Applied Mathematics, 2015, 6, 265-273
Published Online February 2015 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2015.62025

Energy Identities of ADI-FDTD Method with Periodic Structure

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Received 10 January 2015; accepted 28 January 2015; published 4 February 2015

Abstract

In this paper, a new kind of energy identities for the Maxwell equations with periodic boundary conditions is proposed and then proved rigorously by the energy methods. By these identities, several modified energy identities of the ADI-FDTD scheme for the two dimensional (2D) Maxwell equations with the periodic boundary conditions are derived. Also by these identities it is proved that 2D-ADI-FDTD is approximately energy conserved and unconditionally stable in the discrete L^2 and H^1 norms. Experiments are provided and the numerical results confirm the theoretical analysis on stability and energy conservation.

Keywords

Stability, Energy Conservation, ADI-FDTD, Maxwell Equations

1. Introduction

The alternative direction implicit finite difference time domain (ADI-FDTD) methods, proposed in [1] [2], are interesting and efficient methods for numerical solutions of Maxwell equations in time domain, and cause many researchers' work since ADI-FDTD overcomes the stability constraint of the FDTD scheme [3]. For example, it was proved by Fourier methods in [4]-[8] that the ADI-FDTD methods are unconditionally stable and have reasonable numerical dispersion error; Reference [9] studied the divergence property; Reference [10] studied ADI-FDTD in a perfectly matched medium; Reference [11] gave an efficient PML implementation for the ADI-FDTD method. By Poynting's theorem, Energy conservation is an important property for Maxwell equations and good numerical method should conform it. In 2012, Gao [12] proposed several new energy identities of the two dimensional (2D) Maxwell equations with the perfectly electric conducting (PEC) boundary condi-

How to cite this paper: Shi, R.G. and Yang, H.T. (2015) Energy Identities of ADI-FDTD Method with Periodic Structure. *Applied Mathematics*, **6**, 265-273. http://dx.doi.org/10.4236/am.2015.62025

tions and proved that ADI-FDTD is approximately energy conserved and unconditionally in the discrete L^2 and H^1 norms. Is there any other structure which can keep energy conservation for Maxwell equations? Is there any other energy identity for ADI-FDTD method? This two interesting questions promote us to find other energy-conservation structure.

In this paper, we focus our attention on structure with periodic boundary conditions and propose energy identities in L^2 and H^1 norms of the 2D Maxwell equations with periodic boundary conditions. We derive the energy identities of ADI-FDTD for the 2D Maxwell equations (2D-ADI-FDTD) with periodic boundary conditions by a new energy method. Several modified energy identities of 2D-ADI-FDTD in terms of the discrete L^2 and H^1 norms are presented. By these identities it is proved that 2D-ADI-FDTD with the periodic boundary conditions is unconditionally stable and approximately energy conserved under the discrete L^2 and H^1 norms. To test the analysis, experiments to solve a simple problem with exact solution are provided. Computational results of the energy and error in terms of the discrete L^2 and H^1 norms confirm the analysis on the energy conservation and the unconditional stability.

The remaining parts of the paper are organized as follows. In Section 2, energy identities of the 2D Maxwell equations with periodic conditions in L^2 and H^1 norms are first derived. In Section 3, several modified energy identities of the 2D-ADI-FDTD method are derived, the unconditional stability and the approximate energy conservation in the discrete L^2 and H^1 norms are then proved. In Section 4, the numerical experiments are presented.

2. Energy Conservation of Maxwell Equations and 2D-ADI-FDTD

Consider the two-dimensional (2D) Maxwell equations:

$$\varepsilon\frac{\partial E_x}{\partial t}=\frac{\partial H_z}{\partial y},\ \varepsilon\frac{\partial E_y}{\partial t}=-\frac{\partial H_z}{\partial x}\ \text{and}\ \mu\frac{\partial H_z}{\partial t}=\frac{\partial E_x}{\partial x}-\frac{\partial E_y}{\partial y} \tag{2.1}$$

in a rectangular domain with electric permittivity ε and magnetic permeability μ, where ε and μ are positive constants; $\boldsymbol{E}=\left(E_x\left(x,y,t\right),E_y\left(x,y,t\right)\right)$ and $H_z=H_z\left(x,y,t\right)$ denote the electric and magnetic fields, $t\in\left(0,T\right]$, $\left(x,y\right)\in\Omega=\left[0,a\right]\times\left[0,b\right]$.

We assume that the rectangular region Ω is surrounded by periodic boundaries, so the boundary conditions can be written as

$$E_x\left(0,y,t\right)=E_x\left(a,y,t\right),\ E_x\left(x,0,t\right)=E_x\left(x,b,t\right),\ E_y\left(0,y,t\right)=E_y\left(a,y,t\right), \tag{2.2}$$

$$E_y\left(x,0,t\right)=E_y\left(x,b,t\right),\ H_z\left(0,y,t\right)=H_z\left(a,y,t\right),\ H_z\left(x,0,t\right)=H_z\left(x,b,t\right). \tag{2.3}$$

We also assume the initial conditions

$$\boldsymbol{E}\left(x,y,0\right)=\boldsymbol{E}_0\left(x,y\right)=\left(E_{x0}\left(x,y\right),E_{y0}\left(x,y\right)\right),\ H_z\left(x,y,0\right)=H_{z0}\left(x,y\right). \tag{2.4}$$

It can be derived by integration by parts and the periodic boundary conditions (2.2)-(2.3) that the above Maxwell equations have the energy identities:

