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Generalized Spectrum of Steklov-Robin Type Problem for Elliptic Systems

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Abstract

We will study the generalized Steklov-Robin eigenproblem (with possibly matrix weights) in which the spectral parameter is both in the system and on the boundary. The weights may be singular on subsets of positive measure. We prove the existence of an increasing unbounded sequence of eigenvalues. The method of proof makes use of variational arguments.

Keywords

Steklov-Robin, Variational Arguments, Matrix Weights

1. Introduction

We study the generalized Steklov-Robin eigenproblem. This spectrum includes the Steklov, Neumann and Robin spectra. We therefore generalize the results in [1]-[4].

Consider the elliptic system

$$\begin{aligned} -\Delta U + A(x)U &= \mu M(x)U \quad \text{in } \Omega, \\ \frac{\partial U}{\partial \nu} + \Sigma(x)U &= \mu P(x)U \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with boundary $\partial\Omega$ of class $C^{0,1}$, $U = [u_1, \cdots, u_k]^{\mathrm{T}} \in H(\Omega) := \left[H^1(\Omega)\right]^k = H^1(\Omega) \times H^1(\Omega) \times \cdots \times H^1(\Omega)$. Throughout this paper all matrices are symmetric. The matrix

$$A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1k}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}(x) & a_{k2}(x) & \cdots & a_{kk}(x) \end{bmatrix},$$

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verifies the following conditions:

($A1$) The functions $a_{ij}:\Omega\to\mathbb{R}$.

($A2$) $A(x)$ is positive semidefinite a.e. on Ω with $a_{ij}\in L^p(\Omega)\ \forall i,j=1,\cdots,k$, for $p>\frac{N}{2}$ when $N\geq 3$, and $p>1$ when $N=2$.

The matrix

$$M(x)=\begin{bmatrix} m_{11}(x) & m_{12}(x) & \cdots & m_{1k}(x)\\ m_{21}(x) & m_{22}(x) & \cdots & m_{2k}(x)\\ \vdots & \vdots & \ddots & \vdots\\ m_{k1}(x) & m_{k2}(x) & \cdots & m_{kk}(x)\end{bmatrix}$$

satisfies the following conditions:

($M1$) $M(x)$ is positive semidefinite a.e. on Ω The functions $m_{ij}:\Omega\to\mathbb{R}$, for $p\geq\frac{N}{2}$ when $N\geq 3$, and $p>1$ when $N=2$.

$\partial/\partial\nu:=\nu\cdot\nabla$ is the outward (unit) normal derivative on $\partial\Omega$. The matrix

$$\Sigma(x)=\begin{bmatrix} \sigma_{11}(x) & \sigma_{12}(x) & \cdots & \sigma_{1k}(x)\\ \sigma_{21}(x) & \sigma_{22}(x) & \cdots & \sigma_{2k}(x)\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{k1}(x) & \sigma_{k2}(x) & \cdots & \sigma_{kk}(x)\end{bmatrix},$$

verifies the following conditions:

($S1$) The functions $\sigma_{ij}:\partial\Omega\to\mathbb{R}$.

($S2$) $\Sigma(x)$ is positive semidefinite a.e. on $\partial\Omega$ with $\sigma_{ij}\in L^q(\partial\Omega)\ \forall i,j=1,\cdots,k$, for $q\geq N-1$ when $N\geq 3$, and $q>1$ when $N=2$,

and the matrix

$$P(x)=\begin{bmatrix} \rho_{11}(x) & \rho_{12}(x) & \cdots & \rho_{1k}(x)\\ \rho_{21}(x) & \rho_{22}(x) & \cdots & \rho_{2k}(x)\\ \vdots & \vdots & \ddots & \vdots\\ \rho_{k1}(x) & \rho_{k2}(x) & \cdots & \rho_{kk}(x)\end{bmatrix}.$$

($P1$) $P(x)$ is positive semidefinite a.e. on $\partial\Omega$ for $q\geq N-1$ when $N\geq 3$, and $q>1$ when $N=2$.

We assume that $A(x),\Sigma(x),M(x),P(x)$ verify the following assumptions:

Assumption 1. $A(x)$ *is positive definite on a set of positive measure of* Ω,
or $\Sigma(x)$ *is positive definite on a set of positive measure of* $\partial\Omega$.
And $M(x)$ *is positive definite on a set of positive measure of* Ω,
or $P(x)$ *is positive definite on a set of positive measure of* $\partial\Omega$.

