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Ground States for a Class of Nonlinear Schrodinger-Poisson Systems with Positive Potential

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Abstract

Based on Nehari manifold, Schwarz symmetric methods and critical point theory, we prove the existence of positive radial ground states for a class of Schrodinger-Poisson systems in $\mathbb{R}^3$, which doesn't require any symmetry assumptions on all potentials. In particular, the positive potential is interesting in physical applications.

Keywords

Ground States, Schrodinger-Poisson Systems

1. Introduction

In this paper, we consider the following nonlinear Schrodinger-Poisson systems

$$\begin{cases} -\Delta u + V(x)u - \lambda\rho(x)\Phi u + Q(x)|u|^{p-2}u = 0, & x \in \mathbb{R}^3, \\ -\Delta\Phi = \rho(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $2 < p < 4$; $V(x)$, $\rho(x)$ and $Q(x)$ are positive potentials defined in $\mathbb{R}^3$.

In recent years, such systems have been paid great attention by many authors concerning existence, non-existence, multiplicity and qualitative behavior. The systems are to describe the interaction of nonlinear Schrodinger field with an electromagnetic field. When $\lambda = -1$, $V(x) = \rho(x) = 1$, $Q(x) = -1$, the existence of nontrivial solution for the problem (1.1) was proved as $p \in (4,6)$ in [1], and non-existence result for $p \in (0,2]$ or $p \in (6,+\infty)$ was proved in [2]. When $\lambda < 0$, $V(x) = \rho(x) = 1$, $Q(x) = -1$, using critical point theory,

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Ruiz [3] obtained some multiplicity results for $p\in(2,3)$, and existence results for $p\in[3,6)$. Later, Ambrosetti and Ruiz [4], and Ambrosetti [5] generalized some existence results of Ruiz [3], and obtained the existence of infinitely solutions for the problem (1.1).

In particular, Sanchel and Soler [6] considered the following Schrodinger-Poisson-Slater systems

$$\begin{cases}-\Delta u+\omega u+\Phi u-|u|^{\frac{2}{3}}u=0, & x\in\mathbb{R}^3,\\ -\Delta\Phi=u^2, & x\in\mathbb{R}^3,\end{cases}\tag{1.2}$$

where $\omega\in\mathbb{R}$. The problem (1.2) was introduced as the model of the Hartree-Foch theory for a one-component plasma. The solution is obtained by using the minimization argument and ω as a Lagrange multiplier. However, it is not known if the solution for the problem (1.2) is radial. Mugani [7] considered the following generalized Schrodinger-Poisson systems

$$\begin{cases}-\Delta u+\omega u-\lambda\Phi u+W_u(x,u)=0, & x\in\mathbb{R}^3,\\ -\Delta\Phi=u^2, & x\in\mathbb{R}^3,\end{cases}\tag{1.3}$$

where $\omega\in\mathbb{R}$, $\lambda>0$ and $W(x,s)=W(|x|,s)$, and proved the existence of radially symmetric solitary waves for the problem (1.3).

In this paper, without requiring any symmetry assumptions on $V(x)$, $\rho(x)$ and $Q(x)$, we obtain the existence of positive radial ground state solution for the problem (1.1). In particular, the positive potential $Q(x)$ implies that we are dealing with systems of particles having positive mass. It is interesting in physical applications.

The paper is organized as following. In Section 2, we collect some results and state our main result. In Section 3, we prove some lemmas and consider the problem (1.1) at infinity. Section 4 is devoted to our main theorem.

2. Preliminaries and Main Results

Let $L^s(\mathbb{R}^3)$, $1\le s<+\infty$ denotes a Lebesgue space, the norm in $L^s(\mathbb{R}^3)$ is $|u|_s=\left(\int_{\mathbb{R}^3}|u|^s\mathrm{d}x\right)^{\left(\frac{1}{s}\right)}$, $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{1,2}}=\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^{\left(\frac{1}{2}\right)}$$

$H^1(\mathbb{R}^3)$ be the usual Sobolev space with the usual norm

$$\|u\|_{H^1}=\left(\int_{\mathbb{R}^3}\left(|\nabla u|^2+|u|^2\right)\mathrm{d}x\right)^{\left(\frac{1}{2}\right)}.$$

Assume that the potential $V(x)$ satisfies

$H1)$ $V(x)\in C^1(\mathbb{R}^3,\mathbb{R})$, $\inf_{x\in\mathbb{R}^3}V(x)=1$, $V(x)\le V_\infty=\lim_{|x|\to\infty}V(x)<\infty$.

