Applied Mathematics, 2015, 6, 312-318
Published Online February 2015 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2015.62029

Scientific Research Publishing

Idempotent and Regular Elements of the Complete Semigroups of Binary Relations of the Class $\Sigma_3(X,9)$

Barış Albayrak, Neşet Aydın

Çanakkale Onsekiz Mart University, Çanakkale, Turkey
Email: balbayrak77@gmail.com, neseta@comu.edu.tr

Received 13 January 2015; accepted 1 February 2015; published 5 February 2015

Abstract

In this paper, we take Q_{16} subsemilattice of D and we will calculate the number of right unit, idempotent and regular elements α of $B_X(Q_{16})$ satisfied that $V(D,\alpha) = Q_{16}$ for a finite set X. Also we will give a formula for calculate idempotent and regular elements of $B_X(Q)$ defined by an X-semilattice of unions D.

Keywords

Semilattice, Semigroup, Binary Relation

1. Introduction

Let X be a nonempty set and B_X be semigroup of all binary relations on the set X. If D is a nonempty set of subsets of X which is closed under the union then D is called a complete X-semilattice of unions.

Let f be an arbitrary mapping from X into D. Then one can construct a binary relation α_f on X by $\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x))$. The set of all such binary relations is denoted by $B_X(D)$ and called a complete semigroup of binary relations defined by an X-semilattice of unions D.

We use the notations, $y\alpha = \{x\in X \mid y\alpha x\}$, $Y\alpha = \bigcup_{y\in Y} y\alpha$, $V(D,\alpha) = \{Y\alpha \mid Y\in D\}$, $Y_T^{\alpha} = \{y\in X \mid y\alpha = T\}$.

A representation of a binary relation α of the form $\alpha = \bigcup_{T\in V(X^*,\alpha)}(Y_T^{\alpha}\times T)$ is called quasinormal. Note that, if $\alpha = \bigcup_{T\in V(X^*,\alpha)} (Y_T^\alpha \times T)$ is a quasinormal representation of the binary relation α, then $Y_T^\alpha \cap Y_{T'}^\alpha = \varnothing$ for T, $T' \in V(X^*,\alpha)$ and $T \neq T'$.

A complete *X*-semilattice of unions *D* is an *XI*-semilattice of unions if $\Lambda(D, D_t) \in D$ for any $t \in \breve{D}$ and $Z = \bigcup_{t\in Z} \Lambda(D, D_t)$ for any nonempty element *Z* of *D*.

Now, $\alpha \in B_X(D)$ is said to be right unit if $\beta \circ \alpha = \beta$ for all $\beta \in B_X(D)$. Also, $\alpha \in B_X(D)$ is idempotent if $\alpha \circ \alpha = \alpha$. And $\alpha \in B_X(D)$ is said to be regular if $\alpha \circ \beta \circ \alpha = \alpha$ for some $\beta \in B_X(D)$.

Let D', D'' be complete *X*-semilattices of unions and φ be a one-to-one mapping from D' to D''. A mapping $\varphi : D' \to D''$ is a complete isomorphism provided $\varphi(\cup D_1) = \bigcup_{T'\in D_1} \varphi(T')$ for all nonempty subset D_1 of the semilattice D'. Besides that, if $\varphi : V(D,\alpha) \to D'$ is a complete isomorphism where $\alpha \in B_X(D)$, $\varphi(T)\alpha = T$ for all $T \in V(D,\alpha)$, φ is said to be a complete α-isomorphism.

Let *Q* and D' be respectively some *XI* and *X*-subsemilattices of the complete *X*-semilattice of unions *D*. Then

$$R_\varphi(Q, D') = \{\alpha \in B_X(D) | \alpha \text{ regular element, } \varphi \text{ complete } \alpha\text{-isomorphism}\}$$

where $\varphi : Q \to D'$ complete isomorphism and $V(D,\alpha) = Q$. Besides, let us denote

$$R(Q, D') = \bigcup_{\varphi\in\Phi(Q,D')} R_\varphi(Q, D') \quad \text{and} \quad R(D') = \bigcup_{Q'\in\Omega(Q)} R(Q', D')$$

where

$$\Phi(Q, D') = \{\varphi \mid \varphi : Q \to D' \text{ is a complete } \alpha\text{-isomorhism } \exists \alpha \in B_X(D)\}$$

$$\Omega(Q) = \{Q' | Q' \text{ is } XI\text{-subsemilattices of } D \text{ which is complete isomorphic to } Q\}$$

This structure was comprehensively investigated in Diasamidze [1].

