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Idempotent and Regular Elements of the Complete Semigroups of Binary Relations of the Class $\Sigma_3(X,9)$

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Abstract

In this paper, we take Q_{16} subsemilattice of D and we will calculate the number of right unit, idempotent and regular elements α of $B_X(Q_{16})$ satisfied that $V(D,\alpha) = Q_{16}$ for a finite set X. Also we will give a formula for calculate idempotent and regular elements of $B_X(Q)$ defined by an X-semilattice of unions D.

Keywords

Semilattice, Semigroup, Binary Relation

1. Introduction

Let X be a nonempty set and B_X be semigroup of all binary relations on the set X. If D is a nonempty set of subsets of X which is closed under the union then D is called a complete X-semilattice of unions.

Let f be an arbitrary mapping from X into D. Then one can construct a binary relation α_f on X by $\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x))$. The set of all such binary relations is denoted by $B_X(D)$ and called a complete semigroup of binary relations defined by an X-semilattice of unions D.

We use the notations, $y\alpha = \{x \in X \mid y\alpha x\}$, $Y\alpha = \bigcup_{y\in Y} y\alpha$, $V(D,\alpha) = \{Y\alpha \mid Y \in D\}$, $Y_T^\alpha = \{y \in X \mid y\alpha = T\}$.

A representation of a binary relation α of the form $\alpha = \bigcup_{T\in V(X^*,\alpha)} (Y_T^\alpha \times T)$ is called quasinormal. Note that, if $\alpha=\bigcup_{T\in V\left(X^*,\alpha\right)}\left(Y_T^\alpha\times T\right)$ is a quasinormal representation of the binary relation α, then $Y_T^\alpha\cap Y_{T'}^\alpha=\varnothing$ for T, $T'\in V\left(X^*,\alpha\right)$ and $T\neq T'$.

A complete X-semilattice of unions D is an XI-semilattice of unions if $\Lambda\left(D,D_t\right)\in D$ for any $t\in\breve{D}$ and $Z=\bigcup_{t\in Z}\Lambda\left(D,D_t\right)$ for any nonempty element Z of D.

Now, $\alpha\in B_X\left(D\right)$ is said to be right unit if $\beta\circ\alpha=\beta$ for all $\beta\in B_X\left(D\right)$. Also, $\alpha\in B_X\left(D\right)$ is idempotent if $\alpha\circ\alpha=\alpha$. And $\alpha\in B_X\left(D\right)$ is said to be regular if $\alpha\circ\beta\circ\alpha=\alpha$ for some $\beta\in B_X\left(D\right)$.

Let D', D'' be complete X-semilattices of unions and φ be a one-to-one mapping from D' to D''. A mapping $\varphi:D'\to D''$ is a complete isomorphism provided $\varphi\left(\cup D_1\right)=\bigcup_{T'\in D_1}\varphi\left(T'\right)$ for all nonempty subset D_1 of the semilattice D'. Besides that, if $\varphi:V\left(D,\alpha\right)\to D'$ is a complete isomorphism where $\alpha\in B_X\left(D\right)$, $\varphi\left(T\right)\alpha=T$ for all $T\in V\left(D,\alpha\right)$, φ is said to be a complete α-isomorphism.

Let Q and D' be respectively some XI and X-subsemilattices of the complete X-semilattice of unions D. Then

$$R_\varphi\left(Q,D'\right)=\left\{\alpha\in B_X\left(D\right)\middle|\alpha\text{ regular element, }\varphi\text{ complete }\alpha\text{-isomorphism}\right\}$$

where $\varphi:Q\to D'$ complete isomorphism and $V\left(D,\alpha\right)=Q$. Besides, let us denote

$$R\left(Q,D'\right)=\bigcup_{\varphi\in\Phi\left(Q,D'\right)}R_\varphi\left(Q,D'\right)\quad\text{and}\quad R\left(D'\right)=\bigcup_{Q'\in\Omega\left(Q\right)}R\left(Q',D'\right)$$

where

$$\Phi\left(Q,D'\right)=\left\{\varphi\mid\varphi:Q\to D'\text{ is a complete }\alpha\text{-isomorhism }\exists\alpha\in B_X\left(D\right)\right\}$$

$$\Omega\left(Q\right)=\left\{Q'\middle|Q'\text{ is }XI\text{-subsemilattices of }D\text{ which is complete isomorphic to }Q\right\}$$

This structure was comprehensively investigated in Diasamidze [1].