Let $H_V^1(\mathbb{R}^3)$ be the Hilbert subspace of $u\in H^1(\mathbb{R}^3)$ such that

$$\|u\|_{H_V^1}=\left(\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x\right)^{\left(\frac{1}{2}\right)}\tag{2.1}$$

Then $H_V^1(\mathbb{R}^3)\subset H^1(\mathbb{R}^3)\subset L^s(\mathbb{R}^3)$, $2\le s\le 6$ with the corresponding embeddings being continuous (see [8]). Furthermore, assume the potential $\rho(x)$ satisfies

$H2)$ $\rho(x)>0$, $\lim_{|x|\to+\infty}\rho(x)=\rho_\infty>0$, $\rho_0(x)=\rho(x)-\rho_\infty\in L^2(\mathbb{R}^3)$.

It is easy to reduce the problem (1.1) to a single equation with a non-local term. Indeed, for every $v \in D^{1,2}\left(\mathbb{R}^3\right)$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \rho(x) u^2 v \mathrm{d}x\right| &= \left|\int_{\mathbb{R}^3} \left(\rho(x)-\rho_\infty\right) u^2 v \mathrm{d}x + \int_{\mathbb{R}^3} \rho_\infty u^2 v \mathrm{d}x\right| \\ &\le \int_{\mathbb{R}^3} \left|\rho(x)-\rho_\infty\right| u^2 |v| \mathrm{d}x + \int_{\mathbb{R}^3} \rho_\infty u^2 |v| \mathrm{d}x \\ &\le \left(\int_{\mathbb{R}^3} \left(\left|\rho_0(x)\right| u^2\right)^{\left(\frac{6}{5}\right)} \mathrm{d}x\right)^{\left(\frac{6}{5}\right)} \left(\int_{\mathbb{R}^3} (v)^6 \mathrm{d}x\right)^{\left(\frac{1}{6}\right)} + \rho_\infty \left(\int_{\mathbb{R}^3} \left(u^2\right)^{\left(\frac{6}{5}\right)} \mathrm{d}x\right)^{\left(\frac{6}{5}\right)} \left(\int_{\mathbb{R}^3} (v)^6 \mathrm{d}x\right)^{\left(\frac{1}{6}\right)} \\ &\le |v|_6 \left[\left|\rho_0(x)\right|_2 \left(\int_{\mathbb{R}^3} \left(u^2\right)^3 \mathrm{d}x\right)^{\left(\frac{1}{3}\right)} + \rho_\infty \left(\int_{\mathbb{R}^3} \left(u^2\right)^{\left(\frac{6}{5}\right)} \mathrm{d}x\right)^{\left(\frac{6}{5}\right)}\right].\end{aligned} \tag{2.2}$$

Since $\rho_0(x) = \rho(x) - \rho_\infty \in L^2\left(\mathbb{R}^3\right)$, $u \in H^1\left(\mathbb{R}^3\right)$ and (2.1), by the Lax-Milgram theorem, there exists a unique $\Phi[u]$ such that

$$\int_{\mathbb{R}^3} \nabla\Phi[u] \nabla v \mathrm{d}x = \int_{\mathbb{R}^3} \rho(x) u^2 v \mathrm{d}x, \quad \forall v \in D^{1,2}\left(\mathbb{R}^3\right) \tag{2.3}$$

It follows that $\Phi[u]$ satisfies the Poisson equation

$$-\Delta\Phi[u] = \rho(x) u^2$$

and there holds

$$-\Phi[u] = \int_{\mathbb{R}^3} \frac{\rho(x) u^2(y)}{|x-y|} \mathrm{d}y = \frac{1}{|x|} * \rho(x) u^2$$

Because $\rho(x) > 0$, we have $\Phi[u] > 0$ when $u \neq 0$, and $\|\Phi[u]\|_{D^{1,2}} = M \|u\|_{H^1}^2$, M is positive constant. Substituting $\Phi[u]$ in to the problem (1.1), we are lead to the equation with a non-local term

$$-\Delta u + V(x) u - \lambda \rho(x) \Phi[u] u + Q(x) |u|^{p-2} u = 0. \tag{2.4}$$

In the following, we collect some properties of the functional $\Phi[u]$, which are useful to study our problem.

