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On Asymptotic Stability of Linear Control Systems

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Abstract

Asymptotic stability of linear systems is closely related to Hurwitz stability of the system matrices. For uncertain linear systems we consider stability problem through common quadratic Lyapunov functions (CQLF) and problem of stabilization by linear feedback.

Keywords

Common Quadratic Lyapunov Functions, Uncertain System, Gradient Method, Bendixson Theorem

1. Introduction

Let linear uncertain system

$$\dot{x} = Ax, \quad A \in \operatorname{conv}\{A_1, A_2, \cdots, A_N\} \tag{1}$$

be given where $x = x(t) \in \mathbb{R}^n$, A_i $(i = 1, 2, \cdots, N)$ are $n \times n$ real matrices. Consider the following matrix inequalities

$$A_i^{\mathrm{T}} P + P A_i < 0 \quad (i = 1, 2, \cdots, N) \tag{2}$$

where $P > 0$ and the symbol "$>$" stands for positive definiteness. The matrix P is called a common solution to (2).

If the system (2) has a common $P > 0$ solution, then this system is uniformly asymptotically stable [1].

The problem of existence of common positive definite solution P of (2) has been studied in a lot of works (see [1]-[7] and references therein). Numerical solution for common P via nondifferentiable convex optimization has been discussed in [8].

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In the first part of the paper we treat the problem (2) as a nonconvex optimization problem (minimization of a convex function under nonconvex constraints) and apply a modified gradient method. The comparison with [8] shows that our approach gives better result in some cases.

In the second part we consider the stabilization problem, *i.e*. the following question: for the affine family

$$\left\{A(q): q \in R\right\}$$

where $R \subset \mathbb{R}^l$ is a box, is there a stable member? We consider a sufficient condition which follows from the Bendixson theorem [9].

2. Gradient Method

According to [2], let $\mathcal{S}$ be the set (subspace) of $(n \cdot N) \times (n \cdot N)$ dimensional symmetric block-diagonal matrices of the form $R \oplus R \oplus \cdots \oplus R$ where R is symmetric.

Let $Z_1, Z_2, \cdots, Z_r$ be a basis of $\mathcal{S}$, $r = n(n+1)/2$,

$$Q_i = (-Z_i) \oplus \left(A_1^{\mathrm{T}} Z_i + Z_i A_1\right) \oplus \cdots \oplus \left(A_N^{\mathrm{T}} Z_i + Z_i A_N\right)$$

$$\phi(x) = \phi(x_1, x_2, \cdots, x_r) = \lambda_{\max}\left(\sum_{i=1}^{r} x_i Q_i\right) \tag{3}$$

Then $\{A_1, A_2, \cdots, A_N\}$ has CQLF $\Leftrightarrow$ there exists $x_* \in \mathbb{R}^r$ such that $\phi(x_*) < 0$. In this case the matrix $P(x_*)$ is a common solution to (2) where

$$P(x) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_{n+1} & \cdots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n-1} & \cdots & x_r \end{pmatrix}.$$

The function $\phi(x)$ is positive homogenous $\left(\phi(\alpha x) = \alpha\phi(x) \text{ for all } \alpha \geq 0\right)$. Therefore the vector x can be restricted to the condition $\|x\| = 1$. The advantage of the restriction $\|x\| = 1$ shows the following proposition.

Proposition 1. Let $S = \left\{x \in \mathbb{R}^r : x = 1\right\}$ be the unit sphere, let the function $f: \mathbb{R}^r \to \mathbb{R}$ be positive homogeneous $\left(f(\lambda x) = \lambda f(x) \text{ for all } \lambda > 0\right)$ and be differentiable at $a \in S$. Assume that $f(a) > 0$. Then $\langle g, a\rangle < 0$ where $g = -\nabla f(x)\big|_{x=a}$, ∇ denotes the gradient and $\langle\cdot,\cdot\rangle$ denotes the scalar product.

Proof: Since f is positive homogeneous, it increases in the direction of the vector *a*: for $\lambda > 1$,

$$f(\lambda a) = \lambda f(a) > f(a).$$

Therefore the directional derivative of *f* at *a* in the direction of *a* is positive $D_a f(a) > 0$.