Remark 2. *Assumption* 1 *is equivalent to*

$$\int_\Omega\langle A(x)U,U\rangle\mathrm{d}x+\int_{\partial\Omega}\langle\Sigma(x)U,U\rangle\mathrm{d}x>0\quad\forall U\neq 0.$$

Remark 3. *Since* $A(x),\Sigma(x),M(x),P(x)$ *satisfy* (A2), (S2), (M1), (P1) *respectively, then we can write them in the following form* (*i.e.*; *eigen-decomposition of a positive semi-definite matrix or diagonalization*)

$$J(x)=Q_J^{\mathrm{T}}(x)D_J(x)Q_J(x).$$

where $Q_J(x)^{\mathrm{T}}Q_J(x)=I$ ($Q_J^{\mathrm{T}}(x)=Q_J^{-1}(x)$ *i.e.*; *are orthogonal matrices*) *are the normalized eigenvectors, I is the identity matrix,* $D_J(x)$ *is diagonal matrix and in the diagonal of* $D_J(x)$ *are the eigenvalues of J* (*i.e.*; $D(x)_J=\mathrm{diag}\left(\lambda_1^J(x),\cdots,\lambda_k^J(x)\right)$) *and* $J=\{A,\Sigma,M,P\}$.

Remark 4. *The weight matrices* $M(x)$ *and* $P(x)$ *may vanish on subsets of positive measure*.

Definition 1. *The generalized Steklov-Robin eigensystem is to find a pair* $(\mu,\varphi)\in\mathbb{R}\times H(\Omega)$ *with* $\varphi\not\equiv 0$

such that

$$\begin{aligned}&\int_\Omega \nabla\varphi\cdot\nabla U\,\mathrm{d}x+\int_\Omega\langle A(x)\varphi,U\rangle\mathrm{d}x+\int_{\partial\Omega}\langle\Sigma(x)\varphi,U\rangle\mathrm{d}x\\&=\mu\left[\int_\Omega\langle M(x)\varphi,U\rangle\mathrm{d}x+\int_{\partial\Omega}\langle P(x)\varphi,U\rangle\mathrm{d}x\right]\quad\forall U\in H(\Omega).\end{aligned}\tag{2}$$

Remark 5. *Let* $U=\varphi$ *in* (2) *if there is such an eigenpair, then* $\mu>0$ *and*

$$\int_\Omega\langle M(x)\varphi,\varphi\rangle\mathrm{d}x+\int_{\partial\Omega}\langle P(x)\varphi,\varphi\rangle\mathrm{d}x>0.$$

Indeed, if $\int_\Omega\langle M(x)\varphi,\varphi\rangle\mathrm{d}x+\int_{\partial\Omega}\langle P(x)\varphi,\varphi\rangle\mathrm{d}x=0$, or $\mu=0$, then

$$\int_\Omega|\nabla\varphi|^2\,\mathrm{d}x+\int_\Omega\langle A(x)\varphi,\varphi\rangle\mathrm{d}x+\int_{\partial\Omega}\langle\Sigma(x)\varphi,\varphi\rangle\mathrm{d}x=0.$$

We have that $\int_\Omega|\nabla\varphi|^2\,\mathrm{d}x=0$ which implies that $\varphi=\text{constant}$, and $\int_\Omega\langle A(x)\varphi,\varphi\rangle\mathrm{d}x=0$ this implies that $\langle A(x)\varphi,\varphi\rangle=0$, a.e. (with $\varphi\neq0$) in Ω. This implies that $A(x)$ is not positive definite on a subset of Ω of positive measure, and $\int_{\partial\Omega}\langle\Sigma(x)\varphi,\varphi\rangle\mathrm{d}x=0$, then $\langle\Sigma(x)\varphi,\varphi\rangle=0$, a.e. with $(\varphi\neq0)$ on ∂Ω. This implies that $\Sigma(x)$ is not positive definite on subset of $\partial\Omega$ of positive measure. So we have that, φ would be a constant vector function; which would contradict the assumptions (Assumption 1) imposed on $A(x)$ and $\Sigma(x)$.

Remark 6. *If* $A(x)\equiv0$ *and* $\Sigma(x)\equiv0$ *then* $\mu=0$ *is an eigenvalue of the system* (1) *with eigenfunction* $\varphi=\text{constant}$ *vector function on* $\bar{\Omega}$.