Lemma 2.1. [9] For any $u \in H^1\left(\mathbb{R}^3\right)$, we have

1) $\Phi[u]: H^1\left(\mathbb{R}^3\right) \to D^{1,2}\left(\mathbb{R}^3\right)$ is continuous, and maps bounded sets into bounded sets;

2) if $u_n \to u$ weakly in $H^1\left(\mathbb{R}^3\right)$, then $\Phi[u_n] \to \Phi[u]$ weakly in $D^{1,2}\left(\mathbb{R}^3\right)$;

3) $\Phi[tu(x)] = t^2 \Phi[u(x)]$ for all $t \in \mathbb{R}$.

Now, we state our main theorem in this paper.

Theorem 2.2. Assume that $\lambda > 0$, $2 < p < 4$, the potential $V(x)$ satisfies condition $H1$), the potential $\rho(x)$ satisfies condition $H3$) and $\rho(x) \ge \rho_\infty$, the potential $Q(x)$ satisfies

$H3$) $Q(x) > 0$, $\lim_{|x| \to +\infty} Q(x) = Q_\infty > 0$, $Q_0(x) = Q(x) - Q_\infty \in L^{\frac{6}{6-p}}\left(\mathbb{R}^3\right)$

and $Q(x) \le Q_\infty$, $Q(x) - Q_\infty < 0$ on positive measure. Then there exists a positive radial ground state solution for the problem (1.1).

Remark 2.3. If $\lambda \le 0$, $V(x)$, $\rho(x)$ and $Q(x)$ are positive potentials defined in $\mathbb{R}^3$, and $2 < p < 6$, $(u, \Phi) \in H_V^1\left(\mathbb{R}^3\right) \times D^{1,2}\left(\mathbb{R}^3\right)$ be a solution for the problem (1.1). Then $(u, \Phi) = (0, 0)$, Indeed, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x) u^2\right) \mathrm{d}x - \lambda \int_{\mathbb{R}^3} \rho(x) \Phi[u] u^2 \mathrm{d}x + \int_{\mathbb{R}^3} Q(x) |u|^p \mathrm{d}x \\ &\ge \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x) u^2\right) \mathrm{d}x. \end{aligned}$$

Since $V(x)>0$, this implies $u=0$. By Lemma 2.1, we have $\Phi=0$.

3. Some Lemmas and the Problem (1.1) at Infinity

Now, we consider the functional $I_\lambda : H_V^1(\mathbb{R}^3)\to\mathbb{R}$ given by

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)\mathrm{d}x-\frac{\lambda}{4}\int_{\mathbb{R}^3}\rho(x)\Phi[u]u^2\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^3}Q(x)|u|^p\mathrm{d}x \\ &= \frac{1}{2}\|u\|_{H_V^1}^2-\frac{\lambda}{4}\int_{\mathbb{R}^3}\rho(x)\Phi[u]u^2\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^3}Q(x)|u|^p\mathrm{d}x. \end{aligned}$$

Since $\rho(x)$ satisfies condition $H2$), by (2.2), the Holder inequality and Sobolev inequality, we have

$$\int_{\mathbb{R}^3}\rho(x)\Phi[u]u^2\mathrm{d}x\le \bar{S}^{-1}\left[S_6^{-2}|\rho|_2+\rho_\infty S_{\left(\frac{12}{5}\right)}^{-2}\right]\|u\|_{H_V^1}^2\|v\|_{D^{1,2}}, \tag{3.2}$$

where $\bar{S}=\inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}}\frac{\|u\|_{D^{1,2}}}{|u|_6}$ and $S=\inf_{u\in H_V^1(\mathbb{R}^3)\setminus\{0\}}\frac{\|u\|_{H_V^1}}{|u|_6}$. Since the potential $Q(x)$ satisfies condition Q, $2<p<4$, we have

$$\begin{aligned} \int_{\mathbb{R}^3}Q(x)|u|^p\mathrm{d}x &\le \left|\int_{\mathbb{R}^3}\left(Q(x)-Q_\infty\right)|u|^p\mathrm{d}x+\int_{\mathbb{R}^3}Q_\infty|u|^p\mathrm{d}x\right| \\ &\le \int_{\mathbb{R}^3}|Q_0(x)||u|^p\mathrm{d}x+\int_{\mathbb{R}^3}Q_\infty|u|^p\mathrm{d}x \\ &\le \left(\int_{\mathbb{R}^3}|Q_0(x)|^{\left(\frac{6}{6-p}\right)}\mathrm{d}x\right)^{\left(\frac{6-p}{6}\right)}\left(\int_{\mathbb{R}^3}(u)^6\mathrm{d}x\right)^{\left(\frac{p}{6}\right)}+Q_\infty\int_{\mathbb{R}^3}|u|^p\mathrm{d}x. \end{aligned}$$