Lemma 1. [1] *If Q is complete X-semilattice of unions and* $I(Q)$ *is the set all right units of the semigroup* $B_X(Q)$ *then* $I(Q) = R_{id_Q}(Q, Q)$.

Lemma 2. [2] *Let X be a finite set, D be a complete X-semilattice of unions and* $Q = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ *be X-subsemilattice of unions of D satisfies the following conditions*

$$\begin{array}{ll} T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_8, & T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_7 \subset T_8, \\ T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_6 \subset T_8, & T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_7 \subset T_8, \\ T_4 \setminus T_3 \neq \varnothing, T_3 \setminus T_4 \neq \varnothing, & T_6 \setminus T_7 \neq \varnothing, \quad T_7 \setminus T_6 \neq \varnothing, \\ T_3 \cup T_4 = T_5, T_6 \cup T_7 = T_8 & T_1 \neq \varnothing. \end{array}$$

Q is *XI*-semilattice of unions.

Theorem 1. [2] *Let X be a finite set and Q be XI-semilattice. If* $D' = \{\overline{T}_1, \overline{T}_2, \overline{T}_3, \overline{T}_4, \overline{T}_5, \overline{T}_6, \overline{T}_7, \overline{T}_8\}$ *is* α*-isomorphic to Q and* $\Omega(Q) = m_0$, *then*

$$\begin{aligned} |R(D')| = m_0 \cdot 4 \cdot \left(2^{|(\overline{T}_3\cap\overline{T}_4)\setminus\overline{T}_1|}\left(2^{|\overline{T}_2\setminus\overline{T}_1|} - 1\right)\right) \cdot \left(3^{|\overline{T}_4\setminus\overline{T}_3|} - 2^{|\overline{T}_4\setminus\overline{T}_3|}\right) \cdot \left(3^{|\overline{T}_3\setminus\overline{T}_4|} - 2^{|\overline{T}_3\setminus\overline{T}_4|}\right) \cdot 5^{|(\overline{T}_7\cap\overline{T}_6)\setminus\overline{T}_5|} \\ \cdot \left(6^{|\overline{T}_7\setminus\overline{T}_6|} - 5^{|\overline{T}_7\setminus\overline{T}_6|}\right) \cdot \left(6^{|\overline{T}_6\setminus\overline{T}_7|} - 5^{|\overline{T}_6\setminus\overline{T}_7|}\right) \cdot 8^{|X\setminus\overline{T}_8|}. \end{aligned}$$

Theorem 2. [2] *Let* $\alpha \in B_X(Q)$ *be a quasinormal representation of the form* $\alpha = \bigcup_{i=1}^{8}(Y_i^\alpha \times T_i)$ *such that* $V(D,\alpha) = Q$. $\alpha \in B_X(D)$ *is a regular iff for some complete* α*-isomorphism* $\varphi : Q \to D' \subseteq D$, *the following conditions are satisfied*:

$$\begin{array}{l} Y_1^\alpha \supseteq \varphi(T_1), \quad Y_1^\alpha \cup Y_2^\alpha \supseteq \varphi(T_2), \quad Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \supseteq \varphi(T_3), \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4), \quad Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \supseteq \varphi(T_6), \\ Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_7^\alpha \supseteq \varphi(T_7), \quad Y_2^\alpha \cap \varphi(T_2) \neq \varnothing, \\ Y_3^\alpha \cap \varphi(T_3) \neq \varnothing, \quad Y_4^\alpha \cap \varphi(T_4) \neq \varnothing, \quad Y_6^\alpha \cap \varphi(T_6) \neq \varnothing, \quad Y_7^\alpha \cap \varphi(T_7) \neq \varnothing. \end{array}$$