It is therefore appropriate to consider the closed linear subspace (to be shown below) of $H(\Omega)$ under Assumption 1 defined by

$$\mathbb{H}_{(M,P)}(\Omega):=\left\{U\in H(\Omega):\int_\Omega\langle M(x)U,U\rangle\mathrm{d}x+\int_{\partial\Omega}\langle P(x)U,U\rangle\mathrm{d}x=0\right\}.$$

Now all the eigenfunctions associated with (2) must belong to the (A,Σ)-orthogonal complement $H_{(M,P)}(\Omega):=\left[\mathbb{H}_{(M,P)}(\Omega)\right]^\perp$ of this subspace in $H(\Omega)$. We will show that indeed $\mathbb{H}_{(M,P)}(\Omega)$ is subspace of $H(\Omega)$. Let $U,V\in\mathbb{H}_{(M,P)}(\Omega)$ and $\alpha\in\mathbb{R}$ we wish to show that $\alpha U\in\mathbb{H}_{(M,P)}(\Omega)$ and $U+V\in\mathbb{H}_{(M,P)}(\Omega)$.

$$\begin{aligned}&\left(\int_\Omega\langle M(x)(\alpha U),\alpha U\rangle\mathrm{d}x+\int_{\partial\Omega}\langle P(x)(\alpha U),\alpha U\rangle\mathrm{d}x\right)\\&=\alpha^2\left(\int_\Omega\langle M(x)U,U\rangle\mathrm{d}x+\int_{\partial\Omega}\langle P(x)U,U\rangle\mathrm{d}x\right)\overset{U\in\mathbb{H}_{(M,P)}(\Omega)}{=}0.\end{aligned}$$

Therefore $\alpha U\in\mathbb{H}_{(M,P)}(\Omega)$. Now we show that $U+V\in\mathbb{H}_{(M,P)}(\Omega)$.

$$\begin{aligned}&\int_\Omega\langle M(x)(U+V),(U+V)\rangle\mathrm{d}x+\int_{\partial\Omega}\langle P(x)(U+V),(U+V)\rangle\mathrm{d}x\\&=\int_\Omega\langle M(x)U,U\rangle\mathrm{d}x+\int_{\partial\Omega}\langle P(x)U,U\rangle\mathrm{d}x+\int_\Omega\langle M(x)V,V\rangle\mathrm{d}x\\&\quad+\int_{\partial\Omega}\langle P(x)V,V\rangle\mathrm{d}x+2\int_\Omega\langle M(x)U,V\rangle\mathrm{d}x+2\int_{\partial\Omega}\langle P(x)U,V\rangle\mathrm{d}x.\end{aligned}$$

Since $U\in\mathbb{H}_{(M,P)}(\Omega)$, it follows that

$$\begin{aligned}0&=\int_\Omega\langle M(x)U,U\rangle\mathrm{d}x=\int_\Omega\langle Q_M^{\mathrm{T}}(x)D_M(x)Q_M(x)U,U\rangle\mathrm{d}x\\&=\int_\Omega\langle D_M(x)Q_M(x)U,Q_M(x)U\rangle\mathrm{d}x.\end{aligned}$$

By setting $y(x):=Q_M(x)U$, we get

$$0=\int_\Omega\langle D_M(x)y(x),y(x)\rangle\mathrm{d}x=\sum_{i=1}^k\int_\Omega\lambda_i^M(x)y_i^2(x)\mathrm{d}x.$$

Since $\lambda_i^M(x)\geq0$ for *a.e.* $x\in\Omega$, it readily follows that

$$\lambda_i^M(x)y_i(x)=0\quad\text{for } a.e.\ x\in\Omega;$$

that is, the vector $D_M(x)y(x)$ satisfies

$$D_M(x)y(x)=0 \quad \text{for } \ a.e.\, x\in\Omega,$$

or equivalently

$$D_M(x)Q_M(x)U=0 \quad \text{for } \ a.e. \text{ on } \Omega.$$

Hence,

$$2\int_\Omega \langle M(x)U,V\rangle \mathrm{d}x = 2\int_\Omega \langle Q_M^{\mathrm{T}}(x)D_M(x)Q_M(x)U,V\rangle \mathrm{d}x$$
$$= 2\int_\Omega \langle D_M(x)Q_M(x)U,Q_M(x)V\rangle \mathrm{d}x = 0,$$

since $D_M(x)Q_M(x)U=0\; a.e.$ on Ω. A similar arguments shows that

$$2\int_{\partial\Omega}\langle P(x)U,V\rangle \mathrm{d}x = 0.$$

Therefore $U+V\in\mathbb{H}_{(M,P)}(\Omega)$, so we have that $\mathbb{H}_{(M,P)}(\Omega)$ is a subspace of $H(\Omega)$. Thus, one can split the Hilbert space $H(\Omega)$ as a direct (A,Σ)-orthogonal sum in the following way

$$H(\Omega)=\mathbb{H}_{(M,P)}(\Omega)\oplus_{(A,\Sigma)}\left[\mathbb{H}_{(M,P)}(\Omega)\right]^{\perp}.$$