By Sobolev inequality, we obtain that

$$\int_{\mathbb{R}^3}Q(x)|u|^p\mathrm{d}x\le M\|u\|_{H_V^1}^p \tag{3.3}$$

Combining (3.2) and (3.3), we obtain that the functional I_λ is a well defined C^1 functional, and if $u\in H_V^1(\mathbb{R}^3)$ is critical point of it, then the pair $(u,\Phi[u])$ is a weak solution of the problem (1.1).

Now, we define the Nehari manifold ([10]) of the functional I_λ

$$N_\lambda=\left\{u\in H_V^1(\mathbb{R}^3)\setminus\{0\}:H_\lambda(u)=0\right\},$$

where

$$H_\lambda(u)=I'_\lambda(u)[u]=\|u\|_{H_V^1}^2-\lambda\int_{\mathbb{R}^3}\rho(x)\Phi[u]u^2\mathrm{d}x+\int_{\mathbb{R}^3}Q(x)|u|^p\mathrm{d}x$$

Hence, we have

$$\begin{aligned} I'_\lambda(u)\big|_{N_\lambda} &= \frac{1}{4}\|u\|_{H_V^1}^2+\left(\frac{1}{p}-\frac{1}{4}\right)\int_{\mathbb{R}^3}Q(x)|u|^p\mathrm{d}x \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|_{H_V^1}^2+\lambda\left(\frac{1}{p}-\frac{1}{4}\right)\int_{\mathbb{R}^3}\rho(x)\Phi[u]u^2\mathrm{d}x \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^3}Q(x)|u|^p\mathrm{d}x-\frac{\lambda}{4}\int_{\mathbb{R}^3}\rho(x)\Phi[u]u^2\mathrm{d}x. \end{aligned} \tag{3.4}$$

Lemma 3.1. 1) For any $\lambda > 0$, $u \in H_V^1(\mathbb{R}^3)\setminus\{0\}$, there exists a unique $t(u) > 0$ such that $t(u)u \in N_\lambda$. Moreover, we have $I_\lambda(t(u)u) = \max_{t\ge 0} I_\lambda(tu)$.

2) $I_\lambda(u)$ is bounded from below on N_λ by a positive solution.

Proof. 1) Taking any $u \in H_V^1(\mathbb{R}^3)\setminus\{0\}$ and $\|u\|_{H_V^1} = 1$, we obtain that there exists a unique $t(u) > 0$ such that $t(u)u \in N_\lambda$. Indeed, we define the function $g(t) = I_\lambda(tu)$. We note that $g'(t) = (I'_\lambda(tu), v) = 0$ if only if $tu \in N_\lambda$. Since $g'(t) = 0$ is equivalent to

$$t^2\|u\|_{H_V^1}^2 - \lambda t^4 \int_{\mathbb{R}^3} \rho(x)\Phi[u]u^2 \mathrm{d}x + t^p \int_{\mathbb{R}^3} Q(x)|u|^p \mathrm{d}x = 0.$$

By $\rho(x)$, $Q(x) > 0$ and $\Phi[u] > 0$, we have

$$b = \int_{\mathbb{R}^3} \rho(x)\Phi[u]u^2 \mathrm{d}x > 0, \quad c = \int_{\mathbb{R}^3} Q(x)|u|^p \mathrm{d}x > 0.$$

By $\lambda > 0$, $2 < p < 4$, the equation $1 - \lambda b t^2 + c t^{p-2} = 0$ has a unique $t(u) > 0$ and the corresponding point $t(u)u \in N_\lambda$ and $I_\lambda(t(u)u) = \max_{t\ge 0} I_\lambda(tu)$.

2) Let $u \in N_\lambda$, by (3.4) and $2 < p < 4$, we have

$$\begin{aligned} I_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{H_V^1}^2 + \lambda\left(\frac{1}{p} - \frac{1}{4}\right)\int_{\mathbb{R}^3} \rho(x)\Phi[u]u^2 \mathrm{d}x \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{H_V^1}^2 > C > 0. \end{aligned}$$

By the definition of Nehari manifold N_λ of the functional I_λ, we obtain that

u is a critical point of I_λ if and only if u is a critical point of I_λ constrained on N_λ. (3.5)

Now, we set

$$m_\lambda = \inf\{I_\lambda(u) : u \in N_\lambda\}$$

By 2) of Lemma 3.1, we have $m_\lambda > 0$.