Lemma 1. [1] *If Q is complete X-semilattice of unions and $I\left(Q\right)$ is the set all right units of the semigroup $B_X\left(Q\right)$ then $I\left(Q\right)=R_{id_Q}\left(Q,Q\right)$.*

Lemma 2. [2] *Let X be a finite set, D be a complete X-semilattice of unions and $Q=\left\{T_1,T_2,T_3,T_4,T_5,T_6,T_7,T_8\right\}$ be X-subsemilattice of unions of D satisfies the following conditions*

$$\begin{array}{ll}T_1\subset T_2\subset T_3\subset T_5\subset T_6\subset T_8, & T_1\subset T_2\subset T_3\subset T_5\subset T_7\subset T_8,\\ T_1\subset T_2\subset T_4\subset T_5\subset T_6\subset T_8, & T_1\subset T_2\subset T_4\subset T_5\subset T_7\subset T_8,\\ T_4\setminus T_3\neq\varnothing,T_3\setminus T_4\neq\varnothing, & T_6\setminus T_7\neq\varnothing,\ \ T_7\setminus T_6\neq\varnothing,\\ T_3\cup T_4=T_{5,}T_6\cup T_7=T_8 & T_1\neq\varnothing.\end{array}$$

Q is XI-semilattice of unions.

Theorem 1. [2] *Let X be a finite set and Q be XI-semilattice. If $D'=\left\{\overline{T}_1,\overline{T}_2,\overline{T}_3,\overline{T}_4,\overline{T}_5,\overline{T}_6,\overline{T}_7,\overline{T}_8\right\}$ is α-isomorphic to Q and $\Omega\left(Q\right)=m_0$, then*

$$\left|R\left(D'\right)\right|=m_0\cdot4\cdot\left(2^{\left|\left(\overline{T}_3\cap\overline{T}_4\right)\setminus\overline{T}_1\right|}\left(2^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-1\right)\right)\cdot\left(3^{\left|\overline{T}_4\setminus\overline{T}_3\right|}-2^{\left|\overline{T}_4\setminus\overline{T}_3\right|}\right)\cdot\left(3^{\left|\overline{T}_3\setminus\overline{T}_4\right|}-2^{\left|\overline{T}_3\setminus\overline{T}_4\right|}\right)\cdot5^{\left|\left(\overline{T}_7\cap\overline{T}_6\right)\setminus\overline{T}_5\right|}$$
$$\cdot\left(6^{\left|\overline{T}_7\setminus\overline{T}_6\right|}-5^{\left|\overline{T}_7\setminus\overline{T}_6\right|}\right)\cdot\left(6^{\left|\overline{T}_6\setminus\overline{T}_7\right|}-5^{\left|\overline{T}_6\setminus\overline{T}_7\right|}\right)\cdot8^{\left|X\setminus\overline{T}_8\right|}.$$

Theorem 2. [2] *Let $\alpha\in B_X\left(Q\right)$ be a quasinormal representation of the form $\alpha=\bigcup_{i=1}^{8}\left(Y_i^\alpha\times T_i\right)$ such that $V\left(D,\alpha\right)=Q$. $\alpha\in B_X\left(D\right)$ is a regular iff for some complete α-isomorphism $\varphi:Q\to D'\subseteq D$, the following conditions are satisfied*:

$$\begin{array}{l}Y_1^\alpha\supseteq\varphi\left(T_1\right),\quad Y_1^\alpha\cup Y_2^\alpha\supseteq\varphi\left(T_2\right),\quad Y_1^\alpha\cup Y_2^\alpha\cup Y_3^\alpha\supseteq\varphi\left(T_3\right),\\ Y_1^\alpha\cup Y_2^\alpha\cup Y_4^\alpha\supseteq\varphi\left(T_4\right),\quad Y_1^\alpha\cup Y_2^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\cup Y_5^\alpha\cup Y_6^\alpha\supseteq\varphi\left(T_6\right),\\ Y_1^\alpha\cup Y_2^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\cup Y_5^\alpha\cup Y_7^\alpha\supseteq\varphi\left(T_7\right),\quad Y_2^\alpha\cap\varphi\left(T_2\right)\neq\varnothing,\\ Y_3^\alpha\cap\varphi\left(T_3\right)\neq\varnothing,\quad Y_4^\alpha\cap\varphi\left(T_4\right)\neq\varnothing,\quad Y_6^\alpha\cap\varphi\left(T_6\right)\neq\varnothing,\quad Y_7^\alpha\cap\varphi\left(T_7\right)\neq\varnothing.\end{array}$$