Lemma 2.1 Let $\boldsymbol{E}\left(t\right)=\left(E_x\left(x,y,t\right),E_y\left(x,y,t\right)\right)$ and $H_z\left(t\right)=H_z\left(x,y,t\right)$ be the solution of the Maxwell-systems (2.1)-(2.4). Then

$$\left\|\boldsymbol{E}\left(t\right)\right\|^2+\left\|H_z\left(t\right)\right\|^2=\left\|\boldsymbol{E}\left(0\right)\right\|^2+\left\|H_z\left(0\right)\right\|^2, \tag{2.5}$$

where and in what follows, $\left\|\bullet\right\|$ denotes the L^2 norm with the weights ε (corresponding electric field) or μ (magnetic field). For example,

$$\left\|\boldsymbol{E}\left(t\right)\right\|^2=\left\|E_x\left(t\right)\right\|^2+\left\|E_y\left(t\right)\right\|^2,\ \left\|E_x\left(t\right)\right\|^2=\int_0^a\int_0^b\varepsilon E_x^2\left(x,y,t\right)\mathrm{d}y\mathrm{d}x. \tag{2.6}$$

Identity (2.5) is called the Poynting Theorem and can be seen in many classical physics books. Besides the above energy identities, we found new ones below.

Theorem 2.2 Let $\boldsymbol{E}\left(t\right)$ and $H_z\left(t\right)$ be the solution of the Maxwell systems (2.1)-(2.4), the same as those in Lemma 2.1. Then, the following energy identities hold

$$\left\|\frac{\partial \boldsymbol{E}(t)}{\partial u}\right\|^2+\left\|\frac{\partial H_z(t)}{\partial u}\right\|^2=\left\|\frac{\partial \boldsymbol{E}(0)}{\partial u}\right\|^2+\left\|\frac{\partial H_z(0)}{\partial u}\right\|^2, \tag{2.7}$$

$$\left\|\boldsymbol{E}(t)\right\|_1^2+\left\|H(t)\right\|_1^2=\left\|\boldsymbol{E}(0)\right\|_1^2+\left\|H_z(0)\right\|_1^2, \tag{2.8}$$

where $u = x$ or y, and $\|\bullet\|_1$ is the H^1 norm (the H^1 norm of f is defined by $\|f\|_1^2=\|f\|^2+|f|_{1/2}^2$, where $|f|_{1/2}^2=\|f_x\|^2+\|f_y\|^2$, $f_x=\partial f/\partial x$, $f_y=\partial f/\partial y$. $|f|_{1/2}$ is also called the H^1-semi norm of f).

Proof. First, we prove Equation (2.7) with $u = x$. Differentiating each of the Equations in (2.1) with respect to x leads to

$$\varepsilon\frac{\partial^2 E_x}{\partial x\partial t}=\frac{\partial^2 H_z}{\partial x\partial y},\quad \varepsilon\frac{\partial^2 E_y}{\partial x\partial t}=-\frac{\partial^2 H_z}{\partial x\partial x}\ \text{ and }\ \mu\frac{\partial^2 H_z}{\partial x\partial t}=\frac{\partial^2 E_x}{\partial x\partial y}-\frac{\partial^2 E_y}{\partial x\partial x}. \tag{2.9}$$

By the integration by parts and the periodic boundary conditions (2.2)-(2.3), we have

$$\begin{aligned}&\int_0^a\int_0^b\frac{\partial^2 E_x}{\partial x\partial y}\cdot\frac{\partial H_z}{\partial x}\mathrm{d}x\mathrm{d}y=-\int_0^a\int_0^b\frac{\partial E_x}{\partial x}\cdot\frac{\partial^2 H_z}{\partial x\partial y}\mathrm{d}x\mathrm{d}y,\\&\int_0^a\int_0^b\frac{\partial^2 E_y}{\partial x^2}\cdot\frac{\partial H_z}{\partial x}\mathrm{d}x\mathrm{d}y=r(t)-\int_0^a\int_0^b\frac{\partial E_y}{\partial x}\cdot\frac{\partial^2 H_z}{\partial x^2}\mathrm{d}x\mathrm{d}y\end{aligned} \tag{2.10}$$

where

$$r(t)=\int_0^b\left(\frac{\partial E_y}{\partial x}(a,y,t)\frac{\partial H_z}{\partial x}(a,y,t)-\frac{\partial E_y}{\partial x}(0,y,t)\frac{\partial H_z}{\partial x}(0,y,t)\right)\mathrm{d}y. \tag{2.11}$$

Multiplying the Equations (2.9) by $\varepsilon\partial E_x/\partial x$, $\varepsilon\partial E_y/\partial x$ and $\mu\partial H_z/\partial x$ respectively, integrating both sides over $[0,a]\times[0,b]$ and using (2.10), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{\partial E_x}{\partial x}\right\|^2+\left\|\frac{\partial E_y}{\partial x}\right\|^2+\left\|\frac{\partial H_z}{\partial x}\right\|^2\right)=-r(t). \tag{2.12}$$