Since $\lim_{|x|\to\infty} V(x) = V_\infty$, $\lim_{|x|\to+\infty} \rho(x) = \rho_\infty$, $\lim_{|x|\to+\infty} Q(x) = Q_\infty$, we consider the problem (1.1) at infinity

$$\begin{cases} -\Delta u + V_\infty u - \lambda\rho_\infty \Phi u + Q_\infty |u|^{p-2} u = 0, & x \in \mathbb{R}^3, \\ -\Delta\Phi = \rho_\infty u^2, & x \in \mathbb{R}^3. \end{cases} \tag{3.6}$$

Similar to (2.2), we obtain that there exists a unique $\tilde{\Phi}[u]$ such that

$$\int_{\mathbb{R}^3} \nabla\tilde{\Phi}[u]\nabla v \mathrm{d}x = \int_{\mathbb{R}^3} \rho_\infty u^2 v \mathrm{d}x, \quad \forall v \in D^{1,2}(\mathbb{R}^3).$$

It follows that $\tilde{\Phi}[u]$ satisfies the Poisson equation

$$-\Delta\tilde{\Phi}[u] = \rho_\infty u^2 \tag{3.7}$$

Hence substituting $\tilde{\Phi}[u]$ into the first equation of (3.6) we have to study the equivalent problem

$$-\Delta u + V_\infty u - \lambda\rho_\infty\tilde{\Phi}[u]u + Q_\infty |u|^{p-2} u = 0 \tag{3.8}$$

The weak solution of the problem (3.8) is the critical point of the functional

$$\begin{aligned} I_\lambda^\infty(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V_\infty u^2\right)\mathrm{d}x - \frac{\lambda}{4}\int_{\mathbb{R}^3} \rho_\infty\tilde{\Phi}[u]u^2 \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^3} Q_\infty |u|^p \mathrm{d}x \\ &= \frac{1}{2}\|u\|_{H_{V_\infty}^1}^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} \rho_\infty\tilde{\Phi}[u]u^2 \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^3} Q_\infty |u|^p \mathrm{d}x, \end{aligned}$$

where $H^1_{V_\infty}\left(\mathbb{R}^3\right)=H^1\left(\mathbb{R}^3\right)$ is endowed with the norm

$$\|u\|_{H^1_{V_\infty}}=\left(\int_{\mathbb{R}^3}\left(|\nabla u|^2+V_\infty u^2\right)\mathrm{d}x\right)^{\frac{1}{2}}$$

Define the Nehari manifold of the functional I^∞_λ

$$N^\infty_\lambda=\left\{u\in H^1_{V_\infty}\left(\mathbb{R}^3\right)\setminus\{0\}:H^\infty_\lambda(u)=0\right\},$$

where

$$H^\infty_\lambda(u)=I'^\infty_\lambda(u)[u]=\|u\|^2_{H^1_{V_\infty}}-\lambda\int_{\mathbb{R}^3}\rho_\infty\tilde{\Phi}[u]u^2\mathrm{d}x+\int_{\mathbb{R}^3}Q_\infty|u|^p\mathrm{d}x$$

and

$$m^\infty_\lambda=\inf\left\{I^\infty_\lambda(u):u\in N^\infty_\lambda\right\}>0$$

The Nehari manifold N^∞_λ has properties similar to those of N_λ.

Lemma 3.2. The problem (3.8) has a positive radial ground state solution $\omega_\infty\in N^\infty_\lambda$ such that

$$I^\infty_\lambda(\omega_\infty)=m^\infty_\lambda$$

For the proof of Lemma 3.2, we make use of Schwarz symmetric method. We begin by recalling some basic properties.