Remark 7. 1) *If* $M(x)\equiv 0$ *in* Ω*, then the subspace* $\mathbb{H}_{(M,P)}(\Omega)=H_0(\Omega):=\left[H_0^1\right]^k=H_0^1(\Omega)\times H_0^1(\Omega)\times\cdots\times H_0^1(\Omega)$, *provided* $P(x)>0$ *on* $\partial\Omega$.

2) *If* $P(x)\equiv 0$ *in* $\partial\Omega$ *and* $x\in\Omega(M)$, *then the subspace* $\mathbb{H}_{(M,P)}(\Omega)=\{0\}$, *provided* $M(x)>0$ *on* Ω.

- We shall make use in what follows the real Lebesgue space $L_k^q(\partial\Omega)$ for $1\le q\le\infty$, and of the continuity and compactness of the trace operator

  $$\Gamma: H(\Omega)\to L_k^q(\partial\Omega) \quad \text{for } 1\le q<\frac{2(N-1)}{N-2},$$

  is well-defined, it is a Lebesgue integrable function with respect to Hausdorff $N-1$ dimensional measure. Sometimes we will just use U in place of ΓU when considering the trace of a function on $\partial\Omega$. Throughout, this work we denote the $L_k^2(\partial\Omega)$-inner product by

  $$\langle U,V\rangle_\partial := \int_{\partial\Omega} U\cdot V\mathrm{d}x$$

  and the associated norm by

  $$\|U\|_\partial^2 := \int_{\partial\Omega} U\cdot U \quad \forall U,V\in H(\Omega)$$

  (see [5], [6] and the references therein for more details).
- The trace mapping $\Gamma: H(\Omega)\to L_k^2(\partial\Omega)$ is compact (see [7]).

  $$\langle U,V\rangle_{(M,P)} = \int_\Omega \langle M(x)U,V\rangle \mathrm{d}x + \int_{\partial\Omega}\langle P(x)U,V\rangle \mathrm{d}x, \tag{3}$$

  defines an inner product for $H(\Omega)$, with associated norm

  $$\|U\|_{(M,P)}^2 := \int_\Omega \langle M(x)U,U\rangle \mathrm{d}x + \int_{\partial\Omega}\langle P(x)U,U\rangle \mathrm{d}x. \tag{4}$$

Now, we state some auxiliary result, which will be need in the sequel for the proof of our main result. Using the Hölder inequality, the continuity of the trace operator, the Sobolev embedding theorem and lower semicontinuity of $\|.\|_{(A,\Sigma)}$, we deduce that $\|.\|_{(A,\Sigma)}$ is equivalent to the standard $H(\Omega)$-norm. This observation enables us to prove the existence of an unbounded and discrete spectrum for the Steklov-Robin eigenproblem (1) and discuss some of its properties.

Definition 2. *Define the functional*

$$\Lambda_{A,\Sigma}: H(\Omega)\to[0,\infty),$$

$$\Lambda_{A,\Sigma}(U):=\int_\Omega\left[\nabla U\cdot\nabla U+\langle A(x)U,U\rangle\right]\mathrm{d}x+\int_{\partial\Omega}\langle\Sigma(x)U,U\rangle \mathrm{d}x=\|U\|_{(A,\Sigma)}^2, \quad \forall U\in H(\Omega),$$

and

$$\Upsilon_{M,P}: H(\Omega) \to [1,\infty),$$

$$\Upsilon_{M,P}(U) := \int_\Omega \langle M(x)U,U\rangle \mathrm{d}x + \int_{\partial\Omega} \langle P(x)U,U\rangle \mathrm{d}x - 1 = \|U\|^2_{(M,P)} - 1, \ \forall U \in H(\Omega).$$

Lemma 1. *Suppose* (A2), (S2), (M1), (P1) *are met*. *Then the functionals* $\Lambda_{A,\Sigma}$ *and* $\Upsilon_{M,P}$ *are* C^1*-functional* (*i.e.*; *continuously differentiable*).