On the other hand

$$D_a f(a) = \langle \nabla f, a\rangle$$

and

$$\langle \nabla f, a\rangle > 0 \quad \text{or} \quad \langle -\nabla f, a\rangle < 0 \quad \text{or} \quad \langle g, a\rangle < 0. \qquad \square$$

Proposition 1 shows that under its assumption the minus gradient vector at the point *a* is directed into the unit ball (**Figure 1**).

Consider the following optimization problem

$$\begin{gathered} \phi(x) \to \text{minimize} \\ \text{subject to } x = 1. \end{gathered}$$

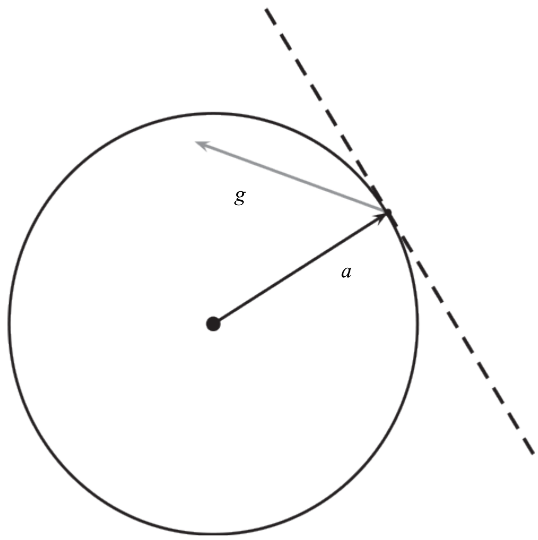

Figure 1. The direction (g) of the minus gradient.

Since the matrix $\sum_{i=1}^{r} x_i Q_i$ is symmetric, the function $\phi(x)$ (3) can be written as

$$\phi(x) = \max_{\|u\|=1} u^{\mathrm{T}} \left(\sum_{i=1}^{r} x_i Q_i \right) u.$$

The gradient vector of $\phi(x)$ at a point a is:

$$\nabla \phi(x)\big|_{x=a} = \left(u^{\mathrm{T}} Q_1 u, u^{\mathrm{T}} Q_2 u, \cdots, u^{\mathrm{T}} Q_r u \right) \tag{4}$$

where u is the unit eigenvector of $\sum_{i=1}^{r} a_i Q_i$ corresponding to the simple maximum eigenvalue [2].

Well-known gradient algorithm in combination with Proposition 1 gives the following.

Algorithm 1.

Step 1. Take an initial point $x^0 = S$. Compute $\phi(x^0)$. If $\phi(x^0) \geq 0$, find t such that the line

$$l(t) = x^0 - t \cdot \nabla \phi(x)\big|_{x=x^0}$$

intersects the unit sphere S (**Figure 2**).

Step 2. Take $x^1 = x^0 - t_* \cdot \nabla \phi(x)\big|_{x=x^0}$ where t_* satisfies the condition $\|l(t_*)\| = 1$. If $\phi(x^1) < 0$, x^1 is required point. Otherwise find t such that the line $l(t) = x^1 - t \cdot \nabla \phi(x)\big|_{x=x^1}$ intersects the unit sphere and repeat the procedure.

Example 1. Consider the switched system

$$\dot{x} \in \{A_1, A_2\} x$$

where

$$A_1 = \begin{pmatrix} -4 & -1 & 3 \\ -3 & -2 & 2 \\ 3 & 0 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -8 & -3 & 1 \\ 9 & 2 & 0 \\ 6 & 3 & -6 \end{pmatrix}$$

