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On a Problem of an Infinite Plate with a Curvilinear Hole inside the Unit Circle

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Abstract

In this work, we used the complex variable methods to derive the Goursat functions for the first and second fundamental problem of an infinite plate with a curvilinear hole C. The hole is mapped in the domain inside a unit circle by means of the rational mapping function. Many special cases are discussed and established of these functions. Also, many applications and examples are considered. The results indicate that the infinite plate with a curvilinear hole inside the unit circle is very pronounced.

Keywords

Complex Variable Method, An Infinite Plate, Curvilinear Hole, Conformal Mapping, Goursat Functions

1. Introduction

Many intangible phenomena can be found in nature-like magnetic field, electricity and heat. These phenomena cannot be presented mathematically in the real plane. The complex plane plays an important role in presenting these intangible phenomena. Also, many mathematical problems cannot be solved in the real plane; their solutions can be found in the complex plane.

The considerable mathematical difficulties which arise during any attempt to solve plane elastic problems necessitate the search for practical methods of solution. The first use and development of the methods of complex function theory in two-dimensional elastic problems were made by Muskhelishvili (see [1]), and their ideas were expounded in their latter books (see [2]-[4]). The development of the theory was based on the complex representation of the general solution of the equations of the plane theory of elasticity. This complex representation has been found very useful for the effective solution of the plane elastic problems.

Contact and mixed problems in the theory of elasticity have been recognized as a rich and challenging subject

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for study (see Popov [5], Sabbah [6] and Atkin and Fox [7]). These problems can be established from the initial value problems or from the boundary value problems, or from the mixed problems (see Colton and Kress [8] and Abdou [9]). Also, many different methods are established for solving the contact and mixed problems in elastic and thermoelastic problems; the books edited by Noda [10], Hetnarski [11], Parkus [12] and Popov [5] contain many different methods to solve the problems in the theory of elasticity in one, two and three dimensions.

Several authors wrote about the boundary value problems and their applications in many different sciences (see [7] [13]-[15]). Form these problems, we established contact and mixed problems (see [8] [16]). Complex variable method used to express the solutions of these problems in the form of power series applied Laurent's theorem (see [8] [17]-[19]). The extensive literature on the topic is now available and we can only mention a few recent interesting investigations in [20]-[24].

The first and second fundamental problems in the plane theory of elasticity are equivalent to finding analytic functions $\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z = x + iy$.

These functions satisfy the boundary conditions

$$k\phi_1(t) - t\overline{\phi_1^{\backslash}(t)} - \overline{\psi_1(t)} = f(t) \tag{1}$$

where $\phi_1(t)$ and $\psi_1(t)$ are two analytic functions; t denotes the affix of a point on the boundary. In the first fundamental problem $k = -1$, $f(t)$ is a given function of stresses, while in the second fundamental problem

$$k = \chi = \frac{(\lambda + 3\mu)}{\lambda + \mu}, \qquad \lambda = \frac{E}{(1-2v)(1+v)} \tag{2}$$

And $f = 2\mu g$ is a given function of the displacement; λ and μ are called the Lame constants.

Let the complex potentials $\phi_1(t)$ and $\psi_1(t)$ take the form

$$\phi_1(\zeta) = -\frac{X + iY}{2\pi(1+\chi)} \ln\zeta + c\Gamma\zeta + \phi(\zeta) \tag{3}$$

$$\psi_1(\zeta) = \chi\frac{(X - iY)}{2\pi(1+\chi)} \ln\zeta + c\Gamma^*\zeta + \psi(\zeta) \tag{4}$$

where X, Y are the components of the resultant vector of all external forces acting on the boundary and Γ, Γ^* are constants; generally complex functions $\phi(\zeta), \psi(\zeta)$ are single-valued analytic functions within the region inside the unit circle γ and $\phi(\infty) = 0$.

Take the conformal mapping which mapped the domain of the curvilinear hole C on the domain inside a unit circle γ by the rational function

$$z = w(\zeta), \ |\zeta| < 1, c > 0. \tag{5}$$

and $w'(\zeta)$ does not vanish or become infinite to conform the curvilinear hole of an infinite elastic plate onto the domain inside a unit circle γ *i.e.*

$$w'(\zeta) \neq 0, \infty. \tag{6}$$

2. Conformal Mapping

Consider the rational mapping on the domain inside a unit circle γ by the rational function

$$z = w(\zeta) = \frac{\zeta^3 + m\zeta}{\zeta - n}, \ |n| < 1, \ |\zeta| < 1, \tag{7}$$

where, m and n are complex number $n = n_1 + in_2, m = m_1 + im_2$, Equation (7) must satisfy the condition Equation (6).

For determining the tax parameters x and y, we put $\zeta = \rho e^{i\theta}$, $|\rho| = 1$ in Equation (7) to get

$$x + iy = \frac{(\cos 3\theta + m_1 \cos\theta - m_2 \sin\theta) + i(\sin 3\theta + m_1 \sin\theta + m_2 \cos\theta)}{(\cos\theta - n_1) + i(\sin\theta - n_2)} \tag{8}$$

Then

$$x = \frac{\cos 2\theta + m_1 - n_1(\cos 3\theta + m_1 \cos\theta - m_2 \sin\theta) + n_1(\sin 3\theta + m_1 \sin\theta + m_2 \cos\theta)}{(\cos\theta - n_1)^2 + (\sin\theta - n_2)^2} \tag{9}$$

$$y = \frac{\sin 2\theta + m_2 + n_2(\cos 3\theta + m_1 \cos\theta - m_2 \sin\theta) - n_1(\sin 3\theta + m_1 \sin\theta + m_2 \cos\theta)}{(\cos\theta - n_1)^2 + (\sin\theta - n_2)^2} \tag{10}$$

Also,

$$z' = w'(\zeta) = \frac{2\zeta^3 - 3n\zeta^2 - mn}{(\zeta - n)^2}.$$

To obtain the critical points, we consider

$$2\zeta^3 - 3n\zeta^2 - mn = 0 \tag{11}$$

this linear equation of three order, the roots of this equation must be under 1.

The following graphs give the different shapes of the rational mapping (7), see **Figure 1**.