From (2.1) and the boundary conditions (2.2)-(2.3) we note that

$$\begin{aligned}\frac{\partial E_y}{\partial x}\frac{\partial H_z}{\partial x}(0,y,t)&=\lim_{x\to 0}\frac{\partial E_y}{\partial x}\frac{\partial H_z}{\partial x}(x,y,t)=-\lim_{x\to 0}\varepsilon\left(\frac{\partial E_x}{\partial y}-\mu\frac{\partial H_z}{\partial t}\right)\frac{\partial E_y}{\partial t}(x,y,t)\\&=-\lim_{x\to a}\varepsilon\left(\frac{\partial E_x}{\partial y}-\mu\frac{\partial H_z}{\partial t}\right)\frac{\partial E_y}{\partial t}(x,y,t)=\lim_{x\to a}\frac{\partial E_y}{\partial x}(x,y,t)\frac{\partial H_z}{\partial x}(x,y,t)\\&=\frac{\partial E_y}{\partial x}(a,y,t)\frac{\partial H_z}{\partial x}(a,y,t).\end{aligned} \tag{2.13}$$

So, $r(t)=0$. Then, by integrating (2.12) with respect to time over $[0,T]$, we get equation (2.7) with $u = x$. Similarly, the identity (2.7) with $u = y$ can be proved. Combining (2.5) and (2.7) leads to (2.8). □

The 2D-ADI-FDTD Scheme

The alternating direction implicit FDTD method for the 2D Maxwell equations (denoted by 2D-ADI-FDTD) was proposed by (Namiki, 1999). For convenience in analysis of this scheme, next we give some notations. Let

$$\begin{aligned}&x_i=i\Delta x, x_{i+\frac{1}{2}}=x_i+\frac{\Delta x}{2}, y_j=j\Delta y, y_{j+\frac{1}{2}}=y_j+\frac{\Delta y}{2},\ t^n=n\Delta t, t^{n+\frac{1}{2}}=t^n+\frac{\Delta t}{2},\\&i=0,1,\cdots,I-1,\ x_I=a,\ j=0,1,\cdots,J-1,\ y_J=b,\ n=0,1,\cdots,N-1,\ N\Delta t=T.\end{aligned}$$

where Δx and Δy are the mesh sizes along x and y directions, Δt is the time step, I, J and N are positive integers. For a grid function $f_{\alpha,\beta}^m=f\left(x_\alpha,y_\beta,t^m\right)$, define

$$\delta_x f^m_{\alpha,\beta} = \frac{f^m_{\alpha+\frac{1}{2},\beta} - f^m_{\alpha-\frac{1}{2},\beta}}{\Delta x}, \delta_y f^m_{\alpha,\beta} = \frac{f^m_{\alpha,\beta+\frac{1}{2}} - f^m_{\alpha,\beta-\frac{1}{2}}}{\Delta y},$$

$$\delta_t f^m_{\alpha,\beta} = \frac{f^{m+\frac{1}{2}}_{\alpha,\beta} - f^{m-\frac{1}{2}}_{\alpha,\beta}}{\Delta t}, \delta_u \delta_v f^m_{\alpha,\beta} = \delta_u \left(\delta_v f^m_{\alpha,\beta}\right), \Delta u^2 = \left(\Delta u\right)^2,$$

where $u = x, y$ or t. For $\boldsymbol{V} = \left(V_{x_{i+1/2,j}}, V_{y_{i,j+1/2}}\right)$, $W_{i+1/2,j+1/2}$, $\Delta v = \Delta x \Delta y$, define some discrete energy norms based on the Yee staggered grids (Yee, 1966),

$$\|V_x\|^2_{E_x} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} \varepsilon \left(V_{x_{i+\frac{1}{2},j}}\right)^2 \Delta v, \|V_y\|^2_{E_y} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} \varepsilon \left(V_{y_{i,j+\frac{1}{2}}}\right)^2 \Delta v, \|W\|^2_{H_z} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} \mu \left(W_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2 \Delta v,$$

$$\|\delta_x \boldsymbol{V}\|^2_{\delta_x \boldsymbol{E}} = \|\delta_x V_x\|^2_{\delta_x E_x} + \|\delta_x V_y\|^2_{\delta_x E_y}, \|\delta_x V_x\|^2_{\delta_x E_x} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} \varepsilon \left(\delta_x V_{x_{i,j}}\right)^2 \Delta v, \|\delta_x V_y\|^2_{\delta_x E_y} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} \varepsilon \left(\delta_x V_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2 \Delta v.$$

Other norms: $\|\delta_y V_x\|_{\delta_y E_x}$, $\|\delta_y V_y\|_{\delta_y E_y}$ and $\|\delta_y H_z\|_{\delta_y H_z}$ are similarly defined. Denote by $E^m_{u_{\alpha,\beta}}$ and $H^m_{z_{\alpha,\beta}}$ the approximations of $E_u\left(x_\alpha, y_\beta, t^m\right)$ ($u = x, y$) and $H_z\left(x_\alpha, y_\beta, t^m\right)$, respectively. Then the 2D-ADI-FDTD scheme for (2.1) is written as

Stage 1:

$$\frac{E^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}} - E^n_{x_{i+\frac{1}{2},j}}}{\Delta t/2} = \frac{\delta_y H^n_{z_{i+\frac{1}{2},j}}}{\varepsilon} \tag{2.14}$$

$$\frac{E^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}} - E^n_{y_{i,j+\frac{1}{2}}}}{\Delta t/2} = -\frac{\delta_x H^{n+\frac{1}{2}}_{z_{i,j+\frac{1}{2}}}}{\varepsilon} \tag{2.15}$$

$$\frac{H^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t/2} = \frac{1}{\mu}\left(\delta_y E^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}} - \delta_x E^{n+\frac{1}{2}}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\right) \tag{2.16}$$