are Hurwitz stable matrices. Let

$$Z_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

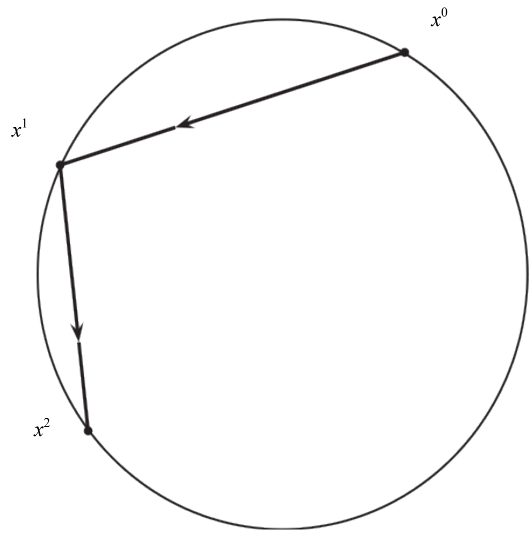

Figure 2. Searching on the unit sphere.

$$Z_4=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad Z_5=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad\text{and}\quad Z_6=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$

For $i=1,2,\cdots,6$

$$Q_i=\left(-Z_i\right)\oplus\left(A_1^{\mathrm{T}}Z_i+Z_iA_1\right)\oplus\left(A_2^{\mathrm{T}}Z_i+Z_iA_2\right).$$

Take the initial point $x^0=\left(1/\sqrt{3},0,0,1/\sqrt{3},0,1/\sqrt{3}\right)^{\mathrm{T}}$, then

$$P\left(x^0\right)=\begin{pmatrix}1/\sqrt{3}&0&0\\0&1/\sqrt{3}&0\\0&0&1/\sqrt{3}\end{pmatrix}$$

is positive definite. Eigenvalues of the matrix

$$\sum_{i=1}^{6}x_i^0Q_i=\frac{1}{\sqrt{3}}\cdot Q_1+0\cdot Q_2+0\cdot Q_3+\frac{1}{\sqrt{3}}\cdot Q_4+0\cdot Q_5+\frac{1}{\sqrt{3}}\cdot Q_6$$

are $-12.507,\ -5.364,\ 4.015,\ -0.224,\ -0.577,\ -8.566,\ -1.601$.

Maximum eigenvalue 4.015 is simple and the corresponding unit eigenvector is

$$v=\left(0,0,0,0,0,0,-0.317,-0.911,-0.261\right)^{\mathrm{T}}.$$

Gradient of the function ϕ at x^0 is

$$\nabla\phi(x)\big|_{x=x^0}=\left(3.189,6.162,0.671,-8.537,-8.049,-1.607\right)^{\mathrm{T}}.$$

The vector $x^1=x^0-t\cdot\nabla\phi(x)\big|_{x=x^0}$ should be on the six dimensional unit sphere. Therefore $t=0.0425$ and

$$x^1=\left(0.7129,0.2620,0.0285,0.2143,-0.3422,0.5090\right)^{\mathrm{T}}.$$

After 9 steps, we get $\phi\left(x^9\right)<0$ where

$$x^9=\left(0.7950,0.2183,-0.0623,0.2185,-0.1254,0.5028\right)^{\mathrm{T}},$$

$$P\left(x^9\right)=\begin{pmatrix} 0.7950 & 0.2183 & -0.0623 \\ 0.2183 & 0.2185 & -0.1254 \\ -0.0623 & -0.1254 & 0.5028 \end{pmatrix}.$$

$P\left(x^9\right)$ is a common positive definite solution for $A_1^{\mathrm{T}}P\left(x^9\right)+P\left(x^9\right)A_1<0$ and $A_2^{\mathrm{T}}P\left(x^9\right)+P\left(x^9\right)A_2<0$.

The same problem solved by the algorithm from [8] gives answer only after 70 steps. We have solved a number of examples using the above gradient algorithm and by the algorithm from [8]. These examples show that this algorithm is faster than the algorithm from [8] in some cases.

As the comparison with the algorithm from [8] is concerned, the algorithm from [8] at each step uses the gradient only one maximum eigenvalue function, *i.e.* at 1 step it uses the gradient of $P\to\lambda_{\max}\left(A_1^{\mathrm{T}}P+PA_1\right)$, at 2 step the gradient of $P\to\lambda_{\max}\left(A_2^{\mathrm{T}}P+PA_2\right)$ and so on. This procedure delays the convergence. In our algorithm we use the function $P\to\max_i\left(\lambda_{\max}\left(A_i^{\mathrm{T}}P+PA_i\right)\right)$ and the corresponding gradient direction decreases the greates maximum eigenvalue.