3. The Components of Stresses

It is known that, the components of stresses are given by, see [1]

$$\sigma_{xx} + \sigma_{yy} = 4\,\mathrm{Re}\{\phi'(z)\} \tag{12}$$

$$\sigma_{yy} - \sigma_{xx} + i\sigma_{xy} = 2\{\bar{z}\phi''(z) + \psi'(z)\} \tag{13}$$

Hence, we have

$$\sigma_{yy} = \mathrm{Re}\{2\phi'(z) + M(z,\bar{z})\},\ M(z,\bar{z}) = \bar{z}\phi''(z) + \psi'(z) \tag{14}$$

$$\sigma_{xx} = \mathrm{Re}\{2\phi'(z) - M(z,\bar{z})\},\ M(z,\bar{z}) = \bar{z}\phi''(z) + \psi'(z) \tag{15}$$

and

$$\sigma_{xy} = 2\,\mathrm{Im}\{\bar{z}\phi''(z) + \psi'(z)\} = 2\,\mathrm{Im}\{M(z,\bar{z})\} \tag{16}$$

4. Goursat Functions

To obtain the tow complex potential functions (Goursat functions) by using the conformal mapping (7) in the boundary condition (6). We write the expression $\dfrac{w(\zeta)}{\overline{w'(\zeta)}}$ in the form,

$$\frac{w(\zeta)}{\overline{w'(\zeta)}} = \alpha(\zeta) + \overline{\beta(\zeta)} \tag{17}$$

where,

$$\alpha(\zeta) = \frac{h}{(\zeta - n)},\quad \overline{\beta(\zeta)} = \frac{w(\zeta)}{\overline{w'(\zeta)}} - \frac{h}{\zeta - n} \tag{18}$$

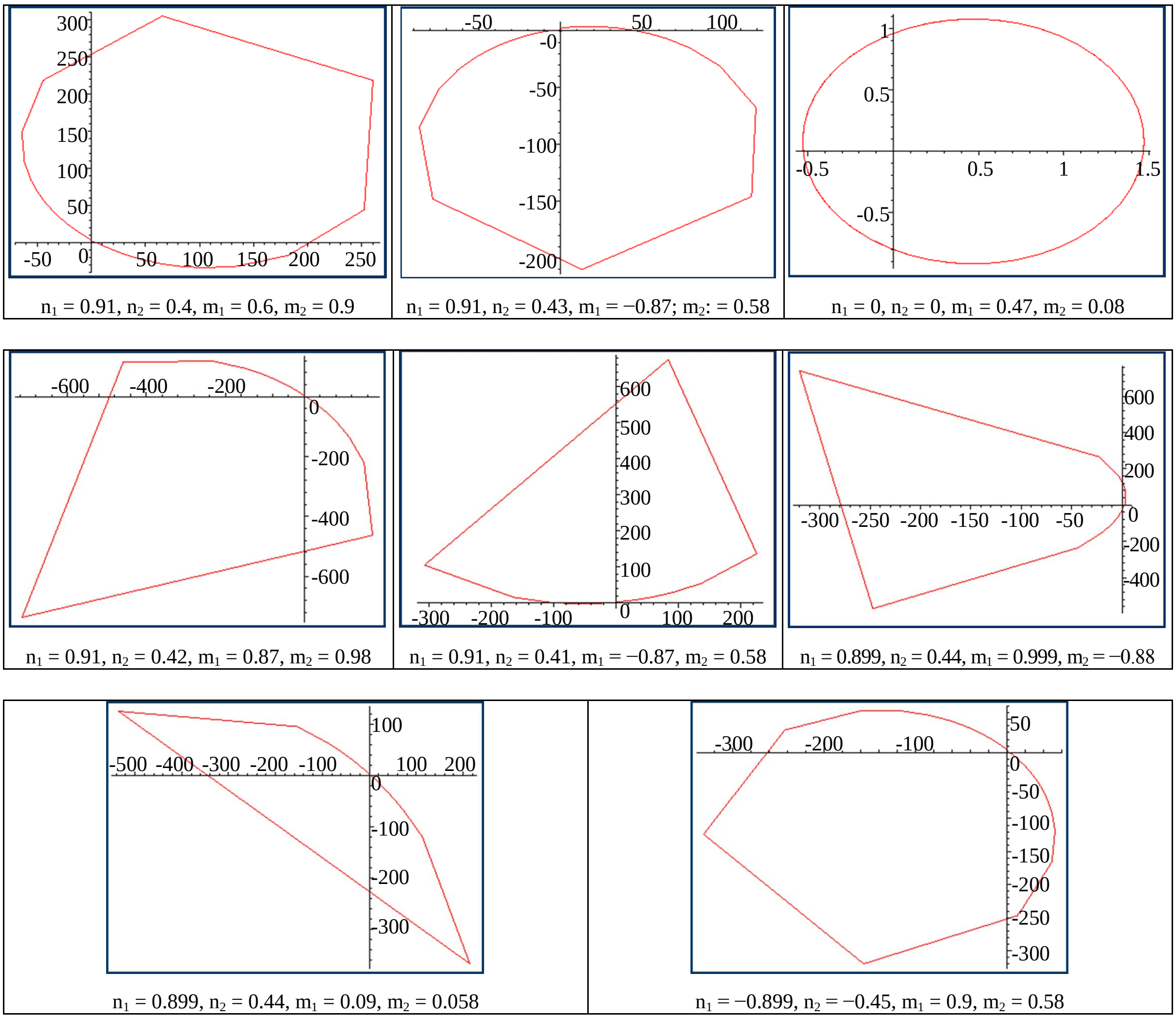

Figure 1. The different shapes of the rational mapping (7).

$\beta\left(\zeta^{-1}\right)$ is a regular function for $|\zeta|<1$.