See [8] *for the proof of Lemma* 1.

Theorem 8. $\Lambda_{A,\Sigma}$ *is G-differentiable and convex*. *Then* $\Lambda_{A,\Sigma}$ *is weakly lower-semi-continuous*.

See [8] *for the proof of Theorem* 8.

2. Main Result

Theorem 9. *Assume Assumption* 1 *as above, then we have the following*.

1) *The eigensystem* (1) *has a sequence of real eigenvalues*

$$0 < \mu_1 \le \mu_2 \le \mu_3 \le \cdots \le \mu_j \le \cdots \to \infty \quad \text{as } j \to \infty,$$

and each eigenvalue has a finite-dimensional eigenspace.

2) *The eigenfunctions* φ_j *corresponding to the eigenvalues* μ_j *from an* (A,Σ)*-orthogonal and* (M,P)*-orthonormal family in* $[\mathbb{H}_{M,P}(\Omega)]^\perp$ (*a closed subspace of* $H(\Omega)$).

3) *The normalized eigenfunctions provide a complete* (A,Σ)*-orthonormal basis of* $[\mathbb{H}_{M,P}(\Omega)]^\perp$. *Moreover, each function* $U \in [\mathbb{H}_{M,P}(\Omega)]^\perp$ *has a unique representation of the from*

$$U = \sum_{j=1}^{\infty} c_j \varphi_j \quad \text{with } c_j := \frac{1}{\mu_j}\langle U,\varphi_j\rangle_{(A,\Sigma)} = \langle U,\varphi_j\rangle_{(M,P)},$$
$$\|U\|^2_{(A,\Sigma)} = \sum_{j=1}^{\infty} \mu_j |c_j|^2. \tag{5}$$

In addition,

$$\|U\|^2_{(M,P)} = \sum_{j=1}^{\infty} |c_j|^2.$$

Proof of Theorem 9. We will prove the existence of a sequence of real eigenvalues μ_j and the eigenfunctions φ_j corresponding to the eigenvalues that from an orthogonal family in $[\mathbb{H}_{M,P}(\Omega)]^\perp$.

We show that $\Lambda_{A,\Sigma}$ attains its minimum on the constraint set

$$W_0 = \left\{U \in [\mathbb{H}_{M,P}(\Omega)]^\perp : \Upsilon_{M,P}(U) = 0\right\}.$$

Let $\alpha := \inf_{U \in W_0} \Lambda_{(A,\Sigma)}(U)$, by using the continuity of the trace operator, the Sobolev embedding theorem and the lower-semi-continuity of $\Lambda_{A,\Sigma}$.

Let $\{U_l\}_{l=1}^{\infty}$ be a minimizing sequence in W_0 for $\Lambda_{A,\Sigma}$, since $\lim_{l\to\infty} \Lambda_{A,\Sigma}(U_l) = \alpha$, we have that $\Lambda_{A,\Sigma}(U_l) = \|U_l\|_{(A,\Sigma)}$, by the definition of α we have that for all $\epsilon > 0$ and for all sufficiently large l, then $\|U_l\|^2_{A,\Sigma} \le \alpha + \epsilon$ by using the equivalent norm we have that, there is exist β, such that

$$\|U_l\|^2_{H(\Omega)} \le \beta \|U_l\|^2_{A,\Sigma},$$

so we have that

$$\|U_l\|^2_{H(\Omega)} \le \beta \|U_l\|^2_{A,\Sigma} \le \beta(\alpha + \epsilon).$$

Therefore, this sequence is bounded in $H(\Omega)$. Thus it has a weakly convergent subsequence $\left\{U_{l_j} : j \ge 1\right\}$

which convergent weakly to $\hat{U}$ in $H(\Omega)$. From Rellich-Kondrachov theorem this subsequence converges strongly to $\hat{U}$ in $L_k^2(\Omega)$, so $\hat{U}$ in W_0. Thus $\Lambda_{A,\Sigma}(\hat{U})=\alpha$ as the functional is weakly l.s.c. (see Theorem 8).