Stage 2:

$$\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}}}{\Delta t/2} = \frac{\delta_y H^{n+1}_{z_{i+\frac{1}{2},j}}}{\varepsilon} \tag{2.17}$$

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}}}{\Delta t/2} = -\frac{\delta_x H^{n+\frac{1}{2}}_{z_{i,j+\frac{1}{2}}}}{\varepsilon} \tag{2.18}$$

$$\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t/2} = \frac{1}{\mu}\left(\delta_y E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} - \delta_x E^{n+\frac{1}{2}}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\right) \tag{2.19}$$

For simplicity in notations, we sometimes omit the subscripts of these field values without causing any ambiguity. By the definition of cross product of vectors, the boundary conditions for (2.2)-(2.3) become

$$\begin{cases} E^m_{x_{i+\frac{1}{2},0}} = E^m_{x_{i+\frac{1}{2},J}}, E^m_{x_{\frac{1}{2},j}} = E^m_{x_{I+\frac{1}{2},j}}, E^m_{x_{-\frac{1}{2},j}} = E^m_{x_{I-\frac{1}{2},j}}, E^m_{y_{0,j+\frac{1}{2}}} = E^m_{y_{I,j+\frac{1}{2}}}, E^m_{y_{i,\frac{1}{2}}} = E^m_{y_{i,J+\frac{1}{2}}}, E^m_{y_{i,-\frac{1}{2}}} = E^m_{y_{i,J-\frac{1}{2}}}, \\ H^m_{z_{\frac{1}{2},j+\frac{1}{2}}} = H^m_{z_{I+\frac{1}{2},j+\frac{1}{2}}}, H^m_{z_{-\frac{1}{2},j+\frac{1}{2}}} = H^m_{z_{I-\frac{1}{2},j+\frac{1}{2}}}, H^m_{z_{i+\frac{1}{2},\frac{1}{2}}} = H^m_{z_{i+\frac{1}{2},J+\frac{1}{2}}}, H^m_{z_{i+\frac{1}{2},-\frac{1}{2}}} = H^m_{z_{i+\frac{1}{2},J-\frac{1}{2}}}, \end{cases} \tag{2.20}$$

where $m = n$ or $m = n + 1/2$. Finally, the initial values $\boldsymbol{E}^0_{\alpha,\beta}$ and $H^0_{z_{\alpha,\beta}}$ of the2D-ADI-FDTD scheme are obtained by the initial condition (2.4).

3. Modified Energy Identities and Stability of 2D-ADI-FDTD in H^1 Norm

In this Section we derive modified energy identities of 2D-ADI-FDTD and prove its energy conservation and unconditional stability in the discrete H^1 norm.

Theorem 3.1 Let $n > 0$, $\boldsymbol{E}^n = \left(E^n_{x_{i+1/2,j}}, E^n_{y_{i,j+1/2}}\right)$ and $H^n_{z_{i+1/2,j+1/2}}$ be the solution of the ADI-FDTD scheme (2.14)-(2.19). Then the following modified energy identities hold,

$$\begin{aligned} &\left\|\delta_x \boldsymbol{E}^n\right\|^2_{\delta_x \boldsymbol{E}} + \left\|\delta_x H^n_z\right\|^2_{\delta_x H_z} + \frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_x\delta_y H^n_z\right\|^2_{\delta_x E_x} + \left\|\delta_x\delta_y E^n_x\right\|^2_{\delta_x H_z}\right) \\ &= \left\|\delta_x \boldsymbol{E}^0\right\|^2_{\delta_x \boldsymbol{E}} + \left\|\delta_x H^0_z\right\|^2_{\delta_x H_z} + \frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_x\delta_y H^0_z\right\|^2_{\delta_x E_x} + \left\|\delta_x\delta_y E^0_x\right\|^2_{\delta_x H_z}\right), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\left\|\delta_y \boldsymbol{E}^n\right\|^2_{\delta_y \boldsymbol{E}} + \left\|\delta_y H^n_z\right\|^2_{\delta_y H_z} + \frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_y\delta_y H^n_z\right\|^2_{\delta_y E_x} + \left\|\delta_y\delta_y E^n_x\right\|^2_{\delta_y H_z}\right) \\ &= \left\|\delta_y \boldsymbol{E}^0\right\|^2_{\delta_y \boldsymbol{E}} + \left\|\delta_y H^0_z\right\|^2_{\delta_y H_z} + \frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_y\delta_y H^0_z\right\|^2_{\delta_y E_x} + \left\|\delta_y\delta_y E^0_x\right\|^2_{\delta_y H_z}\right), \end{aligned} \tag{3.2}$$

where for $u = x, y$, and $m = n$ or 0

$$\left\|\delta_u \boldsymbol{E}^m\right\|^2_{\delta_u \boldsymbol{E}} = \left\|\delta_u E^m_x\right\|^2_{\delta_u E_x} + \left\|\delta_u E^m_y\right\|^2_{\delta_u E_y}.$$