Let X be a finite set and $D=\{T_1,T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$ be a complete X-semilattice of unions which satisfies the following conditions

$$T_1 \subset T_3 \subset T_5 \subset T_6 \subset T_8 \subset T_9,$$
$$T_1 \subset T_3 \subset T_5 \subset T_6 \subset T_7 \subset T_9,$$
$$T_1 \subset T_3 \subset T_4 \subset T_6 \subset T_8 \subset T_9,$$
$$T_1 \subset T_3 \subset T_4 \subset T_6 \subset T_7 \subset T_9,$$
$$T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_8 \subset T_9,$$
$$T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_7 \subset T_9,$$
$$T_2 \subset T_3 \subset T_4 \subset T_6 \subset T_8 \subset T_9,$$
$$T_2 \subset T_3 \subset T_4 \subset T_6 \subset T_7 \subset T_9,$$
$$T_1 \setminus T_2 \neq \varnothing,\ T_2 \setminus T_1 \neq \varnothing,\ T_4 \setminus T_5 \neq \varnothing,$$
$$T_5 \setminus T_4 \neq \varnothing,\ T_7 \setminus T_8 \neq \varnothing,\ T_8 \setminus T_7 \neq \varnothing,$$
$$T_1 \cup T_2 = T_3,\ T_4 \cup T_5 = T_6,$$
$$T_7 \cup T_8 = T_9,\ T_1 \cap T_2 \neq \varnothing$$

The diagram of the D is shown in **Figure 1**. By the symbol $\sum_3(X,9)$ we denote the class of all complete X-semilattice of unions whose every element is isomophic to an X-semilattice of the form D.

All subsemilattice of $D=\{T_1,T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$ are given in **Figure 2**.

In Diasamidze [1], it has shown that subsemilattices 1 - 15 are XI-semilattice of unions and subsemilattices 17 - 24 are not XI-semilattice of unions. In Yeşil Sungur [3] and Albayrak [4], they have shown that subsemilattices 25 and 26 are XI-semilattice of unions if and only if $T_1 \cap T_2 = \varnothing$". Also they found that number of right unit, idempotent and regular elements in subsemilattices.

In this paper, we take in particular, $Q_{16}=\{T,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$ subsemilattice of D. We will calculate the number of right unit, idempotent and regular elements α of $B_X(Q_{16})$ satisfied that $V(D,\alpha)=Q_{16}$ for a finite set X. Also we will give a formula for calculate idempotent and regular elements of $B_X(D)$ defined by an X-semilattice of unions $D=\{T_1,T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$.

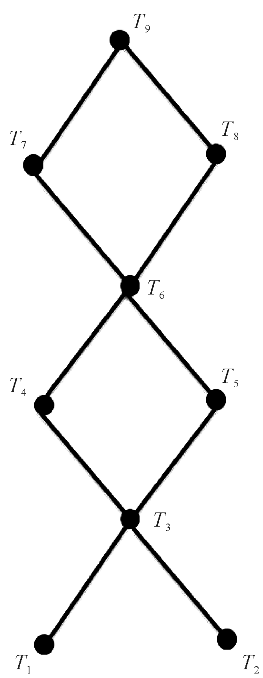

Figure 1. Diagram of D.