Let X be a finite set and $D=\{T_1,T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$ be a complete X-semilattice of unions which satisfies the following conditions

$$
\begin{gathered}
T_1 \subset T_3 \subset T_5 \subset T_6 \subset T_8 \subset T_9, \\
T_1 \subset T_3 \subset T_5 \subset T_6 \subset T_7 \subset T_9, \\
T_1 \subset T_3 \subset T_4 \subset T_6 \subset T_8 \subset T_9, \\
T_1 \subset T_3 \subset T_4 \subset T_6 \subset T_7 \subset T_9, \\
T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_8 \subset T_9, \\
T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_7 \subset T_9, \\
T_2 \subset T_3 \subset T_4 \subset T_6 \subset T_8 \subset T_9, \\
T_2 \subset T_3 \subset T_4 \subset T_6 \subset T_7 \subset T_9, \\
T_1 \setminus T_2 \neq \varnothing,\ T_2 \setminus T_1 \neq \varnothing,\ T_4 \setminus T_5 \neq \varnothing, \\
T_5 \setminus T_4 \neq \varnothing,\ T_7 \setminus T_8 \neq \varnothing,\ T_8 \setminus T_7 \neq \varnothing, \\
T_1 \cup T_2 = T_3,\ T_4 \cup T_5 = T_6, \\
T_7 \cup T_8 = T_9,\ T_1 \cap T_2 \neq \varnothing
\end{gathered}
$$

The diagram of the D is shown in **Figure 1**. By the symbol $\sum_3(X,9)$ we denote the class of all complete X-semilattice of unions whose every element is isomophic to an X-semilattice of the form D.

All subsemilattice of $D=\{T_1,T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$ are given in **Figure 2**.

In Diasamidze [1], it has shown that subsemilattices 1 - 15 are XI-semilattice of unions and subsemilattices 17 - 24 are not XI-semilattice of unions. In Yeşil Sungur [3] and Albayrak [4], they have shown that subsemilattices 25 and 26 are XI-semilattice of unions if and only if $T_1 \cap T_2 = \varnothing$". Also they found that number of right unit, idempotent and regular elements in subsemilattices.

In this paper, we take in particular, $Q_{16}=\{T,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$ subsemilattice of D. We will calculate the number of right unit, idempotent and regular elements α of $B_X(Q_{16})$ satisfied that $V(D,\alpha)=Q_{16}$ for a finite set X. Also we will give a formula for calculate idempotent and regular elements of $B_X(D)$ defined by an X-semilattice of unions $D=\{T_1,T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$.

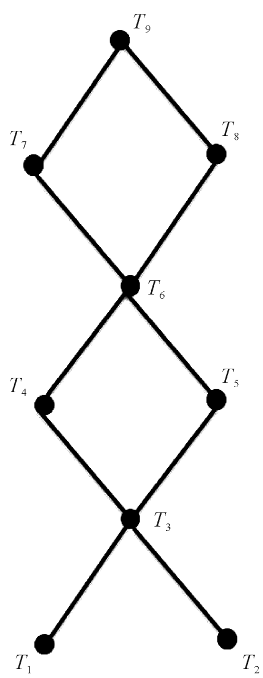

Figure 1. Diagram of D.

2. Results

Let $Q_{16}=\{T,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$ be complete X-subsemilattice of D satisfies the following conditions

$$
\begin{gathered}
T\subset T_3\subset T_4\subset T_6\subset T_7\subset T_9,\\
T\subset T_3\subset T_5\subset T_6\subset T_7\subset T_9,\\
T\subset T_3\subset T_4\subset T_6\subset T_8\subset T_9,\\
T\subset T_3\subset T_5\subset T_6\subset T_8\subset T_9,\\
T_4\setminus T_5\neq\varnothing,\quad T_5\setminus T_4\neq\varnothing,\\
T_7\setminus T_8\neq\varnothing,\quad T_8\setminus T_7\neq\varnothing,\\
T_4\cup T_5=T_6,\quad T_8\cup T_7=T_9\\
T\neq\varnothing.
\end{gathered}
$$

The diagram of the Q_{16} is shown in **Figure 3**. From Lemma 2 Q_{16} is XI-semilattice of unions.