There exists φ_1 such that $\Lambda_{A,\Sigma}(\varphi_1)=\alpha$. Hence, $\Lambda_{A,\Sigma}$ attains its minimum at φ_1 and φ_1 satisfies the following

$$\begin{aligned}&\int_\Omega\langle\nabla\varphi_1\cdot\nabla V\rangle \mathrm{d}x+\int_\Omega\langle A(x)\varphi_1,V\rangle \mathrm{d}x+\int_{\partial\Omega}\langle\Sigma(x)\varphi_1,V\rangle\\&=\mu_1\left(\int_\Omega\langle M(x)\varphi_1,V\rangle \mathrm{d}x+\int_{\partial\Omega}\langle P(x)\varphi_1,V\rangle \mathrm{d}x\right).\end{aligned}\tag{6}$$

for all $V\in\left[\mathbb{H}_{(M,P)}(\Omega)\right]^\perp$. We see that (μ_1,φ_1) satisfies Equation (2) in a weak sense and $\varphi_1\in W_0$ this implies that $\varphi_1\in\left[\mathbb{H}_{(M,P)}(\Omega)\right]^\perp$ by the definition of W_0. Now take $V=\varphi_1$ in Equation (6), we obtain that the eigenvalue μ_1 is the infimum $\alpha=\Lambda_{A,\Sigma}(\varphi_1)=\mu_1$. This means that we could define μ_1 by the Rayleigh quotient

$$\mu_1=\inf_{\substack{U\in W_0\\U\neq0}}\frac{\Lambda_{A,\Sigma}(U)}{\|U\|_{(M,P)}^2}.$$

Clearly, $\mu_1=\Lambda_{A,\Sigma}(\varphi_1)>0$. Indeed assume that $\Lambda_{A,\Sigma}(\varphi_1)=0$ then $|\nabla\varphi_1|=0$ on Ω, hence φ_1 must be a constant and $\langle A(x)\varphi,\varphi\rangle=0$ with $\varphi\neq0$ that contradicts the assumptions imposed on $A(x)$. Thus $\mu_1>0$.

Now we show the existence of higher eigenvalues.

Define

$$\mathbb{F}_1:W_0\to\mathbb{R}\ \text{by}\ \mathbb{F}_1(U):=\langle U,\varphi_1\rangle_{M,P}.$$

We know that the kernel of $\mathbb{F}_1$

$$ker\mathbb{F}_1=\{U\in W_0:\mathbb{F}_1(U)=0\}=:W_1.$$

Since W_1 is the null-space of the continuous functional $\langle .,\varphi_1\rangle_{M,P}$ on $\left[\mathbb{H}_{(M,P)}(\Omega)\right]^\perp$, W_1 is a closed subspace of $\left[\mathbb{H}_{(M,P)}(\Omega)\right]^\perp$, and it is therefore a Hilbert space itself under the same inner product $\langle .,.\rangle_{(M,P)}$. Now we define

$$\mu_2=\inf\{\Lambda_{A,\Sigma}(U):U\in W_1\}=\inf_{\substack{U\in W_1\\U\neq0}}\frac{\Lambda_{A,\Sigma}(U)}{\|U\|_{(M,P)}^2}.$$

Since $W_1\subset W_0$ then we have that $\mu_1\le\mu_2$. Now we define

$$\mathbb{F}_2:W_1\to\mathbb{R}\ \text{by}\ \mathbb{F}_2(U)=\langle U,\varphi_2\rangle_{M,P}$$

we know that the kernel of $\mathbb{F}_2$

$$ker\mathbb{F}_2=\{U\in W_1:\mathbb{F}_2(U)=0\}=:W_2.$$

Since W_2 is the null-space of the continuous functional $\langle .,\varphi_2\rangle_{M,P}$ on $\left[\mathbb{H}_{(M,P)}(\Omega)\right]^\perp$, W_2 is a closed subspace of $\left[\mathbb{H}_{(M,P)}(\Omega)\right]^\perp$, and it is therefore a Hilbert space itself under the same inner product $\langle .,.\rangle_{(M,P)}$. Now we define

$$\mu_3=\inf\{\Lambda_{A,\Sigma}(U):U\in W_2\}=\inf_{\substack{U\in W_2\\U\neq0}}\frac{\Lambda_{A,\Sigma}(U)}{\|U\|_{(M,P)}^2}.$$

Since $W_2\subset W_1$ then we have that $\mu_2\le\mu_3$. Moreover, we can repeat the above arguments to show that μ_3 is achieved at some $\varphi_3\in\left[\mathbb{H}_{(M,P)}(\Omega)\right]^\perp$.