2. Results

Let $Q_{16}=\{T,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$ be complete X-subsemilattice of D satisfies the following conditions

$$T\subset T_3\subset T_4\subset T_6\subset T_7\subset T_9,$$
$$T\subset T_3\subset T_5\subset T_6\subset T_7\subset T_9,$$
$$T\subset T_3\subset T_4\subset T_6\subset T_8\subset T_9,$$
$$T\subset T_3\subset T_5\subset T_6\subset T_8\subset T_9,$$
$$T_4\setminus T_5\neq\varnothing,\quad T_5\setminus T_4\neq\varnothing,$$
$$T_7\setminus T_8\neq\varnothing,\quad T_8\setminus T_7\neq\varnothing,$$
$$T_4\cup T_5=T_6,\quad T_8\cup T_7=T_9$$
$$T\neq\varnothing.$$

The diagram of the Q_{16} is shown in **Figure 3**. From Lemma 2 Q_{16} is XI-semilattice of unions.

Let $Q_{16}\vartheta_{XI}$ denote the set of all XI-subsemilattice of the semilattice D which are isomorphic of the X-semilattice Q_{16}. Then we get

$$Q_{16}\vartheta_{XI}=\{\{T_1,T_3,T_4,T_5,T_6,T_7,T_8,T_9\},\{T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}\}$$

Let $\alpha\in B_X(Q_{16})$ be a idempotent element having a quasinormal representation of the form

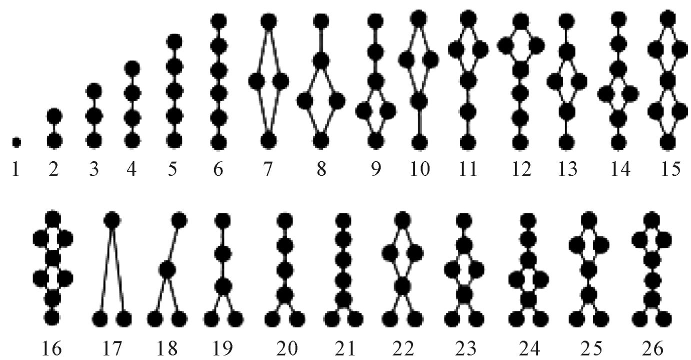

Figure 2. All subsemilattice of D.

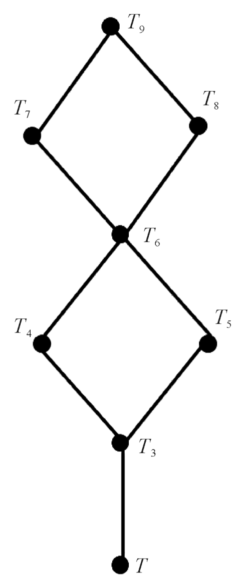

Figure 3. The diagram of the Q_{16}.

$\alpha=\left(Y_T^\alpha \times T\right) \cup \bigcup_{i=3}^{9}\left(Y_i^\alpha \times T_i\right)$, such that $V(D,\alpha)=Q_{16}$. First we calculate number of this idempotent elements in $B_X(Q_{16})$.

Lemma 3. *If X is a finite set and* $I(Q_{16})$ *is the set all right units of the semigroup* $B_X(Q_{16})$, *then the number* $|I(Q_{16})|$ *may be calculated by formula*:

$$\begin{aligned}|I(Q_{16})|&=\left((2^{|T_3\setminus T|}-1)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)\\&\cdot 5^{|(T_7\cap T_8)\setminus T_6|}\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}.\end{aligned}$$

Proof. From Lemma 1 we have $I(Q_{16})=R_{id_{Q_{16}}}(Q_{16},Q_{16})$ where $id_{Q_{16}}$ is identity mapping of the set Q_{16}. For this reason $D'=Q$ in Theorem 1. Then we obtain

$$\begin{aligned}|I(Q_{16})|&=\left(\left(2^{|T_3\setminus T|}-1\right)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)\\&\cdot 5^{|(T_7\cap T_8)\setminus T_6|}\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}.\end{aligned}$$ □

Theorem 3. *If X is a finite set and* $I^*(Q_{16})$ *is the set all idempotent elements of the semigroup* $B_X(Q_{16})$, *then the number* $|I^*(Q_{16})|$ *may be calculated by formula*:

$$\begin{aligned}|I^*(Q_{16})|&=\left(\left(2^{|T_3\setminus T_2|}-1\right)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)\\&\cdot 5^{|(T_7\cap T_8)\setminus T_6|}\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}\\&+\left(\left(2^{|T_3\setminus T_1|}-1\right)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)\\&\cdot 5^{|(T_7\cap T_8)\setminus T_6|}\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}.\end{aligned}$$