In order to separate the singularity, we use the definition of mapping, to have

$$\frac{w(\zeta)}{\overline{\overline{w'(\zeta)}}}=\frac{\zeta^3+m\zeta}{\zeta-n}\cdot\frac{\zeta(1-n\zeta)^2}{\left(2-3n\zeta-mn\zeta^3\right)}=\frac{1}{\zeta-n}\cdot\frac{\zeta^2\left(\zeta^2+m\right)(1-n\zeta)^2}{\left(2-3n\zeta-mn\zeta^3\right)} \tag{19}$$

The term $\left(2-3n\zeta-mn\zeta^3\right)$ in the are has no singular point while $(\zeta-n)$ has a singularity at $\zeta=n$. where

$$\overline{w'(\zeta)}=\frac{\left(2-3n\zeta-mn\zeta^3\right)}{\zeta(1-n\zeta)^2} \tag{20}$$

To determine h form Equation (19), we can write the form

$$\frac{w(\sigma)}{\overline{\overline{w'(\sigma)}}}=\frac{1}{\sigma-n}\cdot\frac{\sigma^2\left(\sigma^2+m\right)(1-n\sigma)^2}{\left(2-3n\sigma-mn\sigma^3\right)}. \tag{21}$$

By using the residues in this equating we have

$$h=\frac{n^2\left(n^2+m\right)\left(1-n^2\right)^2}{\left(2-3n^2-mn^4\right)}. \tag{22}$$

Using Equation (3) and Equation (4) in Equation (1), we get

$$k\phi(\sigma)-\alpha(\sigma)\overline{\phi'(\sigma)}-\overline{\psi_*(\sigma)}=G(\sigma) \tag{23}$$

where

$$\psi_*(\sigma)=\psi(\sigma)+\beta(\sigma)\phi'(\sigma) \tag{24}$$

$$G(\sigma)=F(\sigma)-ck\Gamma\sigma+\frac{c\overline{\Gamma^*}}{\sigma}+N(\sigma)\alpha(\sigma)+N(\sigma)\overline{\beta(\sigma)} \tag{25}$$

$$N(\sigma)=\left[c\overline{\Gamma}-\frac{\sigma(X-iY)}{2\pi(1+\chi)}\right],\quad F(\sigma)=f(t) \tag{26}$$

Assume that the function $F(\sigma)$ with its derivatives must satisfy the Holder condition. Our aim is to determine the functions $\phi(\zeta)$ and $\psi(\zeta)$ for the various boundary value problems. For this multiply both sides of Equation (23) by $\frac{\mathrm{d}\sigma}{2\pi i(\sigma-\zeta)}$, where ζ is any point in the interior of γ and integral over the circle, we obtain

$$\frac{k}{2\pi i}\int_\gamma\frac{\phi(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma-\frac{1}{2\pi i}\int_\gamma\frac{\alpha(\sigma)\overline{\phi'}(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma-\frac{1}{2\pi i}\int_\gamma\frac{\overline{\psi_*}(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma=\frac{1}{2\pi i}\int_\gamma\frac{G(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma \tag{27}$$

Using Equations (24)-(26) in Equation (27) then applying the properties of Cauchy integral, to have

$$\frac{k}{2\pi i}\int_\gamma\frac{\phi(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma=-k\phi(\zeta) \tag{28}$$

and

$$\frac{1}{2\pi i}\int_\gamma\frac{\alpha(\sigma)\overline{\phi'}(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma=\frac{chb}{n-\zeta} \tag{29}$$

$$\frac{1}{2\pi i}\int_\gamma\frac{N(\sigma)\alpha(\sigma)}{(\sigma-\zeta)}\mathrm{d}\sigma=\frac{N(n)h}{n-\zeta} \tag{30}$$

Also,

$$\frac{1}{2\pi i}\int_\gamma\frac{G(\sigma)}{(\sigma-\zeta)}\mathrm{d}\sigma=A(\zeta)-\frac{c\overline{\Gamma^*}}{\zeta}+\frac{hN(n)}{(n-\zeta)} \tag{31}$$

where,

$$A(\zeta)=\frac{1}{2\pi i}\int_\gamma\frac{F(\sigma)}{(\sigma-\zeta)}\mathrm{d}\sigma,\quad N(\sigma)=\left[c\overline{\Gamma}-\frac{\sigma(X-iY)}{2\pi(1+\chi)}\right]. \tag{32}$$

From the above, Equation (27) becomes

$$-k\phi(\zeta)=A(\zeta)+\frac{h}{n-\zeta}\left(cb+N(n)\right)-\frac{c\overline{\Gamma^*}}{\zeta} \tag{33}$$

To determined b, where b are complex constants, differentiating Equation (33) with respect to ζ and substituting in Equation (29), we get

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\alpha(\sigma)}{(\sigma-\zeta)}\left[-\overline{A'(\sigma)}-c\Gamma^{*}\sigma^{2}-\frac{h\sigma^{2}}{(n\sigma-1)^{2}}\left(c\bar{b}+\overline{N(n)}\right)\mathrm{d}\sigma\right]=\frac{ckhb}{(n-\zeta)} \tag{34}$$

Substituting Equation (18) in Equation (34), then using the properties of Cauchy integral and applying the reside theorem at the singular points, we obtain

$$ckb+\overline{A'(n)}+c\Gamma^{*}n^{2}+\upsilon h\left(c\bar{b}+\overline{N(n)}\right)=0 \tag{35}$$

where

$$\upsilon=\frac{n^{2}}{\left(1-n^{2}\right)^{2}} \tag{36}$$

The last equation can be written in the form

$$ckb+\upsilon hc\bar{b}=E \tag{37}$$

where,

$$E=-\overline{A'(n)}-c\Gamma^{*}n^{2}-\upsilon h\overline{N(n)} \tag{38}$$

taking the complex conjugate of Equation (37), we get

$$ck\bar{b}+\upsilon hcb=\bar{E} \tag{39}$$

form Equation (37) and Equation (39), we have

$$b=\frac{kE-\upsilon h\bar{E}}{c\left(k^{2}-\upsilon^{2}h^{2}\right)} \tag{40}$$