Let $f\in L^s\left(\mathbb{R}^3\right)$ such that $f\geq 0$, then there is a unique nonnegative function $f^*\in L^s\left(\mathbb{R}^3\right)$, called the Schwarz symmetric of f, such that it depends only on $|x|$, whose level sets

$$\left\{x\in\mathbb{R}^3:f(x)>t\right\}=\left\{x\in\mathbb{R}^3:f^*(x)>t\right\}.$$

We consider the following Poisson equation

$$-\Delta\phi=f\quad\text{and}\quad-\Delta v=f^*$$

From Theorem 1 of [11], we have

$$\int_{\mathbb{R}^3}|\nabla v|^s\mathrm{d}x\geq\int_{\mathbb{R}^3}|\nabla\phi|^s\mathrm{d}x,\quad\forall 0<s\leq 2.$$

Hence, let $\phi=\tilde{\Phi}[u]$, $f=\rho_\infty u^2$ and $v=\tilde{\Phi}\left[u^*\right]$, $f^*=\rho_\infty\left(u^*\right)^2$, we have

$$\int_{\mathbb{R}^3}\rho_\infty\tilde{\Phi}[u]u^2\mathrm{d}x\leq\int_{\mathbb{R}^3}\rho_\infty\tilde{\Phi}\left[u^*\right]\left(u^*\right)^2\mathrm{d}x.\tag{3.9}$$

The Proof of Lemma 3.2. Let $u_n\in N^\infty_\lambda$ be such that $I^\infty_\lambda(u_n)\to m^\infty_\lambda$. Let $t_n>0$ such that $t_n|u_n|\in N^\infty_\lambda$ then we have

$$\int_{\mathbb{R}^3}\left(|\nabla u_n|^2+V_\infty(u_n)^2\right)\mathrm{d}x-\lambda\int_{\mathbb{R}^3}\rho_\infty\tilde{\Phi}[u_n](u_n)^2\mathrm{d}x+\int_{\mathbb{R}^3}Q_\infty|u_n|^p\mathrm{d}x=0,$$

and

$$(t_n)^2\int_{\mathbb{R}^3}\left(|\nabla u_n|^2+V_\infty(u_n)^2\right)\mathrm{d}x-\lambda(t_n)^4\int_{\mathbb{R}^3}\rho_\infty\tilde{\Phi}[u_n](u_n)^2\mathrm{d}x+(t_n)^p\int_{\mathbb{R}^3}Q_\infty|u_n|^p\mathrm{d}x=0.$$

Hence, we obtain that

$$\left((t_n)^2-(t_n)^4\right)\int_{\mathbb{R}^3}\left(|\nabla u_n|^2+V_\infty(u_n)^2\right)\mathrm{d}x+\left((t_n)^p-(t_n)^4\right)\int_{\mathbb{R}^3}Q_\infty|u_n|^p\mathrm{d}x=0.\tag{3.10}$$

Since $t_n > 0$ and $2 < p < 4$, (3.10) implies that $t_n = 1$. Therefore, we can assume that $u_n \geq 0$. On the other hand, let $(u_n)^*$ be the Schwartz symmetric function associated to u_n, then we have

$$\int_{\mathbb{R}^3} |u_n|^p \mathrm{d}x = \int_{\mathbb{R}^3} \left|(u_n)^*\right|^p \mathrm{d}x, \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x \geq \int_{\mathbb{R}^3} \left|\nabla (u_n)^*\right|^2 \mathrm{d}x \tag{3.11}$$

Let $(t_n)^* > 0$ be such that $(t_n)^*(u_n)^* \in N_\lambda^\infty$, and $u_n \in N_\lambda^\infty$, by (3.9) and (3.11), we have

$$\begin{aligned} 0 &= \left[(t_n)^*\right]^2 \int_{\mathbb{R}^3} \left(\left|\nabla (u_n)^*\right|^2 + V_\infty \left[(u_n)^*\right]^2\right) \mathrm{d}x - \lambda \left[(t_n)^*\right]^4 \int_{\mathbb{R}^3} \rho_\infty \tilde{\Phi}\left[(u_n)^*\right]\left[(u_n)^*\right]^2 \mathrm{d}x + \left[(t_n)^*\right]^p \int_{\mathbb{R}^3} Q_\infty \left|(u_n)^*\right|^p \mathrm{d}x \\ &\leq \left[(t_n)^*\right]^2 \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V_\infty (u_n)^2\right) \mathrm{d}x - \lambda \left[(t_n)^*\right]^4 \int_{\mathbb{R}^3} \rho_\infty \tilde{\Phi}[u_n](u_n)^2 \mathrm{d}x + \left[(t_n)^*\right]^p \int_{\mathbb{R}^3} Q_\infty |u_n|^p \mathrm{d}x \\ &= \left\{\left[(t_n)^*\right]^2 - \left[(t_n)^*\right]^4\right\} \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V_\infty (u_n)^2\right) \mathrm{d}x + \left\{\left[(t_n)^*\right]^p - \left[(t_n)^*\right]^4\right\} \int_{\mathbb{R}^3} Q_\infty |u_n|^p \mathrm{d}x. \end{aligned}$$