Proof. First we prove (3.1). Applying δ_x to the Equations (2.14)-(2.19), and rearranging the terms by the time levels, we have

$$\delta_x E^{n+\frac{1}{2}}_{x_{i,j}} = \delta_x E^n_{x_{i,j}} + \frac{\Delta t}{2\varepsilon}\delta_x\delta_y H^n_{z_{i,j}}, \tag{3.3}$$

$$\delta_x E^{n+\frac{1}{2}}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + \frac{\Delta t}{2\varepsilon}\delta_x\delta_x H^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} = \delta_x E^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}}, \tag{3.4}$$

$$\delta_x H^{n+\frac{1}{2}}_{z_{i,j+\frac{1}{2}}} + \frac{\Delta t}{2\mu}\delta_x\delta_x E^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}} = \delta_x H^n_{z_{i,j+\frac{1}{2}}} + \frac{\Delta t}{2\mu}\delta_x\delta_y E^n_{x_{i,j+\frac{1}{2}}}, \tag{3.5}$$

$$\delta_x E^{n+1}_{x_{i,j}} - \frac{\Delta t}{2\varepsilon}\delta_x\delta_y H^{n+1}_{z_{i,j}} = \delta_x E^{n+\frac{1}{2}}_{x_{i,j}}, \tag{3.6}$$

$$\delta_x E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} = \delta_x E^{n+\frac{1}{2}}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} - \frac{\Delta t}{2\varepsilon}\delta_x\delta_x H^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}, \tag{3.7}$$

$$\delta_x H^{n+1}_{z_{i,j+\frac{1}{2}}} - \frac{\Delta t}{2\mu}\delta_x\delta_y E^{n+1}_{x_{i,j+\frac{1}{2}}} = \delta_x H^{n+\frac{1}{2}}_{z_{i,j+\frac{1}{2}}} - \frac{\Delta t}{2\mu}\delta_x\delta_x E^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}}. \tag{3.8}$$

Multiplying both sides of the equations, (3.3)-(3.4) by $\sqrt{\varepsilon}$ respectively, and those of (3.5) by $\sqrt{\mu}$, and taking the square of the updated equations lead to

$$\varepsilon\left(\delta_x E_{x_{i,j}}^{n+\frac{1}{2}}\right)^2=\varepsilon\left(\delta_x E_{x_{i,j}}^{n}\right)^2+\frac{\Delta t^2}{4\mu\varepsilon}\mu\left(\delta_x\delta_y H_{z_{i,j}}^{n}\right)^2+\Delta t\delta_x E_{x_{i,j}}^{n}\delta_x\delta_x H_{z_{i,j}}^{n},\tag{3.9}$$

$$\varepsilon\left(\delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}\right)^2+\frac{\Delta t^2}{4\mu\varepsilon}\mu\left(\delta_x\delta_x H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}\right)^2+\Delta t\delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}\delta_x\delta_x H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}=\varepsilon\left(\delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\right)^2,\tag{3.10}$$

$$\begin{aligned}&\mu\left(\delta_x H_{z_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}}\right)^2+\frac{\Delta t^2}{4\mu\varepsilon}\varepsilon\left(\delta_x\delta_x E_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}}\right)^2+\Delta t\delta_x H_{z_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}}\delta_x\delta_x E_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}}\\&=\mu\left(\delta_x H_{z_{i,j+\frac{1}{2}}}^{n}\right)^2+\frac{\Delta t^2}{4\mu\varepsilon}\varepsilon\left(\delta_x\delta_y E_{x_{i,j+\frac{1}{2}}}^{n}\right)^2+\Delta t\delta_x H_{z_{i,j+\frac{1}{2}}}^{n}\delta_x\delta_y E_{x_{i,j+\frac{1}{2}}}^{n},\end{aligned}\tag{3.11}$$

Applying summation by parts, we see that

$$\begin{aligned}\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\delta_x H_z^n\cdot\delta_x\delta_y E_x^n\Big|_{i,j+\frac{1}{2}}&=\frac{1}{\Delta y}\sum_{i=0}^{I-1}\delta_x E_{x_{i,0}}^{n}\left(\delta_x H_{z_{i,J-\frac{1}{2}}}^{n}-\delta_x H_{z_{i,-\frac{1}{2}}}^{n}\right)-\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\delta_x E_x^n\cdot\delta_x\delta_y H_z^n\Big|_{i,j}\\&=-\sum_{i=0}^{I-1}\delta_x E_{x_{i,0}}^{m}\delta_x\delta_y H_{z_{i,0}}^{m}-\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\delta_x E_x^m\cdot\delta_x\delta_y H_z^m\Big|_{i,j}\\&=-\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\delta_x E_x^m\cdot\delta_x\delta_y H_z^m\Big|_{i,j},\end{aligned}\tag{3.12}$$

where we have used that $\delta_x E_{x_{i,0}}^{n}=\delta_x E_{x_{i,J}}^{n}$ and that $\delta_x H_{x_{i,J-\frac{1}{2}}}^{n}=\delta_x H_{x_{i,-\frac{1}{2}}}^{n}$, which can be obtained from the periodic boundary conditions. Similarly, we get that

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\delta_x E_y^{n+\frac{1}{2}}\cdot\delta_x\delta_x H_z^{n+\frac{1}{2}}\Big|_{i+\frac{1}{2},j+\frac{1}{2}}=-\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\delta_x H_z^{n+\frac{1}{2}}\cdot\delta_x\delta_x E_y^{n+\frac{1}{2}}\Big|_{i,j+\frac{1}{2}}.\tag{3.13}$$