Let $Q_{16}\vartheta_{XI}$ denote the set of all XI-subsemilattice of the semilattice D which are isomorphic of the X-semilattice Q_{16}. Then we get

$$Q_{16}\vartheta_{XI}=\{\{T_1,T_3,T_4,T_5,T_6,T_7,T_8,T_9\},\{T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}\}$$

Let $\alpha\in B_X(Q_{16})$ be a idempotent element having a quasinormal representation of the form

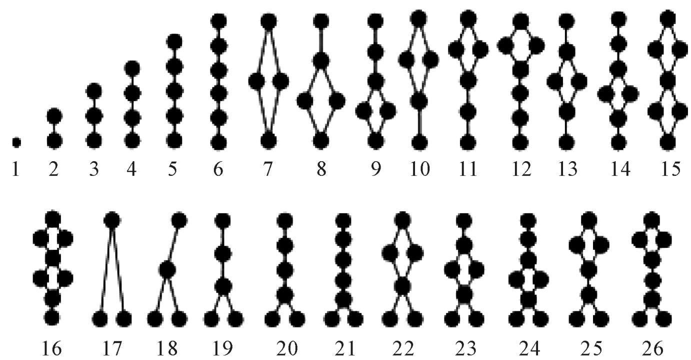

Figure 2. All subsemilattice of D.

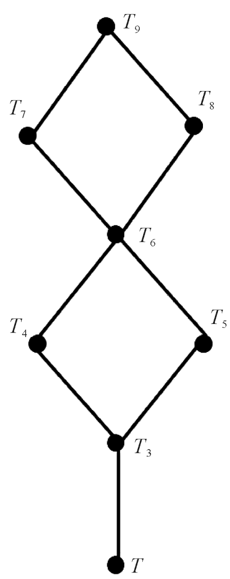

Figure 3. The diagram of the Q_{16}.

$\alpha=\left(Y_T^{\alpha}\times T\right)\cup\bigcup_{i=3}^{9}\left(Y_i^{\alpha}\times T_i\right)$, such that $V\left(D,\alpha\right)=Q_{16}$. First we calculate number of this idempotent elements in $B_X\left(Q_{16}\right)$.

Lemma 3. *If X is a finite set and* $I\left(Q_{16}\right)$ *is the set all right units of the semigroup* $B_X\left(Q_{16}\right)$, *then the number* $\left|I\left(Q_{16}\right)\right|$ *may be calculated by formula*:

$$\left|I\left(Q_{16}\right)\right|=\left((2^{|T_3\setminus T|}-1)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)$$
$$\cdot 5^{|(T_7\cap T_8)\setminus T_6|}\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}.$$

Proof. From Lemma 1 we have $I\left(Q_{16}\right)=R_{id_{Q_{16}}}\left(Q_{16},Q_{16}\right)$ where $id_{Q_{16}}$ is identity mapping of the set Q_{16}. For this reason $D'=Q$ in Theorem 1. Then we obtain

$$\left|I\left(Q_{16}\right)\right|=\left(\left(2^{|T_3\setminus T|}-1\right)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)$$
$$\cdot 5^{|(T_7\cap T_8)\setminus T_6|}\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}.$$
□

Theorem 3. *If X is a finite set and* $I^*\left(Q_{16}\right)$ *is the set all idempotent elements of the semigroup* $B_X\left(Q_{16}\right)$, *then the number* $\left|I^*\left(Q_{16}\right)\right|$ *may be calculated by formula*:

$$\left|I^*\left(Q_{16}\right)\right|=\left(\left(2^{|T_3\setminus T_2|}-1\right)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)$$
$$\cdot 5^{|(T_7\cap T_8)\setminus T_6|}\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}$$
$$+\left(\left(2^{|T_3\setminus T_1|}-1\right)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)$$
$$\cdot 5^{|(T_7\cap T_8)\setminus T_6|}\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}.$$