This implies that $(t_n)^* \leq 1$. Therefore, we have $I_\lambda^\infty\left([u_n]^*\right) \leq I_\lambda^\infty(u_n)$, and we can suppose that u_n is radial in $H_{V_\infty}^1(\mathbb{R}^3)$. Since $H_{V_\infty,r}^1(\mathbb{R}^3)$ is compactly embedded into $L^p(\mathbb{R}^3)$ for $2 < p < 4$, we obtain that m_λ^∞ is achieved at some $\omega_\infty \in N_\lambda^\infty$ which is positive and radial. Therefore, Lemma 3.2 is proved.

4. The Proof of Main Theorem

In this section, we prove Theorem 2.2. Firstly, we consider a compactness result and obtain the behavior of the (PS) sequence of the functional I_λ.

Lemma 4.1. Let u_n be a $(\mathrm{PS})_d$ sequence of the functional I_λ constrained on N_λ, that is

$$u_n \in N_\lambda, \ I_\lambda(u_n) \to d \quad \text{and} \quad I'_\lambda(u_n)\big|_{N_\lambda} \to 0, \ \text{as } n \to \infty \tag{4.1}$$

Then there exists a solution u of the problem (2.4), a number $k \in \mathbb{N} \cup \{0\}$, k functions $u^1, u^2, \cdots, u^k$ of $H_V^1(\mathbb{R}^3)$ and k sequences of points y_n^j, $0 \leq j \leq k$ such that

1) $\left|y_n^j\right| \to +\infty$, $\left|y_n^j - y_n^i\right| \to +\infty$, if $i \neq j$, $n \to \infty$;

2) $u_n - \sum_{j=1}^k u^j\left(\cdot - y_n^j\right) \to \bar{u}$;

3) $I_\lambda(u_n) \to I_\lambda(\bar{u}) + \sum_{j=1}^k I_\lambda^\infty\left(u^j\right)$;

4) u^j are non-trivial weak solution of the problem (3.8).

Proof. The proof is similar to that of Lemma 4.1 in [9].

By Lemma 4.1, taking into account that $I_\lambda^\infty\left(u^j\right) \geq m_\lambda$ for all j and $d \in (0, m_\lambda)$, we obtain that $k = 0$ and $u_n \to \bar{u}$ in $H_V^1(\mathbb{R}^3)$ (strongly), *i.e.* u_n is relatively compact for all $d \in (0, m_\lambda)$. Hence we only need to prove that the energy of a solution of the problem (2.4) cannot overcome the energy of a ground state solution of the problem (3.8).

The proof of Theorem 2.2. By Lemma 4.1, we only prove that $m_\lambda < m_\lambda^\infty$. Indeed, let $\omega_\infty \in N_\lambda^\infty$ such that $I_\lambda^\infty(\omega_\infty) = m_\lambda^\infty$, and let $t > 0$ such that $t\omega_\infty \in N_\lambda$. Since $V(x) \leq V_\infty$, $\rho(x) \geq \rho_\infty$ and $Q(x) \leq Q_\infty$, we have

$$\begin{aligned} m_\lambda \leq I_\lambda(t\omega_\infty) &= \frac{t^2}{2} \int_{\mathbb{R}^3} \left(\left|\nabla \omega_\infty^2\right| + V(x)\omega_\infty^2\right) \mathrm{d}x - \frac{\lambda t^4}{4} \int_{\mathbb{R}^3} \rho(x) \Phi[\omega_\infty] \omega_\infty^2 \mathrm{d}x + \frac{t^p}{p} \int_{\mathbb{R}^3} Q(x) |\omega_\infty|^p \mathrm{d}x \\ &\leq \frac{t^2}{2} \int_{\mathbb{R}^3} \left(|\nabla \omega_\infty|^2 + V_\infty \omega_\infty^2\right) \mathrm{d}x - \frac{\lambda t^4}{4} \int_{\mathbb{R}^3} \rho_\infty \tilde{\Phi}[\omega_\infty] \omega_\infty^2 \mathrm{d}x + \frac{t^p}{p} \int_{\mathbb{R}^3} Q_\infty |\omega_\infty|^p \mathrm{d}x. \end{aligned} \tag{4.2}$$

