\lambda>\lambda^*$.

Theorem 7. *Let n be odd. Then*

(1) $f_\lambda(x)$ *has a unique fixed point* $x_\lambda^* = x_\lambda^*(b,n)$ *for any* $n\in\mathbb{N}, \lambda>0, b>0, b\ne 1$. *The fixed point* x_λ^* *is negative if* $0<b<1,$ *and positive if* $b>1$. *Moreover, we have*

$$\lim_{\lambda\to 0^+} = 0,\ \lim_{\lambda\to\infty} x_\lambda^* = \begin{cases}\infty, & \text{if } b>1\\ -\infty, & \text{if } 0<b<1\end{cases}. \tag{2-28}$$

(2) *Let the parameter n be fixed. Then* x_λ^* *is increasing if* $b>1,$ *and decreasing if* $0<b<1$ *as* λ *increases.*

(3) *Let the parameter* λ *be fixed. If* $b>e,$ *then* x_λ^* *is increasing as n increases. If* $1<b<e,$ *then there exists a unique number* x_0 *such* $f(x_0)=1$ *for any odd n, and* x_λ^* *is increasing if* $x_\lambda^*<x_0$, *and decreasing if* $x_\lambda^*\ge x_0$ *as n increases. If* $0<b<\frac{1}{e}$, *then* x_λ^* *is increasing as n increases. If* $\frac{1}{e}<b<1$, *then there exists a unique* x_0 *such that* $f(x_0)=-1$ *for any odd n. Furthermore,* x_λ^* *is decreasing if it is less than* x_0, *and increasing if it is greater than* x_0 *as n increases.*

(4) *There exists* $\lambda^*>0$ *such that the fixed point* x_λ^* *of the function* $f_\lambda(x)$ *is* (i) *attracting, i.e.,* $\left|f_\lambda'(x_\lambda^*)\right|<1,$ *for* $0<\lambda<\lambda^*$, (ii) *rationally indifferent, i.e.,* $\left|f_\lambda'(x_\lambda^*)\right|=1,$ *at* $\lambda=\lambda^*$, *and* (iii) *repelling, i.e.,* $\left|f_\lambda'(x_\lambda^*)\right|>1,$ *for* $\lambda>\lambda^*$.

Proof. The proof of Theorem 7 is similar to that of Theorem 5. We just also mention some key points. The function f is decreasing if $b>0, b\ne 1,$ and f is positive, concave upward if $b>1$, and negative, concave downward if $0<b<1$. Let x_λ^* be the fixed point of $f_\lambda(x)$. Then $f_\lambda'(x_\lambda^*) = n\cdot\frac{b^{x_\lambda^*}-1-x_\lambda^* b^{x_\lambda^*}\ln b}{b^{x_\lambda^*}-1}$ by (2-26).

Let $y_\lambda = b^{x_\lambda^*}$. Then $y_\lambda\in(1,\infty)$ for $b>0, b\ne 1$ by statement (1). Therefore, $f_\lambda'(x_\lambda^*) = n\cdot\left(1-\frac{y_\lambda\cdot\ln(y_\lambda)}{y_\lambda-1}\right)<0,$ for $y_\lambda>1$. Lemma 3 and (2-28) imply that there exists a unique λ^* such that $f_\lambda'(x_{\lambda^*}^*)=-1$, and $0<f_\lambda'(x_\lambda^*)<-1$ if $0<\lambda<\lambda^*$ and $f_\lambda'(x_\lambda^*)>-1$ if $\lambda>\lambda^*$.

3. Discussion

The Sarkovskii's theorem said that let the function $f:\mathbb{R}\to\mathbb{R}$ be continuous and it has points of prime period

3, then the function f has points of period k for all positive integer k. We also know that the dynamical behavior of $f_\lambda(x) = x^2 + \lambda$ is very complicated, see Scheinerman [1]. In our problem,

$$\mathcal{T} = \left\{ f_\lambda(x; b, n) = \lambda \left(\frac{x}{b^x - 1} \right)^n \text{ if } x \neq 0,\ f_\lambda(0; b, n) = \frac{1}{(\ln b)^n} : n \in \mathbb{N}, \lambda > 0, b > 0, b \neq 1 \right\}$$

has points of prime period 3 if $b = 0.5$, $n = 2$, and $\lambda = 0.02$. We anticipate the problem we studied will have complicated dynamical behavior as that of $f_\lambda(x) = x^2 + \lambda$. On the other hand, the bifurcation of $\mathcal{T}$ will be interesting.

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