Proof. By using Lemma 3 we have number of right units of the semigroup $B_X\left(Q_{16}\right)$ defined by $Q_{16}=\{T,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$ for $T\in\{T_1,T_2\}$. Then number of idempotent elements of $I^*\left(Q_{16}\right)$ calculated by formula $I^*\left(Q_{16}\right)=\sum_{D'\in Q_{16}\vartheta_{XI}}\left|I\left(D'\right)\right|$. By using

$$Q_{16}\vartheta_{XI}=\left\{\{T_1,T_3,T_4,T_5,T_6,T_7,T_8,T_9\},\{T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}\right\}$$

we obtain above formula. □

Now we will calculate number of regular elements $\alpha\in B_X\left(Q_{16}\right)$ having a quasinormal representation of the form $\alpha=\left(Y_T^{\alpha}\times T\right)\cup\bigcup_{i=3}^{9}\left(Y_i^{\alpha}\times T_i\right)$ such that $V\left(D,\alpha\right)=Q_{16}$. Let $R^*\left(Q_{16}\right)$ be the set all regular elements of the semigroup $B_X\left(Q_{16}\right)$. By using $Q_{16}\vartheta_{XI}=\left\{\{T_1,T_3,T_4,T_5,T_6,T_7,T_8,T_9\},\{T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}\right\}$ we get $\left|\Omega\left(Q_{16}\right)\right|=2$. The number of all automorphisms of the semilattice Q_{16} is $q=4$. These are

$$I_Q=\begin{pmatrix}T & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9\\ T & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9\end{pmatrix}\quad \varphi=\begin{pmatrix}T & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9\\ T & T_3 & T_5 & T_4 & T_6 & T_7 & T_8 & T_9\end{pmatrix}$$
$$\theta=\begin{pmatrix}T & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9\\ T & T_3 & T_4 & T_5 & T_6 & T_8 & T_7 & T_9\end{pmatrix}\quad \tau=\begin{pmatrix}T & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9\\ T & T_3 & T_5 & T_4 & T_6 & T_8 & T_7 & T_9\end{pmatrix}$$

Then $\left|\Phi\left(Q_{16}\right)\right|=4$. Also by using

$$D_1'=\{T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\},\quad D_2'=\{T_2,T_3,T_5,T_4,T_6,T_7,T_8,T_9\}$$
$$D_3'=\{T_2,T_3,T_4,T_5,T_6,T_8,T_7,T_9\},\quad D_4'=\{T_2,T_3,T_5,T_4,T_6,T_8,T_7,T_9\}$$
$$D_5'=\{T_1,T_3,T_4,T_5,T_6,T_7,T_8,T_9\},\quad D_6'=\{T_1,T_3,T_5,T_4,T_6,T_7,T_8,T_9\}$$
$$D_7'=\{T_1,T_3,T_4,T_5,T_6,T_8,T_7,T_9\},\quad D_8'=\{T_1,T_3,T_5,T_4,T_6,T_8,T_7,T_9\}$$

we get $R^*(Q_{16}) = \bigcup_{i=1}^{8} R(D_i)$.

Theorem 4. *If X is a finite set and* $R^*(Q_{16})$ *is the set all regular elements of the semigroup* $B_X(Q_{16})$, *then the number* $|R^*(Q_{16})|$ *may be calculated by formula*:

$$\begin{aligned}|R^*(Q_{16})| = & 4\cdot 2\left(\left(2^{|T_3\setminus T_2|}-1\right)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot 5^{|(T_7\cap T_8)\setminus T_6|} \\ &\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|} \\ &+4\cdot 2\cdot\left(\left(2^{|T_3\setminus T_1|}-1\right)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot 5^{|(T_7\cap T_8)\setminus T_6|} \\ &\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}.\end{aligned}$$

Proof. To account for the elements that are in $R^*(Q_{16})$, we first subtract out intersection of $R(D_i')$'s. Let $\alpha \in R(D_1')\cap R(D_2')$. By using Theorem 2 and $Q_{16} = \{T, T_3, T_4, T_5, T_6, T_7, T_8, T_9\}$