So, if summing each of the Equalities (3.9)-(3.11) over their subscripts, adding the updated equations, multiplying both sides by $\Delta x\Delta y$, and using the two identities, (3.12) and (3.13), together with the norms defined in Subsection 2.2, we arrive at

$$\begin{aligned}&\left\|\delta_x E_x^{n+\frac{1}{2}}\right\|_{\delta_x E_x}^2+\left\|\delta_x E_y^{n+\frac{1}{2}}\right\|_{\delta_x E_y}^2+\left\|\delta_x H_z^{n+\frac{1}{2}}\right\|_{\delta_x H_z}^2+\frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_x\delta_y H_z^{n+\frac{1}{2}}\right\|_{\delta_x E_x}^2+\left\|\delta_x\delta_y E_x^{n+\frac{1}{2}}\right\|_{\delta_x H_z}^2\right)\\&=\left\|\delta_x E_x^{n}\right\|_{\delta_x E_x}^2+\left\|\delta_x E_y^{n}\right\|_{\delta_x E_y}^2+\left\|\delta_x H_z^{n}\right\|_{\delta_x H_z}^2+\frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_x\delta_y H_z^{n}\right\|_{\delta_x E_x}^2+\left\|\delta_x\delta_y E_x^{n}\right\|_{\delta_x H_z}^2\right).\end{aligned}\tag{3.14}$$

Similar argument is applied to the second Stage (3.6)-(3.8), we have

$$\begin{aligned}&\left\|\delta_x E_x^{n+1}\right\|_{\delta_x E_x}^2+\left\|\delta_x E_y^{n+1}\right\|_{\delta_x E_y}^2+\left\|\delta_x H_z^{n+1}\right\|_{\delta_x H_z}^2+\frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_x\delta_y H_z^{n+1}\right\|_{\delta_x E_x}^2+\left\|\delta_x\delta_y E_x^{n+1}\right\|_{\delta_x H_z}^2\right)\\&=\left\|\delta_x E_x^{n+\frac{1}{2}}\right\|_{\delta_x E_x}^2+\left\|\delta_x E_y^{n+\frac{1}{2}}\right\|_{\delta_x E_y}^2+\left\|\delta_x H_z^{n+\frac{1}{2}}\right\|_{\delta_x H_z}^2+\frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_x\delta_y H_z^{n+\frac{1}{2}}\right\|_{\delta_x E_x}^2+\left\|\delta_x\delta_y E_x^{n+\frac{1}{2}}\right\|_{\delta_x H_z}^2\right).\end{aligned}\tag{3.15}$$

Combination of (3.14) and (3.15) leads to the identity (3.1). Identity (3.2) is similarly derived by repeating the above argument from the operated Equations (2.14)-(2.19) by δ_y. This completes the proof of Theorem 3.1. □

In the above proof, if taking δ_x as the identity operator, we obtain that

Theorem 3.2 Let $n>0$, $\boldsymbol{E}^n$ and H_z^n be the solution of 2D-ADI-FDTD. Then, the following energy identities hold

$$\begin{aligned}&\left\|E_x^n\right\|_{E_x}^2+\left\|E_y^n\right\|_{E_y}^2+\left\|H_z^n\right\|_{H_z}^2+\frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_y E_x^n\right\|_{H_z}^2+\left\|\delta_y H_z^n\right\|_{E_x}^2\right)\\&=\left\|E_x^0\right\|_{E_x}^2+\left\|E_y^0\right\|_{E_y}^2+\left\|H_z^0\right\|_{H_z}^2+\frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_y E_x^0\right\|_{H_z}^2+\left\|\delta_y H_z^0\right\|_{E_x}^2\right),\end{aligned}\tag{3.16}$$

Combining the results in Theorems 3.1 and 3.2 we have

Theorem 3.3 If the discrete H^1 semi-norm and H^1 norm of the solution of 2D-ADI-FDTD are denoted respectively by

$$\begin{aligned}&\left\|\boldsymbol{E}^n\right\|_{1/2}^2=\left\|\delta_x E_x^n\right\|_{\delta_x E_x}^2+\left\|\delta_x E_y^n\right\|_{\delta_x E_y}^2+\left\|\delta_y E_x^n\right\|_{\delta_y E_x}^2+\left\|\delta_y E_y^n\right\|_{\delta_y E_y}^2,\\&\left\|\boldsymbol{E}^n\right\|^2=\left\|E_x^n\right\|_{E_x}^2+\left\|E_y^n\right\|_{E_y}^2,\left\|\boldsymbol{E}^n\right\|_1^2=\left\|\boldsymbol{E}^n\right\|_{1/2}^2+\left\|\boldsymbol{E}^n\right\|_{\boldsymbol{E}}^2,\\&\left\|\delta_y E_x^n\right\|_1^2=\left\|\delta_x\delta_y E_x^n\right\|_{\delta_x H_z}^2+\left\|\delta_y\delta_y E_x^n\right\|_{\delta_y H_z}^2+\left\|\delta_y E_x^n\right\|_{H_z}^2,\\&\left\|H_z^n\right\|_1^2=\left\|\delta_x H_z^n\right\|_{\delta_x H_z}^2+\left\|\delta_y H_z^n\right\|_{\delta_y H_z}^2+\left\|H_z^n\right\|_{H_z}^2,\\&\left\|\delta_y H_z^n\right\|_1^2=\left\|\delta_x\delta_y H_z^n\right\|_{\delta_x E_x}^2+\left\|\delta_y\delta_y H_z^n\right\|_{\delta_y E_x}^2+\left\|\delta_y H_z^n\right\|_{E_x}^2,\end{aligned}$$