To obtain the complex function $\psi(\zeta)$ we have form Equation (23) after substituting the expression of $\psi(\sigma)$ and $G(\sigma)$, and taking the complex conjugate of the resulting equation after using the expression of $\overline{\beta(\sigma)}$ to yields,

$$\psi(\sigma)=-\overline{F(\sigma)}+ck\bar{\Gamma}\sigma^{-1}-c\Gamma^{*}\sigma+k\overline{\phi(\sigma)}-\overline{\alpha(\sigma)}\phi_{*}(\sigma)-\frac{\overline{w(\sigma)}}{w'(\sigma)}\phi_{*}(\sigma)+\frac{h\sigma}{(1-n\sigma)}\phi_{*}(\sigma) \tag{41}$$

where,

$$\phi_{*}(\sigma)=\phi'(\sigma)+\overline{N(\sigma)},\quad \overline{N(\sigma)}=\left[c\Gamma-\frac{\sigma^{-1}(X+iY)}{2\pi(1+\chi)}\right] \tag{42}$$

and calculate sum residue, we obtain multiplying both sides of Equation (41) by $\frac{1}{2\pi i(\sigma-\zeta)}$, where ζ is any point in the interior of γ and integrating over the circle, then using the properties of Cauchy's integral and calculating the sum residue, we obtain

$$\psi(\zeta)=ck\bar{\Gamma}\zeta^{-1}-\frac{\overline{w(\zeta)}}{w'(\zeta)}\phi_{*}(\zeta)+\frac{h\zeta}{(1-n\zeta)}\phi_{*}\left(n^{-1}\right)+B(\zeta)-B \tag{43}$$

where,

$$B(\zeta) = \frac{1}{2\pi i} \int_{\gamma} \frac{\overline{F(\sigma)}}{(\sigma - \zeta)} \mathrm{d}\sigma, \tag{44}$$

and

$$B = \frac{1}{2\pi i} \int_{\gamma} \frac{\overline{F(\sigma)}}{\sigma} \mathrm{d}\sigma. \tag{45}$$

5. Special Cases

Now, we are in a position to consider several cases:

1) Let $m = 0, n \neq 0$, we get the mapping function represent of the hole is an ellipse, see **Figure 2**

$$z = w(\zeta) = \frac{\zeta^3}{\zeta - n} \tag{46}$$

by let

$$z' = 0 \Rightarrow 2\zeta^3 - 3n\zeta^2 = 0.$$

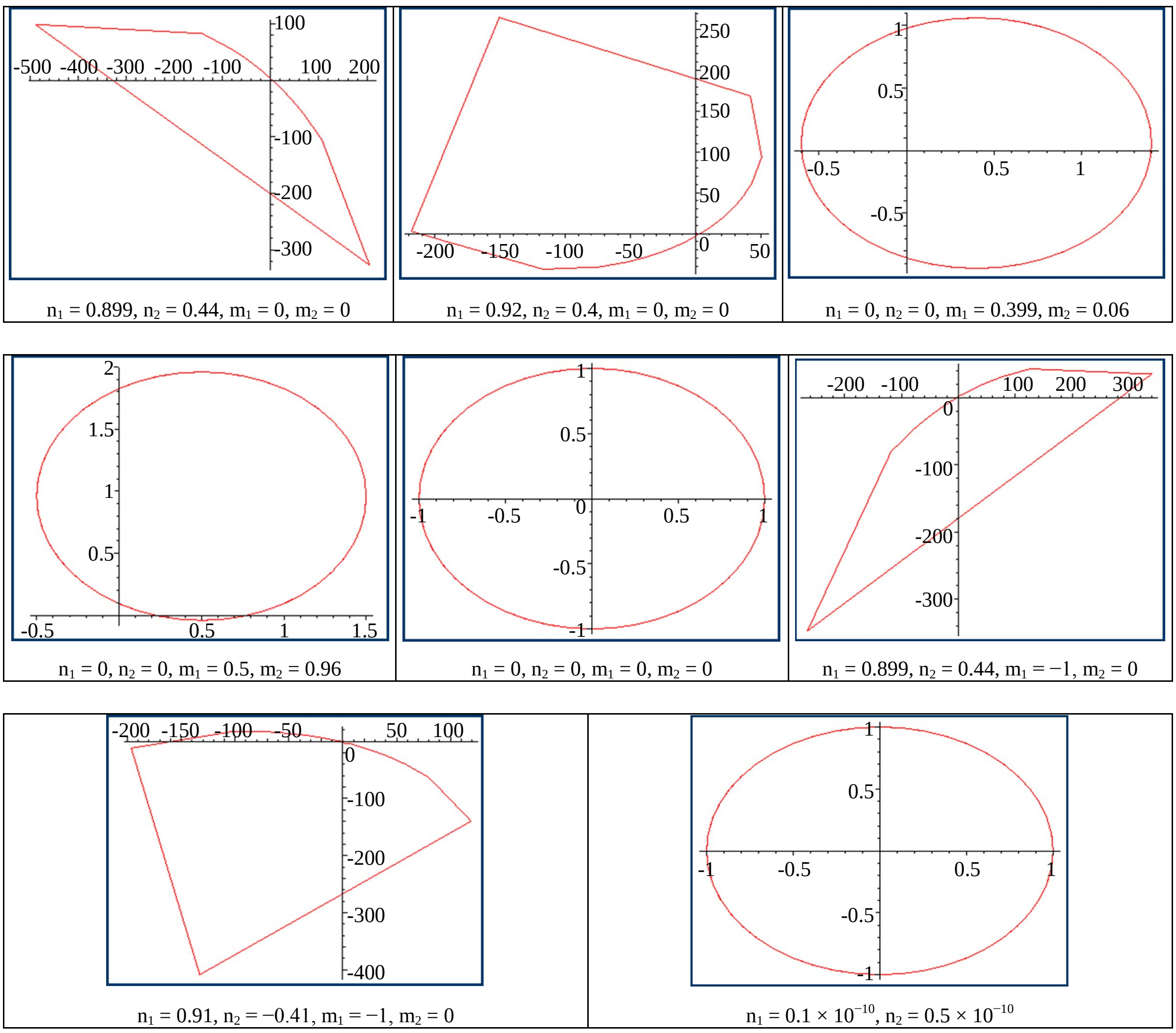

Figure 2. The different shapes of the rational mapping for special cases.