We let

$$W_3=\left\{u\in W_2:\langle U,\varphi_3\rangle_{(M,P)}=0\right\}$$

and

$$\mu_4 = \inf\left\{\Lambda_{A,\Sigma}\left(U\right): U \in W_3\right\} = \inf_{\substack{U\in W_3\\ U\neq 0}} \frac{\Lambda_{A,\Sigma}\left(U\right)}{\left\|U\right\|^2_{(M,P)}}.$$

Since $W_3 \subset W_2$ then we have that $\mu_3 \le \mu_4$. Moreover, we can repeat the above arguments to show that μ_4 is achieved at some $\varphi_4 \in \left[\mathbb{H}_{(M,P)}\left(\Omega\right)\right]^{\perp}$.

Proceeding inductively, in general we can define

$$\mathbb{F}_j : W_{j-1} \to \mathbb{R} \text{ by } \mathbb{F}_j\left(U\right) = \left\langle U, \varphi_j \right\rangle_{M,P},$$

we know that the kernel of $\mathbb{F}_2$

$$ker\mathbb{F}_j = \left\{U \in W_{j-1} : \mathbb{F}_j\left(U\right) = 0\right\} =: W_j.$$

Since W_j is the null-space of the continuous functional $\left\langle ., \varphi_j \right\rangle_{M,P}$ on $\left[\mathbb{H}_{(M,P)}\left(\Omega\right)\right]^{\perp}$, W_j is a closed subspace of $\left[\mathbb{H}_{(M,P)}\left(\Omega\right)\right]^{\perp}$, and it is therefore a Hilbert space itself under the same inner product $\left\langle .,. \right\rangle_{(M,P)}$. Now we define

$$\mu_{j+1} = \inf\left\{\Lambda_{A,\Sigma}\left(U\right): U \in W_j\right\} = \inf_{\substack{U\in W_j\\ U\neq 0}} \frac{\Lambda_{A,\Sigma}\left(U\right)}{\left\|U\right\|^2_{(M,P)}}.$$

In this way, we generate a sequence of eigenvalues

$$0 < \mu_1 \le \mu_2 \le \mu_3 \le \cdots \le \mu_j \le \cdots$$

whose associated φ_j are *c*-orthogonal and $\left(M,P\right)$-orthonormal in $\left[H_0^1\left(\Omega\right)\right]^{\perp}$.

Claim 1 $\mu_j \to \infty$ as $j \to \infty$.

Proof of claim 1. By way of contradiction, assume that the sequence is bounded above by a constant. Therefore, the corresponding sequence of eigenfunctions φ_j is bounded in $H\left(\Omega\right)$. By Rellich-Kondrachov theorem and the compactness of the trace operator, there is a Cauchy subsequence (which we again denote by φ_j), such that

$$\left\|\varphi_j - \varphi_k\right\|^2_{(M,P)} \to 0. \tag{7}$$

Since the φ_j are $\left(M,P\right)$-orthonormal, we have that $\left\|\varphi_j - \varphi_k\right\|^2_{(M,P)} = \left\|\varphi_j\right\|^2_{(M,P)} + \left\|\varphi_k\right\|^2_{(M,P)} = 2 > 0$, if $j \neq k$, which contradicts Equation (7). Thus, $\mu_j \to \infty$. We have that each μ_j occurs only finitely many times.

Claim 2

Each eigenvalue μ_j has a finite-dimensional eigenspace.

See [8] *for the proof of claim* 2.

We will now show that the normalized eigenfunctions provide a complete orthonormal basis of $\left[H_0^1\left(\Omega\right)\right]^{\perp}$. Let

$$\psi_j = \frac{1}{\sqrt{\mu_j}}\varphi_j,$$

so that $\left\|\psi_j\right\|^2_{(A,\Sigma)} = 1.$

Claim 3

The sequence $\left\{\psi_j\right\}_{j\ge 1}$ is a maximal $\left(A,\Sigma\right)$-orthonormal family of $\left[\mathbb{H}_{(M,P)}\left(\Omega\right)\right]^{\perp}$. (We know that the set is maximal $\left(A,\Sigma\right)$-orthonormal if and only if it is a complete orthonormal basis).