$$\begin{aligned}\alpha \in R(D_1')\cap R(D_2') \Rightarrow \alpha \in R(D_1') \text{ and } \alpha \in R(D_2') \\ \Rightarrow Y_T^\alpha \supseteq T_2,\ Y_T^\alpha \cup Y_3^\alpha \supseteq T_3,\ Y_T^\alpha \cup Y_3^\alpha \cup Y_5^\alpha \supseteq T_5 \\ Y_T^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \supseteq T_4,\ Y_T^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \cup Y_8^\alpha \supseteq T_8, \\ Y_T^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \cup Y_7^\alpha \supseteq T_7,\ Y_3^\alpha \cap T_3 \neq \varnothing, \\ Y_4^\alpha \cap T_4 \neq \varnothing,\ Y_5^\alpha \cap T_5 \neq \varnothing,\ Y_7^\alpha \cap T_7 \neq \varnothing,\ Y_8^\alpha \cap T_8 \neq \varnothing, \\ Y_T^\alpha \supseteq T_2,\ Y_T^\alpha \cup Y_3^\alpha \supseteq T_3,\ Y_T^\alpha \cup Y_3^\alpha \cup Y_5^\alpha \supseteq T_4, \\ Y_T^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \supseteq T_5,\ Y_T^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \cup Y_8^\alpha \supseteq T_8, \\ Y_T^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \cup Y_7^\alpha \supseteq T_7,\ Y_3^\alpha \cap T_3 \neq \varnothing, \\ Y_5^\alpha \cap T_4 \neq \varnothing,\ Y_4^\alpha \cap T_5 \neq \varnothing,\ Y_7^\alpha \cap T_7 \neq \varnothing,\ Y_8^\alpha \cap T_8 \neq \varnothing.\end{aligned}$$

We get $\varnothing \neq Y_4^\alpha \cap T_4 \subseteq Y_4^\alpha \cap (Y_T^\alpha \cup Y_3^\alpha \cup Y_5^\alpha)$ which is a contradiction with Y_4^α, Y_T^α, Y_3^α, Y_5^α are disjoint sets. Then $R(D_1')\cap R(D_2') = \varnothing$. Smilarly $R(D_i')\cap R(D_j') = \varnothing$ for $i, j = 1, \cdots, 6$. Thus we obtain

$$|R^*(Q_{16})| = |R(D_1')| + |R(D_2')| + |R(D_3')| + |R(D_4')| + |R(D_5')| + |R(D_6')| + |R(D_7')| + |R(D_8')|$$

From Theorem 1 we get above formula. □

Corollary 1. *If X is a finite set,* I_D *is the set all idempotent elements of the semigroup* $B_X(D)$ *and* R_D *is the set all regular elements of the semigroup* $B_X(D)$, *then the number* $|I_D|$ *and* $|R_D|$ *may be calculated by formula*:

$$|(I_D)| = \sum_{i=1}^{16} |I^*(Q_i)|, \quad |(R_D)| = \sum_{i=1}^{16} |R^*(Q_i)|$$

Proof. Let I_D be the set of all idempotent elements of the semigroup $B_X(D)$. Then number of idempotent element of $B_X(D)$ is equal to sum of idempotent elements of the subsemigroup defined by *XI*-subsemilattice of *D*. $|I^*(Q_i)|$ is given in Diasamidze [1] for $(i = 1, 2, \cdots, 15)$. From Theorem 3 we have number of idempotent elements of the subsemigroup $B_X(Q_{16})$. Then the number $|I_D|$ may be calculated by formula $|(I_D)| = \sum_{i=1}^{16} |I^*(Q_i)|$. Similarly the number $|R_D|$ may be calculated by formula $|(R_D)| = \sum_{i=1}^{16} |R^*(Q_i)|$. □

References

[1] Diasamidze, Ya. and Makharadze, Sh. (2013) Complete Semigroups of Binary Relations. Kriter Yayınevi, İstanbul, 524 p.

[2] Albayrak, B., Aydin, N. and Diasamidze, Ya. (2013) Reguler Elements of the Complete Semigroups of Binary Relations of the Class $\sum_7(X,8)$. *International Journal of Pure and Applied Mathematics*, **86**, 199-216. http://dx.doi.org/10.12732/ijpam.v86i1.13

[3] Yeşil Sungur, D. and Aydin, N. (2014) Reguler Elements of the Complete Semigroups of Binary Relations of the Class $\sum_8(X,7)$. *General Mathematics Notes*, **21**, 27-42.

[4] Albayrak, B., Aydin, N. and Yeşil Sungur, D. (2014) Regular Elements of Semigroups $B_X(D)$ Defined by the Generalized X-Semilattice. *General Mathematics Notes*, **23**, 96-107.