Then (33) and (43) becomes

$$h=\frac{n^{4}\left(1-n^{2}\right)^{2}}{2-3n^{2}}$$

$$-k\phi(\zeta)=A(\zeta)-\frac{c\overline{\Gamma^{*}}}{\zeta}+\frac{n^{4}\left(1-n^{2}\right)^{2}}{\left(2-3n^{2}\right)(n-\zeta)}\left[N(n)+\frac{kE-h\upsilon\overline{E}}{k^{2}-h^{2}\upsilon^{2}}\right] \tag{47}$$

$$-k\phi(\zeta)=A(\zeta)-\frac{c\overline{\Gamma^{*}}}{\zeta}+\frac{n^{4}\left(1-n^{2}\right)^{2}}{\left(2-3n^{2}\right)(n-\zeta)}\left[N(n)+\frac{kE-\frac{n^{6}\overline{E}}{\left(2-3n^{2}\right)}}{k^{2}-\frac{n^{12}}{\left(2-3n^{2}\right)^{2}}}\right].$$

Also,

$$\psi(\zeta)=B(\zeta)+\frac{ck\overline{\Gamma}}{\zeta}-\frac{w\left(\zeta^{-1}\right)}{w'(\zeta)}\phi_{*}(\zeta)+\frac{n^{4}\left(1-n^{2}\right)^{2}\zeta}{\left(2-3n^{2}\right)(1-n\zeta)}\phi_{*}\left(n^{-1}\right)-B. \tag{48}$$

where

$$E=-\overline{A'(n)}-c\Gamma^{*}n^{2}-\frac{n^{6}}{2-3n^{2}}\overline{N(n)}$$

2) For $n=0, 0\le m\le 1$, we get the mapping function represent of the hole is an ellipse, see **Figure 2**

$$z=\frac{\zeta^{3}+m\zeta}{\zeta}=\zeta^{2}+m \qquad z'=0\Rightarrow 2\zeta=0 \tag{49}$$

then

$$\zeta=e^{i\alpha},\quad e^{i\alpha}=\cos\alpha+i\sin\alpha.$$

Then (33) and (43) becomes

$$h=0$$

$$-k\phi(\zeta)=A(\zeta)-\frac{c\overline{\Gamma^{*}}}{\zeta} \tag{50}$$

$$\psi(\zeta)=B(\zeta)+\frac{ck\overline{\Gamma}}{\zeta}-\frac{1+m\zeta^{2}}{2\zeta^{3}}\phi_{*}(\zeta)-B \tag{51}$$

where

$$E=-\overline{A'(n)}-c\Gamma^{*}n^{2},\quad n=0.$$

3) Let $m=n=0$, we get the mapping function represent of the hole is an ellipse, see **Figure 2**

$$z=\frac{\zeta^{3}}{\zeta}=\zeta^{2} \tag{52}$$

$$z'=0\Rightarrow 2\zeta=0$$

$$\zeta=e^{i\alpha},\quad e^{i\alpha}=\cos\alpha+i\sin\alpha$$

Then (33) and (43) becomes

$$h = 0$$

$$-k\phi(\zeta) = A(\zeta) - \frac{c\overline{\Gamma^*}}{\zeta} \tag{53}$$

$$\psi(\zeta) = B(\zeta) + \frac{ck\overline{\Gamma}}{\zeta} - \frac{1}{2\zeta^3}\phi_*(\zeta) - B \tag{54}$$

$$E = -\overline{A'(n)} - c\Gamma^* n^2,\ n = 0.$$

4) Let $m = -1$, where $m_1 = -1, m_2 = 0$ we get the mapping function represent of the hole is an ellipse, see **Figure 2**

$$z = \frac{\zeta^3 - \zeta}{\zeta - n} \tag{55}$$

$$z' = 0 \Rightarrow 2\zeta^3 - 3n\zeta^2 + n = 0$$

Then (33) and (43) becomes

$$h = \frac{n^2(n^2-1)(1-n^2)^2}{2-3n^2+n^4}$$

$$-k\phi(\zeta) = A(\zeta) - \frac{c\overline{\Gamma^*}}{\zeta} + \frac{n^2(n^2-1)(1-n^2)^2}{(2-3n^2+n^4)(n-\zeta)}\left[N(n) + \frac{kE - \frac{n^4(n^2-1)\bar{E}}{(2-3n^2+n^4)}}{k^2 - \frac{n^8(n^2-1)^2}{(2-3n^2+n^4)^2}}\right] \tag{56}$$

Also,

$$\psi(\zeta) = B(\zeta) + \frac{ck\overline{\Gamma}}{\zeta} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) + \frac{n^2(n^2-1)(1-n^2)^2\zeta}{(2-3n^2+n^4)(1-n\zeta)}\phi_*(n^{-1}) - B \tag{57}$$

where

$$E = -\overline{A'(n)} - c\Gamma^* n^2 + \frac{n^4(n^2-1)}{2-3n^2+n^4}\overline{N(n)}$$

5) Let $m = -n^2$,we get the mapping function represent of the hole is an ellipse, see **Figure 2**

$$z = \frac{\zeta^3 - n^2\zeta}{\zeta - n} = \zeta^2 + n\zeta \tag{58}$$

$$z' = 0 \Rightarrow 2\zeta + n = 0$$

$$\zeta = -\frac{n}{2}$$

Then (33) and (43) becomes

$$h = 0$$

$$-k\phi(\zeta) = A(\zeta) - \frac{c\overline{\Gamma^*}}{\zeta} \tag{59}$$

Also,

$$\psi(\zeta) = B(\zeta) + \frac{ck\overline{\Gamma}}{\zeta} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) - B \tag{60}$$

$$E = -\overline{A'(n)} - c\Gamma^* n^2 + \frac{n^4(n^2+1)}{2-3n^2+n^4}\overline{N(n)}$$

6. Applications

In this section we study some applications:

1) For $k=-1, \Gamma = \frac{p}{4}, \Gamma^* = -\frac{1}{2}pe^{-2i\theta}$ and $X = Y = f = 0$, we have the case of infinite plate stretched at infinity by the application of a uniform tensile stress of intensity p, making an angle θ with the x-axis. The plate weakened by the curvilinear hole C which is free from stresses (see **Figure 3**, **Figure 4** ($n_1 = 0.001$, $n_2 = 0.002l$, $m_1 = 0.025$, $m_2 = 0.03I$, $c = 2$, $p = 0.25$)). Then the functions in (33) and (43) become

$$f = 0 \Rightarrow A(\zeta) = 0 \tag{61}$$

$$N(n) = \left[c\overline{\Gamma} - \frac{n(X-iY)}{2\pi(1+\chi)}\right] = \frac{cp}{4} \tag{62}$$

$$E = \frac{2cn^2 pe^{-2i\theta} - \upsilon hcp}{4},\quad \overline{E} = \frac{2cn^2 pe^{2i\theta} - \upsilon hcp}{4} \tag{63}$$

$$cb = \frac{kE - \upsilon h\overline{E}}{(k^2 - \upsilon^2 h^2)} = \frac{-E - \upsilon h\overline{E}}{(1-\upsilon^2 h^2)} \tag{64}$$

$$\phi(\zeta) = \frac{cpe^{2i\theta}}{2\zeta} + \frac{chp}{4(n-\zeta)}\left(1 + \frac{ch\upsilon - 2cn^2 e^{-2i\theta} - 2h\upsilon cn^2 e^{2i\theta} + ch^2\upsilon^2}{1-h^2\upsilon^2}\right). \tag{65}$$

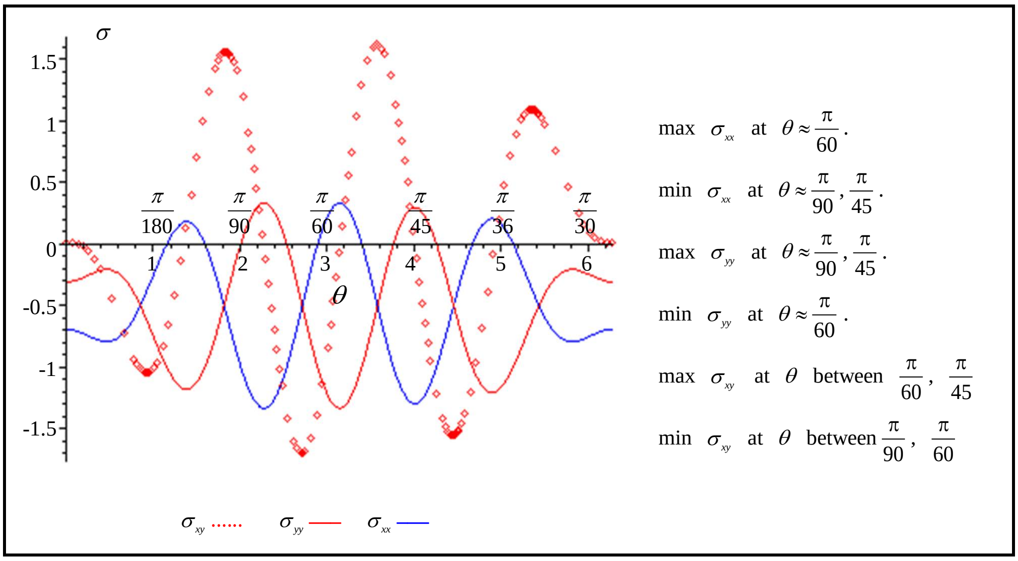

Figure 3. The relation between components of stresses and the angle made on the x-axis.

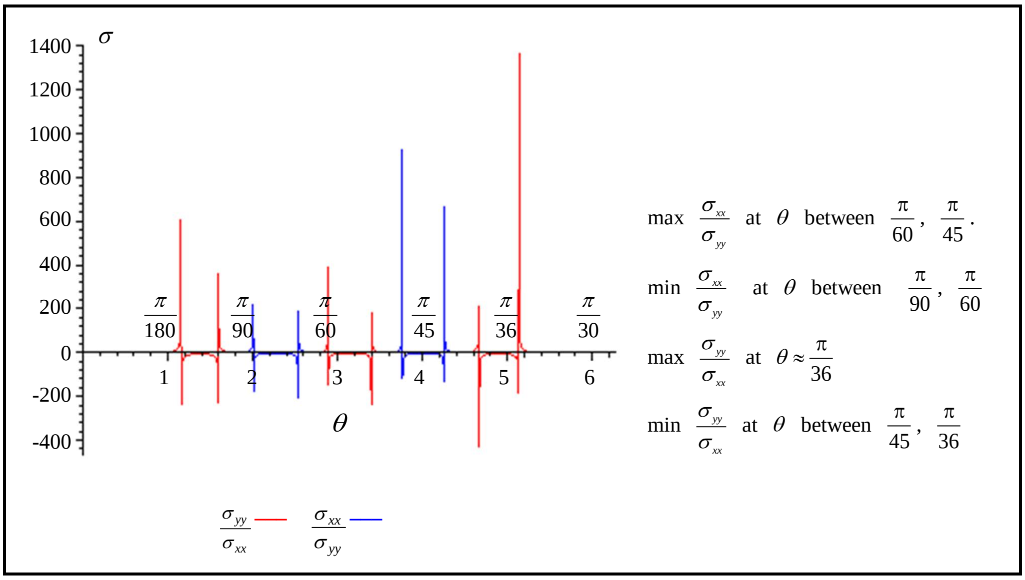

Figure 4. The ratio of vertical to horizontal stresses.

$$\psi(\zeta) = -\frac{cp}{4\zeta} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) + \frac{h\zeta}{(1-n\zeta)}\phi_*(n^{-1}) \tag{66}$$

where

$$\phi_*(\zeta) = \phi'(\zeta) + \frac{cp}{4}.$$

2) For $k=-1$, $\Gamma = \Gamma^* = X = Y = 0$ and $f = Pt$, where P is a real constant (see **Figure 5**, **Figure 6** ($n_1 = 0.001, n_2 = 0.002I, m_1 = 0.025, m_2 = 0.03I, c = 2, p = 0.25$)).