Proof. By using Lemma 3 we have number of right units of the semigroup $B_X(Q_{16})$ defined by $Q_{16}=\{T,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$ for $T\in\{T_1,T_2\}$. Then number of idempotent elements of $I^*(Q_{16})$ calculated by formula $I^*(Q_{16})=\sum_{D'\in Q_{16}\vartheta_{XI}}|I(D')|$. By using

$$Q_{16}\vartheta_{XI}=\{\{T_1,T_3,T_4,T_5,T_6,T_7,T_8,T_9\},\{T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}\}$$

we obtain above formula. □

Now we will calculate number of regular elements $\alpha\in B_X(Q_{16})$ having a quasinormal representation of the form $\alpha=\left(Y_T^\alpha\times T\right)\cup\bigcup_{i=3}^{9}\left(Y_i^\alpha\times T_i\right)$ such that $V(D,\alpha)=Q_{16}$. Let $R^*(Q_{16})$ be the set all regular elements of the semigroup $B_X(Q_{16})$. By using $Q_{16}\vartheta_{XI}=\{\{T_1,T_3,T_4,T_5,T_6,T_7,T_8,T_9\},\{T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}\}$ we get $|\Omega(Q_{16})|=2$. The number of all automorphisms of the semilattice Q_{16} is $q=4$. These are

$$I_Q=\begin{pmatrix}T & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9\\ T & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9\end{pmatrix}\quad \varphi=\begin{pmatrix}T & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9\\ T & T_3 & T_5 & T_4 & T_6 & T_7 & T_8 & T_9\end{pmatrix}$$

$$\theta=\begin{pmatrix}T & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9\\ T & T_3 & T_4 & T_5 & T_6 & T_8 & T_7 & T_9\end{pmatrix}\quad \tau=\begin{pmatrix}T & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9\\ T & T_3 & T_5 & T_4 & T_6 & T_8 & T_7 & T_9\end{pmatrix}$$

Then $|\Phi(Q_{16})|=4$. Also by using

$$\begin{aligned}&D_1'=\{T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}, && D_2'=\{T_2,T_3,T_5,T_4,T_6,T_7,T_8,T_9\}\\&D_3'=\{T_2,T_3,T_4,T_5,T_6,T_8,T_7,T_9\}, && D_4'=\{T_2,T_3,T_5,T_4,T_6,T_8,T_7,T_9\}\\&D_5'=\{T_1,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}, && D_6'=\{T_1,T_3,T_5,T_4,T_6,T_7,T_8,T_9\}\\&D_7'=\{T_1,T_3,T_4,T_5,T_6,T_8,T_7,T_9\}, && D_8'=\{T_1,T_3,T_5,T_4,T_6,T_8,T_7,T_9\}\end{aligned}$$

we get $R^*(Q_{16}) = \bigcup_{i=1}^{8} R(D_i)$.

Theorem 4. *If X is a finite set and* $R^*(Q_{16})$ *is the set all regular elements of the semigroup* $B_X(Q_{16})$, *then the number* $|R^*(Q_{16})|$ *may be calculated by formula*:

$$\begin{aligned}|R^*(Q_{16})| &= 4\cdot 2\left(\left(2^{|T_3\setminus T_2|}-1\right)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot 5^{|(T_7\cap T_8)\setminus T_6|}\\ &\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}\\ &+4\cdot 2\cdot\left(\left(2^{|T_3\setminus T_1|}-1\right)\cdot 2^{|(T_5\cap T_4)\setminus T_3|}\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot 5^{|(T_7\cap T_8)\setminus T_6|}\\ &\cdot\left(3^{|T_4\setminus T_5|}-2^{|T_4\setminus T_5|}\right)\cdot\left(6^{|T_8\setminus T_7|}-5^{|T_8\setminus T_7|}\right)\cdot\left(6^{|T_7\setminus T_8|}-5^{|T_7\setminus T_8|}\right)\cdot 8^{|X\setminus T_9|}.\end{aligned}$$