Proof of Claim 3. By way of contradiction, assume that the sequence $\left\{\psi_j\right\}_{j\ge 1}$ is not maximal, then there exists a $\xi \in \left[\mathbb{H}_{(M,P)}\left(\Omega\right)\right]^{\perp}$, and $\xi \notin \left\{\psi_j\right\}_{j\ge 1}$, such that $\left\|\xi\right\|^2_{(A,\Sigma)} = 1$ and $\left\langle \xi, \psi_j \right\rangle_{(A,\Sigma)} = 0$ $\forall j$, *i.e.*;

$$0=\left\langle \xi,\psi_j \right\rangle_{(A,\Sigma)}=\left\langle \xi,\frac{1}{\sqrt{\mu_j}}\varphi_j \right\rangle_{(A,\Sigma)}=\frac{1}{\sqrt{\mu_j}}\left\langle \xi,\varphi_j \right\rangle_{(A,\Sigma)}$$

$$=^{(\text{by } (6))}\frac{\mu_j}{\sqrt{\mu_j}}\left\langle \xi,\varphi_j \right\rangle_{(M,P)}=\mu_j\left\langle \xi,\frac{1}{\sqrt{\mu_j}}\varphi_j \right\rangle_{(M,P)}=\mu_j\left\langle \xi,\psi_j \right\rangle_{\partial},$$

since $\mu_j>0\ \forall j$. Therefore $\left\langle \xi,\psi_j \right\rangle_{(M,P)}=0$. We have that $\xi\in W_j\ \ \forall j\geq 1$. It follows from the definition of μ_j that

$$\mu_j\leq\frac{\|\xi\|^2_{(A,\Sigma)}}{\|\xi\|^2_{(M,P)}}=\frac{1}{\|\xi\|^2_{(M,P)}}\quad\forall j\geq 1.$$

Since we know from Claim 1 that $\mu_j\to\infty$ as $j\to\infty$, we have that $\|\xi\|^2_{(M,P)}=0.$ Therefore $\xi=0$ a.e in Ω, which contradicts the definition of ξ. Thus the sequence $\{\psi_j\}_{j\geq 1}$ is a maximal (A,Σ)-orthonormal family of $\left[H_{(M,P)}(\Omega)\right]^{\perp}$, so the sequence $\{\psi_j\}_{j\geq 1}$ provides a complete orthonormal basis of $\left[H_{(M,P)}(\Omega)\right]^{\perp}$; that is, for any $U\in\left[\mathbb{H}_{(A,\Sigma)}(\Omega)\right]^{\perp}$, $U=\sum_{j=1}^{\infty}d_j\psi_j$ with $d_j=\left\langle U,\psi_j \right\rangle_{(A,\Sigma)}=\frac{1}{\sqrt{\mu_j}}\left\langle U,\kappa_j \right\rangle_{(A,\Sigma)}$, and

$$\|U\|^2_{(A,\Sigma)}=\sum_{j=1}^{\infty}\left|d_j\right|^2,$$

$$U=\sum_{j=1}^{\infty}d_j\frac{1}{\sqrt{\mu_j}}\varphi_j.$$

Now let

$$c_j=d_j\frac{1}{\sqrt{\mu_j}}=\frac{1}{\mu_j}\left\langle U,\varphi_j \right\rangle_{(A,\Sigma)}=^{(6)}\left\langle U,\varphi_j \right\rangle_{(M,P)}.$$

Therefore,

$$U=\sum_{j=1}^{\infty}c_j\varphi_j$$

and

$$\|U\|^2_{(A,\Sigma)}=\sum_{j=1}^{\infty}\left|c_j\right|^2\left\|\varphi_j\right\|^2_{(A,\Sigma)}=\sum_{j=1}^{\infty}\mu_j\left|c_j\right|^2.$$

Claim 4

We shall show that

$$\|U\|^2_{(M,P)}=\sum_{j=1}^{\infty}\left|c_j\right|^2.$$

Proof of Claim 4.

$$\|U\|^2_{(M,P)}=\left\langle u,u \right\rangle_{(M,P)}=\left\langle \sum_{j=1}^{\infty}c_j\varphi_j,\sum_{k=1}^{\infty}c_k\varphi_k \right\rangle_{(M,P)}$$

$$=\sum_{j=1}^{\infty}c_j\sum_{k=1}^{\infty}c_k\left\langle \varphi_j,\varphi_k \right\rangle_{(M,P)}=\sum_{j=1}^{\infty}\left|c_j\right|^2.$$

Thus

$$\|U\|_{(M,P)}^2 = \sum_{j=1}^{\infty} |c_j|^2 .$$

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