Since $\omega_\infty \in N_\lambda^\infty$ and $t\omega_\infty \in N_\lambda$, we have

$$
\begin{aligned}
t^4 \int_{\mathbb{R}^3} \left(|\nabla \omega_\infty|^2 + V_\infty \omega_\infty^2 \right) dx + t^4 \int_{\mathbb{R}^3} Q_\infty |\omega_\infty|^p dx &= \lambda t^4 \int_{\mathbb{R}^3} \rho_\infty \tilde{\Phi}[\omega_\infty] \omega_\infty^2 dx \\
&\le \lambda t^4 \int_{\mathbb{R}^3} \rho(x) \Phi[\omega_\infty] \omega_\infty^2 dx \\
&= t^2 \int_{\mathbb{R}^3} \left(|\nabla \omega_\infty^2| + V(x) \omega_\infty^2 \right) dx + t^p \int_{\mathbb{R}^3} Q(x) |\omega_\infty|^p dx \\
&\le t^2 \int_{\mathbb{R}^3} \left(|\nabla \omega_\infty|^2 + V_\infty \omega_\infty^2 \right) dx + t^p \int_{\mathbb{R}^3} Q_\infty |\omega_\infty u|^p dx.
\end{aligned}
$$

Therefore, we have

$$
\left(t^4 - t^2 \right) \int_{\mathbb{R}^3} \left(|\nabla \omega_\infty|^2 + V_\infty \omega_\infty^2 \right) dx + \left(t^4 - t^p \right) \int_{\mathbb{R}^3} Q_\infty |\omega_\infty u|^p dx \le 0
$$

By $2 < p < 4$, we have $t \le 1$. If $t = 1$, we have $\omega_\infty \in N_\lambda^\infty$ and $\omega_\infty \in N_\lambda$. Hence, by $\omega_\infty \in N_\lambda^\infty$, we have

$$
\int_{\mathbb{R}^3} \left(|\nabla \omega_\infty|^2 + V_\infty \omega_\infty^2 \right) dx + \int_{\mathbb{R}^3} Q_\infty |\omega_\infty|^p dx = \lambda \int_{\mathbb{R}^3} \rho_\infty \tilde{\Phi}[\omega_\infty] \omega_\infty^2 dx \tag{4.3}
$$

and by $\omega_\infty \in N_\lambda$, we have

$$
\int_{\mathbb{R}^3} \left(|\nabla \omega_\infty|^2 + V(x) \omega_\infty^2 \right) dx + \int_{\mathbb{R}^3} Q(x) |\omega_\infty|^p dx = \lambda \int_{\mathbb{R}^3} \rho(x) \Phi[\omega_\infty] \omega_\infty^2 dx. \tag{4.4}
$$

Combining (4.3) and (4.4), we have

$$
\int_{\mathbb{R}^3} \left(V_\infty - V(x) \right) \omega_\infty^2 dx + \int_{\mathbb{R}^3} \left(Q_\infty - Q(x) \right) |\omega_\infty|^p dx - \lambda \int_{\mathbb{R}^3} \rho_\infty \tilde{\Phi}[\omega_\infty] \omega_\infty^2 dx + \lambda \int_{\mathbb{R}^3} \rho(x) \Phi[\omega_\infty] \omega_\infty^2 dx = 0
$$

Since $V(x) \le V_\infty$, $\rho(x) \ge \rho_\infty$, $Q(x) \le Q_\infty$, and $Q(x) - Q_\infty < 0$ on a positive measure, we have

$$
\int_{\mathbb{R}^3} \left(Q_\infty - Q(x) \right) |\omega_\infty|^p dx
$$

which is not identically zero, and is contradiction. Hence, we have $t < 1$. By (4.2), we have

$$
\begin{aligned}
m_\lambda &< \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla \omega_\infty|^2 + V_\infty \omega_\infty^2 \right) dx - \frac{\lambda}{4} \int_{\mathbb{R}^3} \rho_\infty \tilde{\Phi}[\omega_\infty] \omega_\infty^2 dx + \frac{1}{p} \int_{\mathbb{R}^3} Q_\infty |\omega_\infty|^p dx \\
&= I_\lambda^\infty(\omega_\infty) = m_\lambda^\infty.
\end{aligned}
$$

Then there exists a positive radial ground state solution for the problem (1.1).

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