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Integral Inequalities of Gronwall-Bellman Type

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Abstract

The goal of the present paper is to establish some new approach on the basic integral inequality of Gronwall-Bellman type and its generalizations involving function of one independent variable which provides explicit bounds on unknown functions. The inequalities given here can be used as tools in the qualitative theory of certain partial differential and integral equations.

Keywords

Integral Inequalities, One Independent Variable, Partial Differential Equations, Nondecreasing, Nonincreasing

1. Introduction

The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of the various types. Some applications of this result can be used to the study of existence, uniqueness theory of differential equations and the stability of the solution of linear and nonlinear differential equations. During the past few years, several authors have established several Gronwall type integral inequalities in one or two independent real variables [1]-[15]. Of course, such results have application in the theory of partial differential equations and Volterra integral equations.

Closely related to the foregoing first-order ordinary differential operators is the following result of Bellman [11]: If the functions $g(t)$ and $u(t)$ are nonnegative for $t \ge 0$, and if $c \ge 0$, the inequality

$$u(t) \le c + \int_0^t g(s)u(s)\mathrm{d}s, \quad t \ge 0$$

implies that

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$$u(t) \le c\exp\left(\int_0^t g(s)\,\mathrm{d}s\right), \quad t \ge 0 \tag{1.1}$$

Our aim in this paper is to establish new explicit bounds on some basic integral inequalities of one independent variable which will be equally important in handling the inequality (1.1). Given application in this paper is also illustrating the usefulness of our result.

2. Main Results

Lemma 2.1: Let $u(t)$ and $g(t)$ be nonnegative continuous functions defined for $I = [0,\infty)$. Let $k(t) > 1$ defined for $I = [0,\infty)$ and also $k'(t)$ be nonnegative continuous functions defined for $I = [0,\infty)$. If

$$u(t) \le k(t) + \int_0^t g(s)u(s)\,\mathrm{d}s, \quad \forall t \in I \tag{2.1}$$

Then

$$u(t) \le k(0)\exp\left(k(t) - k(0) + \int_0^t g(s)\,\mathrm{d}s\right), \quad \forall t \in I \tag{2.2}$$

Proof: Define a function $m(t)$ by the right-hand side of (2.1), such that

$$m(t) = k(t) + \int_0^t g(s)u(s)\,\mathrm{d}s \tag{2.3}$$

where

$$m(0) = k(0) \tag{2.4}$$

Then $m(t) > 1$. From (2.1) and (2.3), we observe that

$$u(t) \le m(t) \tag{2.5}$$

Differentiating both sides of (2.3) with respect to *t*, we get

$$m'(t) = k'(t) + g(t)u(t)$$

By using (2.5) and since $k(t) > 1$, the above equation can be restated as

$$\frac{m'(t)}{m(t)} \le k'(t) + g(t) \tag{2.6}$$

Integrating both sides of (2.6) from 0 to *t* and also using (2.4), we observe that

$$m(t) \le k(0)\exp\left(k(t) - k(0) + \int_0^t g(s)\,\mathrm{d}s\right) \tag{2.7}$$

From (2.5) and (2.7), we get the required inequality (2.2).

Theorem 2.2: Let $u(t)$, $f(t)$ and $g(t)$ be nonnegative continuous functions defined for $I = [0,\infty)$. Let $k(t) > 1$ defined for $I = [0,\infty)$ and also $k'(t)$ be nonnegative continuous functions defined for $I = [0,\infty)$. If

$$u(t) \le k(t) + \int_0^t f(s)u(s)\,\mathrm{d}s + \int_0^t f(s)\left(\int_0^s g(\partial)u(\partial)\,\mathrm{d}\partial\right)\mathrm{d}s, \quad \forall t \in I \tag{2.8}$$

Then

$$u(t) \le k(t) + k(0)\int_0^t f(s)\exp\left(k(s) - k(0) + \int_0^s \left(f(\partial) + g(\partial)\right)\mathrm{d}\partial\right)\mathrm{d}s, \quad \forall t \in I \tag{2.9}$$

Proof: Define a function $m(t)$ by the right-hand side of (2.8), such that

$$m(t)=k(t)+\int_0^t f(s)u(s)\mathrm{d}s+\int_0^t f(s)\left(\int_0^s g(\partial)u(\partial)\mathrm{d}\partial\right)\mathrm{d}s,\quad \forall t\in I \tag{2.10}$$