Then the functions in (33) and (43) become

$$f = Pt \Rightarrow f = \frac{Pc(\sigma^3 + m\sigma)}{(\sigma - n)} \tag{67}$$

$$\bar{f} = \frac{cP(1+m\sigma^2)}{\sigma^2(1-n\sigma)} \tag{68}$$

$$A(\zeta) = \frac{cP}{2\pi i}\int_\gamma \frac{\sigma^3 + m\sigma}{(\sigma-n)(\sigma-\zeta)}\mathrm{d}\sigma = \frac{cP(n^3+mn)}{(n-\zeta)} \tag{69}$$

$$A'(\zeta) = \frac{cP(n^3+mn)}{(n-\zeta)^2}, \quad \overline{A'(\zeta)} = \frac{cP\zeta^2(n^3+mn)}{(n\zeta-1)^2}$$

$$\overline{A'(n)} = \frac{cPn^2(n^3+mn)}{(n^2-1)^2}, \quad N(n) = 0 \tag{70}$$

$$E = -\frac{cPn^2(n^3+mn)}{(n^2-1)^2} = \bar{E} \tag{71}$$

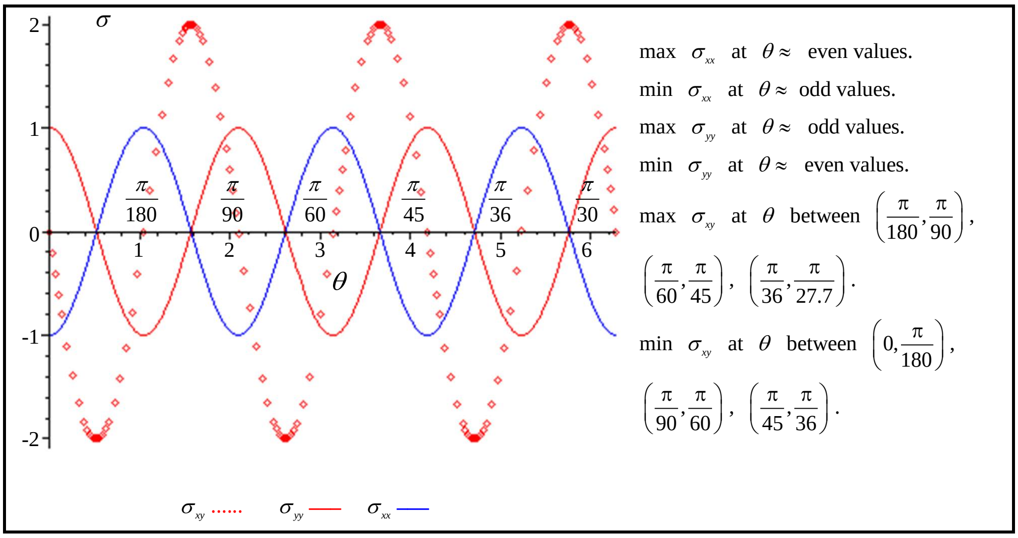

Figure 5. The relation between components of stresses and the angle made on the x-axis.

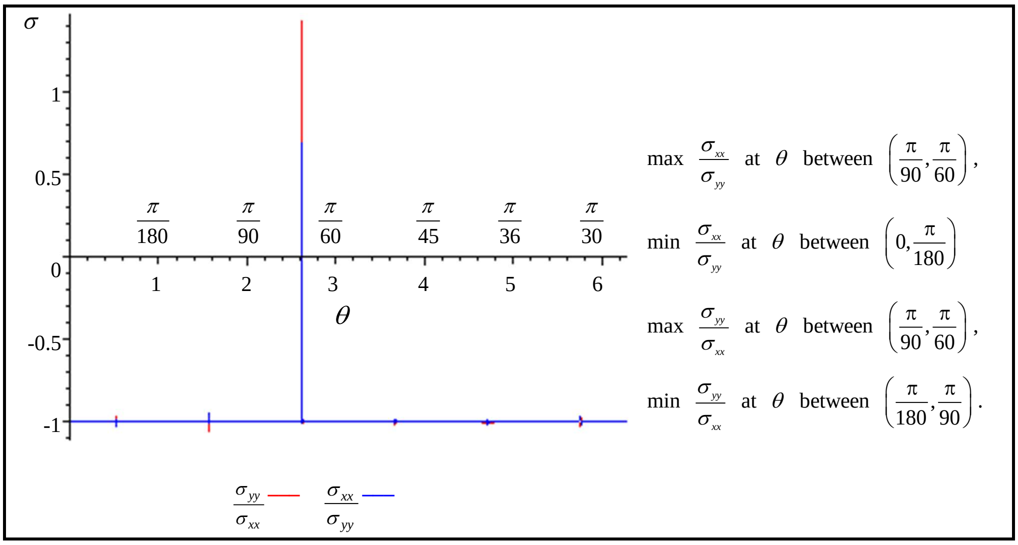

Figure 6. The ratio of vertical to horizontal stresses.

$$cb = \frac{cPn^2\left(n^3+mn\right)}{\left(1-\upsilon h\right)\left(n^2-1\right)^2} \tag{72}$$

$$\phi(\zeta) = \frac{cP\left(n^3+mn\right)}{\left(n-\zeta\right)} + \frac{hcPn^2\left(n^3+mn\right)}{\left(n-\zeta\right)\left(1-\upsilon h\right)\left(n^2-1\right)^2} \tag{73}$$

$$\psi(\zeta) = -cP\left(n+\frac{1}{\zeta^2}\right) - \frac{w\left(\zeta^{-1}\right)}{w'\left(\zeta\right)}\phi'\left(\zeta\right) + \frac{h\zeta}{\left(1-n\zeta\right)}\phi'\left(n^{-1}\right) \tag{74}$$

where

$$B(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{F(\sigma)}}{\left(\sigma-\zeta\right)}\mathrm{d}\sigma = \frac{cP}{2\pi i}\int_\gamma \frac{\left(1+m\sigma^2\right)}{\sigma^2\left(1-n\sigma\right)\left(\sigma-\zeta\right)}\mathrm{d}\sigma = -\frac{cP\left(1+n\zeta\right)}{\zeta^2}$$

$$B=\frac{cP}{2\pi i}\int_{\gamma}\frac{\left(1+m\sigma^{2}\right)}{\sigma^{3}\left(1-n\sigma\right)}\mathrm{d}\sigma=2cP\left(m+n^{2}\right),\ \ \phi_{*}\left(\zeta\right)=\phi'\left(\zeta\right).$$