Proof. To account for the elements that are in $R^*(Q_{16})$, we first subtract out intersection of $R(D_i')$'s. Let $\alpha\in R(D_1')\cap R(D_2')$. By using Theorem 2 and $Q_{16}=\{T,T_3,T_4,T_5,T_6,T_7,T_8,T_9\}$

$$\begin{aligned}\alpha\in R(D_1')\cap R(D_2') &\Rightarrow \alpha\in R(D_1') \text{ and } \alpha\in R(D_2')\\ &\Rightarrow Y_T^\alpha\supseteq T_2,\ Y_T^\alpha\cup Y_3^\alpha\supseteq T_3,\ Y_T^\alpha\cup Y_3^\alpha\cup Y_5^\alpha\supseteq T_5\\ &\quad Y_T^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\supseteq T_4,\ Y_T^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\cup Y_5^\alpha\cup Y_6^\alpha\cup Y_8^\alpha\supseteq T_8,\\ &\quad Y_T^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\cup Y_5^\alpha\cup Y_6^\alpha\cup Y_7^\alpha\supseteq T_7,\ Y_3^\alpha\cap T_3\neq\varnothing,\\ &\quad Y_4^\alpha\cap T_4\neq\varnothing,\ Y_5^\alpha\cap T_5\neq\varnothing,\ Y_7^\alpha\cap T_7\neq\varnothing,\ Y_8^\alpha\cap T_8\neq\varnothing,\\ &\quad Y_T^\alpha\supseteq T_2,\ Y_T^\alpha\cup Y_3^\alpha\supseteq T_3,\ Y_T^\alpha\cup Y_3^\alpha\cup Y_5^\alpha\supseteq T_4,\\ &\quad Y_T^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\supseteq T_5,\ Y_T^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\cup Y_5^\alpha\cup Y_6^\alpha\cup Y_8^\alpha\supseteq T_8,\\ &\quad Y_T^\alpha\cup Y_3^\alpha\cup Y_4^\alpha\cup Y_5^\alpha\cup Y_6^\alpha\cup Y_7^\alpha\supseteq T_7,\ Y_3^\alpha\cap T_3\neq\varnothing,\\ &\quad Y_5^\alpha\cap T_4\neq\varnothing,\ Y_4^\alpha\cap T_5\neq\varnothing,\ Y_7^\alpha\cap T_7\neq\varnothing,\ Y_8^\alpha\cap T_8\neq\varnothing.\end{aligned}$$

We get $\varnothing\neq Y_4^\alpha\cap T_4\subseteq Y_4^\alpha\cap\left(Y_T^\alpha\cup Y_3^\alpha\cup Y_5^\alpha\right)$ which is a contradiction with Y_4^α, Y_T^α, Y_3^α, Y_5^α are disjoint sets. Then $R(D_1')\cap R(D_2')=\varnothing$. Smilarly $R(D_i')\cap R(D_j')=\varnothing$ for $i,j=1,\cdots,6$. Thus we obtain

$$|R^*(Q_{16})| = |R(D_1')|+|R(D_2')|+|R(D_3')|+|R(D_4')|+|R(D_5')|+|R(D_6')|+|R(D_7')|+|R(D_8')|$$

From Theorem 1 we get above formula. □

Corollary 1. *If X is a finite set,* I_D *is the set all idempotent elements of the semigroup* $B_X(D)$ *and* R_D *is the set all regular elements of the semigroup* $B_X(D)$, *then the number* $|I_D|$ *and* $|R_D|$ *may be calculated by formula*:

$$|(I_D)| = \sum_{i=1}^{16}|I^*(Q_i)|,\quad |(R_D)| = \sum_{i=1}^{16}|R^*(Q_i)|$$

Proof. Let I_D be the set of all idempotent elements of the semigroup $B_X(D)$. Then number of idempotent element of $B_X(D)$ is equal to sum of idempotent elements of the subsemigroup defined by *XI*-subsemilattice of *D*. $|I^*(Q_i)|$ is given in Diasamidze [1] for $(i=1,2,\cdots,15)$. From Theorem 3 we have number of idempotent elements of the subsemigroup $B_X(Q_{16})$. Then the number $|I_D|$ may be calculated by formula $|(I_D)| = \sum_{i=1}^{16}|I^*(Q_i)|$. Similarly the number $|R_D|$ may be calculated by formula $|(R_D)| = \sum_{i=1}^{16}|R^*(Q_i)|$. □

References

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