On the other hand an obviously advantage of the method from [8] is the choose of the step size, which is given by an exact formula, whereas our step size is determined by the intersection of the corresponding rays with the unit sphere.

3. Sufficient Condition for a Stable Member

In this section we consider a sufficient condition for a stable member which is obtained by using Bendixson's theorem.

If a matrix is symmetric then it is stable if and only if it is negative definite. Therefore if a family consists of symmetric matrices then searching for stable element is equivalent to the searching for negative definite one.

On the other hand every real $n\times n$ matrix A can be decomposed

$$A=B+C,$$
$$B=\frac{1}{2}\left(A+A^{\mathrm{T}}\right),$$
$$C=\frac{1}{2}\left(A+A^{\mathrm{T}}\right).$$

where B is symmetric and C is skew-symmetric. Bendixson's theorem gives important inequalities for the eigenvalues of A, B and C.

Theorem 1. ([9], p. 40) If A is an $n\times n$ matrix, $B=\frac{1}{2}\left(A+A^{\mathrm{T}}\right)$ and $\lambda_1,\lambda_2,\cdots,\lambda_n$ $\left(\left|\lambda_1\right|\ge\left|\lambda_2\right|\ge\cdots\ge\left|\lambda_n\right|\right)$, $\mu_1\ge\mu_2\ge\cdots\ge\mu_n$ are the eigenvalues of A, B then

$$\mu_n\le\operatorname{Re}\left(\lambda_i\right)\le\mu_1\quad\left(i=1,2,\cdots,n\right).$$

Bendixson's theorem leads to the following.

Proposition 2. Let the family $\left\{A(q):q\in R\right\}$ be given and $B(q)$ is the symmetric part of $A(q)$. Then

1) If there exists $q_*\in R$ such that $B\left(q_*\right)$ is Hurwitz stable then $A\left(q_*\right)$ is also Hurwitz stable,

2) If there exists $q_*\in R$ such that $B\left(q_*\right)$ is positive stable (all eigenvalues lie in the open right half plane) then $A\left(q_*\right)$ is also positive stable.

Proposition 2 gives a sufficient condition for the existence of a stable element.

In the case of affine family

$$A(q)=A_0+q_1A_1+q_2A_2+\cdots+q_lA_l$$

where $q=\left(q_1,q_2,\cdots,q_l\right)^{\mathrm{T}}\in R$, R is a box or $R=\mathbb{R}^l$, the searching procedure for stable element in $B(q)$ can be effectively solved by powerful tools of Linear Matrix Inequalities (Matlab's LMI Toolbox).

In the non-affine case of the family $A(q)$ the gradient algorithm for a stable element in $B(q)$ is applicable.

Example 2. Consider affine family

$$A(q)=\begin{pmatrix} 6-3q_1-q_2-q_3 & 2+q_1-4q_3 & -2-5q_1-q_2-q_3 \\ 5+q_1+3q_2-q_3 & 8-2q_1-2q_2+2q_3 & 3+q_1-3q_3 \\ 5+5q_1-q_2+2q_3 & -4q_1-5q_2+q_3 & -2q_1-q_2 \end{pmatrix}$$

$q_i \in [-10,10] \quad (i=1,2,3)$. Then

$$B(q)=\begin{pmatrix} 6-3q_1-q_2-q_3 & (7+2q_1+3q_2-5q_3)/2 & (3-2q_2+q_3)/2 \\ (7+2q_1+3q_2-5q_3)/2 & 8-2q_1-2q_2+2q_3 & (3-3q_1-5q_2-2q_3)/2 \\ (3-2q_2+q_3)/2 & (3-3q_1-5q_2-2q_3)/2 & -2q_1-q_2 \end{pmatrix}.$$

LMI method applied to the matrix inequality problem $B(q)<0$ gives the value within a few seconds

$$q_*=(9.4591,-3.5180,-0.0354)^{\mathrm{T}}$$

and $B(q_*)$, and consequently $A(q_*)$ is stable.