where

$$m(0)=k(0) \tag{2.11}$$

Then $m(t)>1$. From (2.9) and (2.10), we observe that

$$u(t)\le m(t) \tag{2.12}$$

Differentiating both sides of (2.10) with respect to *t*, we get

$$m'(t)=k'(t)+f(t)\left[u(t)+\int_0^t g(s)u(s)\mathrm{d}s\right]$$

By using (2.12), the above equation can be restated as

$$m'(t)\le k'(t)+f(t)v(t) \tag{2.13}$$

where

$$v(t)=m(t)+\int_0^t g(s)m(s)\mathrm{d}s \tag{2.14}$$

and

$$v(0)=m(0)=k(0) \tag{2.15}$$

Again differentiating both sides of (2.14) with respect to *x* and using (2.13) and using the fact that $m(t)\le v(t)$, we get

$$\frac{v'(t)}{v(t)}\le k'(t)+\left[f(t)+g(t)\right] \tag{2.16}$$

By applying Lemma 2.1 implies the estimation of $v(t)$ as

$$v(t)\le k(0)\exp\left(k(t)-k(0)+\int_0^t\left[f(0)+g(0)\right]\mathrm{d}s\right) \tag{2.17}$$

By substituting (2.17) in (2.13), we have

$$m'(t)\le k'(t)+k(0)f(t)\exp\left(k(t)-k(0)+\int_0^t\left[f(s)+g(s)\right]\mathrm{d}s\right)$$

Integrating both sides of the above inequality from 0 to *t* and also using (2.11), we observe that

$$m(t)\le k(t)+k(0)\int_0^t f(s)\exp\left(k(s)-k(0)+\int_0^s\left[f(\partial)+g(\partial)\right]\mathrm{d}\partial\right)\mathrm{d}s \tag{2.18}$$

From (2.12) and (2.18), we get the required inequality (2.9). This completes the proof.

Theorem 2.3: Let $u(t)$, $f(t)$ and $g(t), k(t)$ and $k'(t)$ be defined as in Theorem 2.2. If

$$u(t)\le k(t)+\int_0^t f(s)u(s)\mathrm{d}s+\int_0^t f(s)\left(\int_0^s f(\tau)\left(\int_0^\tau g(\partial)u(\partial)\mathrm{d}\partial\right)\mathrm{d}\tau\right)\mathrm{d}s,\quad \forall t\in I \tag{2.19}$$

Then

$$u(t)\le k(t)+\int_0^t f(s)\left[k(s)+k(0)\int_0^s f(\tau)\exp\left(k(\tau)-k(0)+\int_0^\tau\left(f(\partial)+g(\partial)\right)\mathrm{d}\partial\right)\mathrm{d}\tau\right]\mathrm{d}s,\quad \forall t\in I$$

Proof: The proof of Theorem 2.3 is the same as the proof of Theorem 2.2 and by applying the Lemma 2.1

with suitable modifications.

3. Application

As an application, let us consider the bound for the solution of Volterra integral equation of the form

$$x(t) = f(t) + \int_0^t p(t,s) g(t, x(s), Tx(s)) \mathrm{d}s \tag{3.1}$$

where x, f and g are the elements of R^n, $p(t,s)$ is a $n \times n$ matrix, $g \in C\left[I \times R^n \times R^n, R^n\right]$ and $x \in C\left[I, R^n\right]$ and T be a continuous operator such that T maps $C(I)$ into $C(I)$.

Define

$$|p(t,s)| \le 1 \tag{3.2}$$

and

$$|g(t,x,y)| \le f(t)\left[|x| + |y|\right], \quad t \in I \tag{3.3}$$

$$\text{Also let } |f(t)| \le k(t), \text{ where } k(t) > 1 \tag{3.4}$$

$$|Tx(t)| \le \int_0^t g(s)|x(s)| \mathrm{d}s, \quad t \in I \tag{3.5}$$

Then

$$|x(t)| \le k(t) + k(0) \int_0^t f(s) \exp\left(k(s) - k(0) + \int_0^s \left(f(\partial) + g(\partial)\right) \mathrm{d}\partial\right) \mathrm{d}s, \quad \forall t \in I$$

Proof: Taking absolute value of the both sides of (3.1), we get

$$|x(t)| \le |f(t)| + \int_0^t \left| p(t,s) g(s, x(s), Tx(s)) \right| \mathrm{d}s \tag{3.6}$$

By substituting from (3.2), (3.3), (3.4) and (3.5) in (3.6), we have

$$|x(t)| \le k(t) + \int_0^t f(s)|x(s)| \mathrm{d}s + \int_0^t f(s) \left(\int_0^s g(\partial) |x(\partial)| \mathrm{d}\partial \right) \mathrm{d}s, \quad \forall t \in I$$

The remaining proof will be the same as the proof of Theorem 2.2 with suitable modifications. We note that Theorem 2.2 can be used to study the stability, boundedness and continuous dependence of the solutions of (3.1).

4. Conclusion

We finally mention that the integral inequalities obtained in this paper allow us to study the stability, boundedness and asymptotic behavior of the solutions of a class of more general partial differential and integral equations.

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