3) For $k=\chi,\ \Gamma=\Gamma^{*}=f=0$ (see **Figure 7**, **Figure 8** ($n_1=0.001,\ n_2=0.002I,\ m_1=0.025,\ m_2=0.03I,\ c=2,\ x=0.25,\ X=2,\ Y=2$)). Then the functions in (33) and (43) become

$$f=0\Rightarrow A\left(\zeta\right)=0 \tag{75}$$

$$N\left(n\right)=-\frac{n\left(X-iY\right)}{2\pi\left(1+\chi\right)} \tag{76}$$

$$E=\frac{\upsilon hn\left(X+iY\right)}{2\pi\left(1+\chi\right)},\ \ \bar{E}=\frac{\upsilon hn\left(X-iY\right)}{2\pi\left(1+\chi\right)} \tag{77}$$

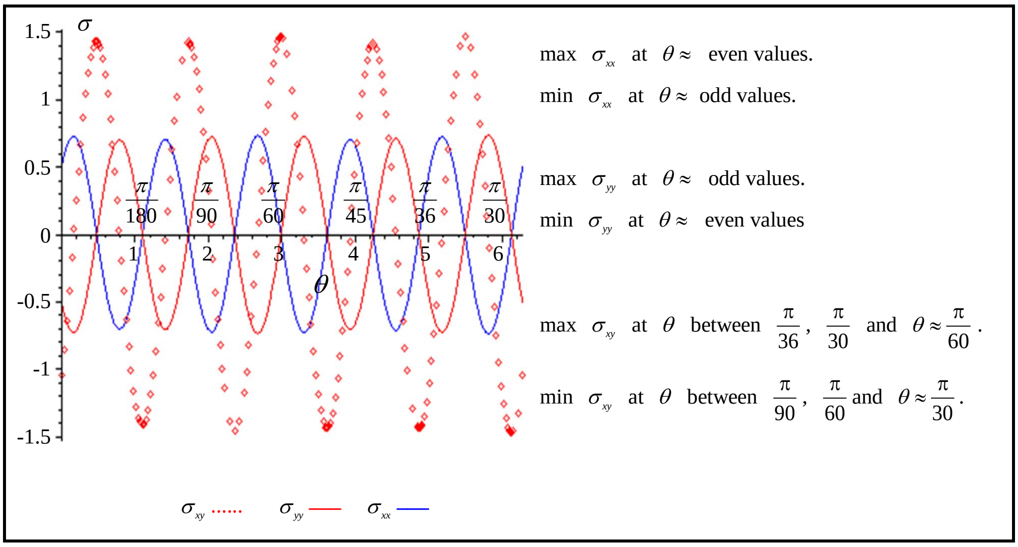

Figure 7. The relation between components of stresses and the angle made on the x-axis.

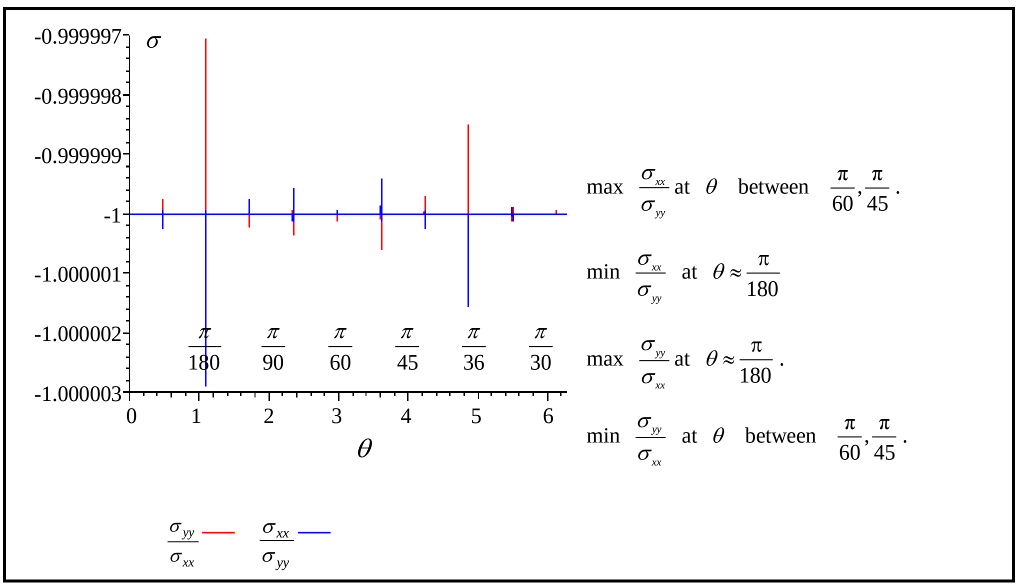

Figure 8. The ratio of vertical to horizontal stresses.

$$cb=\frac{\chi h\upsilon n\left(X+iY\right)-nh^{2}\upsilon^{2}\left(X-iY\right)}{2\pi\left(1+\chi\right)\left(\chi^{2}-h^{2}\upsilon^{2}\right)} \tag{78}$$

$$-\chi\phi\left(\zeta\right)=\frac{h}{\left(n-\zeta\right)}\left(\frac{\chi h\upsilon n\left(X+iY\right)-nh^{2}\upsilon^{2}\left(X-iY\right)}{2\pi\left(1+\chi\right)\left(\chi^{2}-h^{2}\upsilon^{2}\right)}-\frac{n\left(X-iY\right)}{2\pi\left(1+\chi\right)}\right)$$

$$\phi\left(\zeta\right)=\frac{-1}{\chi}\frac{hn}{2\pi\left(1+\chi\right)\left(n-\zeta\right)}\left(\frac{\chi h\upsilon\left(X+iY\right)}{\left(\chi^{2}-h^{2}\upsilon^{2}\right)}-\left(X-iY\right)\left(1+\frac{h^{2}\upsilon^{2}}{\left(\chi^{2}-h^{2}\upsilon^{2}\right)}\right)\right) \tag{79}$$

$$\psi\left(\zeta\right)=-\frac{w\left(\zeta^{-1}\right)}{w'\left(\zeta\right)}\phi_{*}\left(\zeta\right)+\frac{h\zeta}{\left(1-n\zeta\right)}\phi_{*}\left(n^{-1}\right) \tag{80}$$

where

$$\phi_{*}\left(\zeta\right)=\phi'\left(\zeta\right)-\frac{\left(X+iY\right)}{2\pi\left(1+\chi\right)\zeta}.$$

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