LMI method applied to the inequality $B(q)>0$ gives also

$$\tilde{q}=(-2.6549,1.3609,0.9393)^{\mathrm{T}}$$

so the family $A(q)$ contains positive stable matrix $A(\tilde{q})$.

We have investigated Example 2 by the algorithm from [10] and positive answer is obtained after about 100 seconds.

Example 3. Consider non-affine family

$$A(q)=\begin{pmatrix} q_1q_2+q_2-2 & -q_1q_3+q_1-q_3-9 & -3q_2q_3+3q_2+3q_3-10 \\ -17-q_1+q_3 & -q_1-4 & q_2-4 \\ q_1+5 & q_1+11 & q_2-6 \end{pmatrix}$$

$q_i \in [-10,10] \quad (i=1,2,3)$. Here

$$B(q)=\begin{pmatrix} q_1q_2+q_2-2 & -\frac{q_1q_3}{2}-13 & \frac{q_1-3q_2(q_3-1)+3q_3-5}{2} \\ -\frac{q_1q_3}{2}-13 & -q_1-4 & \frac{q_1+q_2+7}{2} \\ \frac{q_1-3q_2(q_3-1)+3q_3-5}{2} & \frac{q_1+q_2+7}{2} & q_2-6 \end{pmatrix}.$$

Consider the function

$$G(q)=\lambda_{\max}(B(q))=\max_{\|v\|=1} v^{\mathrm{T}}B(q)v.$$

We are looking for q satisfying $G(q)<0$. If for some q the maximal eigenvalue $\lambda_{\max}(B(q))$ is simple then $G(q)$ is differentiable at q and its gradient can be easily calculated (by the analogy with (4)).

For this example, gradient method gives solution after 7 steps:

$$q^0=(0,0,0)^{\mathrm{T}},\ \cdots,\ q^7=(5.270,-6.252,0.959)^{\mathrm{T}}$$

(see **Table 1**). The step size *t* is chosen from the decreasing condition of the function $G(q)$: *t* must be chosen such that

$$G(q^{k+1})=G\left(q^k-t\nabla G|_{q^k}\right)<G(q^k).$$

This example has been solved by the algorithm from [10] as well. Positive answer has been obtained only after

Table 1. Gradient algorithm for example 3.

k	q^k	$\lambda_{\max}$	multiplicity	$\nabla G\|_{q^k}$
0	$(0,0,0)$	11.079	1	$(-0.452, 0.208, -0.508)$
1	$(0.411, -0.189, 0.462)$	10.632	1	$(-0.332, 0.655, -0.355)$
2	$(0.714, -0.785, 0.786)$	9.910	1	$(-0.482, 0.930, -0.383)$
3	$(1.153, -1.632, 1.135)$	8.634	1	$(-0.719, 1.173, -0.184)$
4	$(1.808, -2.700, 1.303)$	6.712	1	$(-1.061, 1.303, 0.291)$
5	$(2.774, -3.886, 1.038)$	3.840	1	$(-1.391, 1.360, 0.060)$
6	$(4.040, -5.123, 0.983)$	0.444	1	$(-1.352, 1.240, 0.267)$
7	$(5.270, -6.252, 0.959)$	−2.404		

55 steps. We start with $q^0 = (0,0,0)$ and the algorithm from [10] gives another stabilizing point

$$q^{55} = (3.2721, -2.3853, 2.3818)^{\mathrm{T}}.$$

The eigenvalues of $A(q^{55})$ are $\lambda_1 = -27.8402$, $\lambda_{2,3} = -0.004 \pm j0.2326$.

4. Conclusion

In the first part of the paper, we consider the stability problem of a matrix polytope through common quadratic Lyapunov functions. We suggest a modified gradient algorithm. In the second part by using Bendixson's theorem a sufficient condition for a stable member is given.

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