then, the following energy identities for 2D-ADI-FDTD hold

$$\left\|\boldsymbol{E}^n\right\|_{1/2}^2+\left\|H_z^n\right\|_{1/2}^2+\frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_y E_x^n\right\|_{1/2}^2+\left\|\delta_y H_z^n\right\|_{1/2}^2\right)=\left\|\boldsymbol{E}^0\right\|_{1/2}^2+\left\|H_z^0\right\|_{1/2}^2+\frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_y E_x^0\right\|_{1/2}^2+\left\|\delta_y H_z^0\right\|_{1/2}^2\right).\tag{3.17}$$

$$\left\|\boldsymbol{E}^n\right\|_1^2+\left\|H_z^n\right\|_1^2+\frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_y E_x^n\right\|_1^2+\left\|\delta_y H_z^n\right\|_1^2\right)=\left\|\boldsymbol{E}^0\right\|_1^2+\left\|H_z^0\right\|_1^2+\frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_y E_x^0\right\|_1^2+\left\|\delta_y H_z^0\right\|_1^2\right).\tag{3.18}$$

Remark 3.4 It is easy to see that the identities in Theorems 3.1, 3.2 and 3.3converge to those in Lemma 2.1 and Theorem 2.2 as the discrete step sizes approach zero. This means that2D-ADI-FDTD is approximately energy-conserved and unconditionally stable in the modified discrete form of the L^2 and H^1 norms**.**

4. Numerical Experiments

In this section we solve a model problem by 2D-ADI-FDTD, and then test the analysis of the stability and energy conservation in Section 3 by comparing the numerical solution with the exact solution of the model. The model considered is the Maxwell equations (2.1) with $\varepsilon=\mu=1$, $\Omega=[0,1]\times[0,1]$, $t\in(0,T]$, and its exact solution is: $E_x=E_x(t)=\cos\left(2\pi(x+y)-2\sqrt{2}\pi t\right)$, $E_y=E_y(t)=-E_x$, $H_z=H_z(t)=-\sqrt{2}E_x$.

It is easy to compute the norms of this solution are

$$\begin{aligned}&\left\|\left(E(t),H_z(t)\right)\right\|=\sqrt{\left\|\boldsymbol{E}(t)\right\|^2+\left\|H_z(t)\right\|^2}=\sqrt{2},\\&\left\|\left(E(t),H_z(t)\right)\right\|_1=\left(\left\|\boldsymbol{E}(t)\right\|_1^2+\left\|H_z(t)\right\|_1^2\right)^{1/2}=\left(2+8\pi^2\right)^{1/2}.\end{aligned}$$

4.1. Simulation of the Error and Stability

To show the accuracy of 2D-ADI-FDTD, we define the errors:

$$\mathcal{E}_{x_{i+1/2,j}}^n=E_x\left(t^n\right)_{i+1/2,j}-E_{x_{i+1/2,j}}^n,\ \mathcal{E}_{y_{i,j+1/2}}^n=E_y\left(t^n\right)_{i,j+1/2}-E_{y_{i,j+1/2}}^n\ \text{and}\ \mathcal{H}_{z_{i+1/2,j+1/2}}^n=H_z\left(t^n\right)_{i+1/2,j+1/2}-H_{z_{i+1/2,j+1/2}}^n,$$

where $E_x\left(t^n\right)$, $E_x\left(t^n\right)$, $H_z\left(t^n\right)$ are the true values of the exact solution. Denote the error and relative error in the norms defined in Section 3 by ErL_2, $R\text{-}ErL_2$, ErH_1 and $R\text{-}ErH_1$, *i.e.*

$$ErL_2=\sqrt{\left\|\mathcal{E}_x^n\right\|_{E_x}^2+\left\|\mathcal{E}_y^n\right\|_{E_y}^2+\left\|\mathcal{H}_z^n\right\|_{H_z}^2},\ R\text{-}ErL_2=\frac{ErL_2}{\left\|\left(\boldsymbol{E}(t),H_z(t)\right)\right\|}$$

$$ErH_1 = \left(\left(ErL_2\right)^2 + \left\|\delta_x \mathcal{E}_x^n\right\|_{\delta_x E_x}^2 + \left\|\delta_x \mathcal{E}_y^n\right\|_{\delta_x E_y}^2 + \left\|\delta_x \mathcal{H}_z^n\right\|_{\delta_x H_z}^2 + \left\|\delta_y \mathcal{E}_x^n\right\|_{\delta_y E_x}^2 + \left\|\delta_y \mathcal{E}_y^n\right\|_{\delta_y E_y}^2 + \left\|\delta_y \mathcal{H}_z^n\right\|_{\delta_y H_z}^2 \right)^{1/2},$$

$$R\text{-}ErH_1 = \frac{ErH_1}{\left\|\left(\boldsymbol{E}(t), H_z(t)\right)\right\|_1}, \quad \text{Rate} = \log_2 \frac{\text{Error}(h)}{\text{Error}(h/2)},$$

where log is the logarithmic function.

Table 1 gives the error and relative error of the numerical solution of the model problem computed by 2D-ADI-FDTD in the norms, and the convergence rates with different time step sizes $\Delta t = 4h$, $2h$ and h, when $\Delta x = \Delta y = h = 0.01$ is fixed and $T = 1$. From these results we see that the convergence rate of 2D-ADI-FDTD with respect to time is approximately 2 and that 2D-ADI-FDTD is unconditionally stable (when $\Delta t = \Delta x = \Delta y = h$, the CFL number $c\Delta t\sqrt{1/\Delta x^2 + 1/\Delta y^2} = \sqrt{2} > 1$).

Table 2 lists the similar results to **Table 1** when $\Delta t = 0.1h$ is fixed, $\Delta x = \Delta y$ varies from $2h$, h and $0.5h$, and the time length $T = 1$. From the columns "Rate" we see that 2D-ADI-FDTD is of second order in space under the discrete L^2 and H^1 norm.

4.2. Simulation of the Energy Conservation of 2D-ADI-FDTD

In this subsection we check the energy conservation of 2D-ADI-FDTD by computing the modified energy norms derived in Section 3 for the solution to the scheme. Denote these modified energy norms by

$$I_u\left(\boldsymbol{E}^n, H_z^n\right) = \sqrt{\left\|\delta_u \boldsymbol{E}^n\right\|_{\delta_u \boldsymbol{E}}^2 + \left\|\delta_u H_z^n\right\|_{\delta_u H_z}^2 + \frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_u \delta_y H_z^n\right\|_{\delta_u E_x}^2 + \left\|\delta_u \delta_y E_x^n\right\|_{\delta_u H_z}^2\right)},$$

$$I_0\left(\boldsymbol{E}^n, H_z^n\right) = \sqrt{\left\|\boldsymbol{E}^n\right\|_{\boldsymbol{E}}^2 + \left\|H_z^n\right\|_{H_z}^2 + \frac{\Delta t^2}{4\mu\varepsilon}\left(\left\|\delta_y E_x^n\right\|_{H_z}^2 + \left\|\delta_y H_z^n\right\|_{E_x}^2\right)},$$

$$I_1\left(\boldsymbol{E}^n, H_z^n\right) = \sqrt{\left|I_x\left(\boldsymbol{E}^n, H_z^n\right)\right|^2 + \left|I_y\left(\boldsymbol{E}^n, H_z^n\right)\right|^2 + \left|I_0\left(\boldsymbol{E}^n, H_z^n\right)\right|^2}.$$

In **Table 3** are presented the energy norms $I_u\left(\boldsymbol{E}^n, H_z^n\right)(u = x, y, 0, 1)$ of the solution of the 2D-ADI-FDTD scheme at the time levels $n = 0$, $n = 1000$ and $n = 4000$ (the third to fifth rows), and the absolute values of their difference (the last two rows), where the sizes of the spatial and time steps are $\Delta x = \Delta y = 0.01$, $\Delta t = 0.04$. The second row shows the four kind of energies of the exact solution computed by using the definitions of $I_u(u = 0, x, y)$. From these value we see that 2D-ADI-FDTD is approximately energy-conserved.

Table 1. Error of $\left(\boldsymbol{E}^n, H_z^n\right)$ in L^2 and H^1 with $\Delta x = \Delta y = h$ and different Δt.

Δt	$R\text{-}ErL_2$	ErL_2	Rate	$R\text{-}ErH_1$	ErH_1	Rate
$4h$	6.0284e−2	8.5254e−2		6.0287e−2	7.6675e−1	
$2h$	1.6264e−2	2.3001e−2	1.8901	1.6265e−2	2.0595e−1	1.8901
h	5.1571e−3	7.2932e−3	1.6571	5.1571e−3	6.5229e−2	1.3182

Table 2. Error of $\left(\boldsymbol{E}^n, H_z^n\right)$ in L^2 and H^1 with $\Delta t = 0.1h$ and different spatial step sizes.

$\Delta x = \Delta y$	$R\text{-}ErL_2$	ErL_2	Rate	$R\text{-}ErH_1$	ErH_1	Rate
$2h$	5.0019e−3	8.3182e−3		5.0019e−3	7.4333e−3	
h	1.4981e−3	2.1186e−3	1.7393	1.4981e−3	1.8942e−3	1.7393
$0.5h$	4.0200e−4	5.6851e−4	1.8979	4.0200e−4	5.0834e−4	1.8978

Table 3. Energy of $\left(\boldsymbol{E}^n, H_z^n\right)$ and its error when $\Delta x = \Delta y = h = 0.01$, $\Delta t = 4h$ and $n = 0, 1000, 4000$.

Fields\Norms	$I_x(\cdot)$	$I_y(\cdot)$	$I_0(\cdot)$	$I_1(\cdot)$
$\left(\boldsymbol{E}(t), H_z(t)\right)$	8.9367	8.9367	1.4226	12.7183
$\left(\boldsymbol{E}^0, H_z^0\right)$	8.9367	8.9367	1.4226	12.7183
$\left(\boldsymbol{E}^{1000}, H_z^{1000}\right)$	8.9367	8.9367	1.4226	12.7183
$\left(\boldsymbol{E}^{4000}, H_z^{4000}\right)$	8.9367	8.9367	1.4226	12.7183
$\left(\boldsymbol{E}^{1000}, H_z^{1000}\right)-\left(\boldsymbol{E}^0, H_z^0\right)$	3.2685e−13	3.2685e−13	5.2403e−14	4.6718e−13
$\left(\boldsymbol{E}^{4000}, H_z^{4000}\right)-\left(\boldsymbol{E}^0, H_z^0\right)$	3.2685e−13	3.2685e−13	5.2403e−14	4.6718e−13

5. Conclusion

In this paper, the modified energy identities of the 2D-ADI-FDTD scheme with the periodic boundary conditions in the discrete L^2 and H^1 norms are established which show that this scheme is approximately energy conserved in terms of the two energy norms. By the deriving methods for the energy identities, new kind of energy identities of the Maxwell equations are proposed and proved by the new energy method. Numerical experiments are provided and confirm the analysis of 2D-ADI-FDTD.

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