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Fixed Points and Common Fixed Points of Quasi-Contractive Mappings on Partially Ordered-Cone Metric Spaces

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Abstract

In this paper, we use the mappings with quasi-contractive conditions, defined on a partially ordered set with cone metric structure, to construct convergent sequences and prove that the limits of the constructed sequences are the unique (common) fixed point of the mappings, and give their corollaries. The obtained results improve and generalize the corresponding conclusions in references.

Keywords

Common Fixed Point, Cone Metric Space, Complete

1. Introduction

Huang and Zhang [1] recently have introduced the concept of cone metric spaces and have established fixed point theorems for a contractive type map in a normal cone metric space. Subsequently, some authors [2]-[7] have generalized the results in [1] and have studied the existence of common fixed points of a finite self maps satisfying a contractive condition in the framework of normal or non-normal cone metric spaces. On the other hand, some authors discussed (common) fixed point problems for contractive maps defined on a partially ordered set with cone metric structure [8]-[13]. These results improved and generalized many corresponding (common) fixed point theorems of contractive maps on cone metric spaces. Here, we will obtain (common) fixed point theorems of maps with certain quasi-contractive conditions on a partially ordered set with cone metric structure.

Let E be a real Banach space. A subset P_0 of E is called a cone if and only if:

i) P_0 is closed, nonempty, and $P_0 \neq \{0\}$;

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ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P_0$ implies $ax + by \in P_0$;

iii) $P_0 \cap (-P_0) = \{0\}$.

Given a cone $P_0 \subset E$, we define a partial ordering $\leq$ on E with respect to P_0 by $x \leq y$ if and only if $y - x \in P_0$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int} P_0$ (interior of P_0).

The cone P_0 is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above is called the normal constant of P_0. It is clear that $K \geq 1$.

In the following we always suppose that E is a real Banach space, P_0 is a cone in E with $\text{int} P_0 \neq \varnothing$ and $\leq$ is a partial ordering with respect to P_0.

Let X be a nonempty set. Suppose that the mapping $d: X \times X \to E$ satisfies

d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

d3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, z, y \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Let (X, d) be a cone metric space. We say that a sequence $\{x_n\}$ in X is

e) Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is an N such that for all $n, m > N$, $d(x_m, x_n) \ll c$;

g) convergent sequence if for every $c \in E$ with $0 \ll c$, there is an N such that for all $n > N$ such that $d(x_n, x) \ll c$ for some $x \in E$. Let $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

(X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

Let (X, d) be a cone metric space, $f: X \to X$ and $x_0 \in X$. f is said to be continuous [13] at x_0 if for any sequence $x_n \to x_0$, we have $fx_n \to fx_0$.

Lemma 1 [14] Let (X, d) be a cone metric space. Then the following properties hold:

1) if $u \leq v$ and $v \ll w$, then $u \ll w$; if $0 \leq a \ll c$ for all $c \in \text{int} P_0$, then $a = 0$;

2) if $a \leq \lambda a$ where $a \in P_0$ and $0 < \lambda < 1$, then $a = 0$.

Lemma 2 [15] Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $\{a_n\}$ a sequence in P_0 and $a_n \to 0$. If $d(x_n, x_m) \leq a_n$ for any $m > n > 1$, then $\{x_n\}$ is Cauchy.

2. Main Results

At first, we give an example to show that there exists a self-map f on a partially ordered set $(X, \sqsubseteq)$ such that for each $x \in X$ there exists y satisfying $x = fy$ and $x \sqsubseteq y$.

Example Let $X = R$ be a real space. Define $f: X \to X$ by

$$f(x) = \begin{cases} \frac{x}{2}, & \text{for } x \leq 1 \\ 1 - \frac{x}{2}, & \text{for } 1 < x \leq 2, \\ 3x - 6, & \text{for } 2 < x \leq 3, \\ \frac{x}{3} + 2, & \text{for } x > 3. \end{cases}$$

Then obviously, for each $x \in X$, there exists $y \in X$ satisfying $x = fy$ and $x \sqsubseteq y$.

$(X, \sqsubseteq, d)$ is said to be a partially order-cone metric space if $(X, \sqsubseteq)$ is a partially ordered set and (X, d) is a cone metric space.

Theorem 1 Let $(X, \sqsubseteq, d)$ be a complete partially ordered-cone metric space. Suppose that a map $f: X \to X$ is continuous and the following two assertions hold:

i) there exist $A, B, C, D, E \geq 0$ with $A + B + C + 2E < 1$ and for $x, y \in X$ with $x \sqsubseteq y$, such that

$$d(x, y) \leq Ad(fx, fy) + Bd(x, fx) + Cd(y, fy) + Dd(x, fy) + Ed(fx, y);$$

ii) for each $x \in X$, there exists $y \in X$ such that $x = fy$ and $x \sqsubseteq y$.

Then f has a fixed point $x^* \in X$. Furthermore, if any two elements x and y in $Fix(f)$ are comparative and

$A+D+E<1$, then *f* has a unique fixed point in *X*.

Proof Take any $x_0 \in X$, then by ii), we obtain a sequence $\{x_n\}$ as follows: $x_{n-1} = fx_n$ for all $n \in \mathbb{N}$ and $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$.

For any fixed $n \in \mathbb{N}$, since $x_n \sqsubseteq x_{n+1}$, by i),

$$\begin{aligned} d(x_n, x_{n+1}) &\le Ad(fx_n, fx_{n+1}) + Bd(x_n, fx_n) + Cd(x_{n+1}, fx_{n+1}) + Dd(x_n, fx_{n+1}) + Ed(fx_n, x_{n+1}) \\ &= Ad(x_{n-1}, x_n) + Bd(x_n, x_{n-1}) + Cd(x_{n+1}, x_n) + Ed(x_{n-1}, x_{n+1}) \\ &\le Ad(x_{n-1}, x_n) + Bd(x_n, x_{n-1}) + Cd(x_{n+1}, x_n) + E\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right], \end{aligned}$$

so $d(x_n, x_{n+1}) \le \left(\dfrac{A+B+E}{1-C-E}\right) d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}$.

Let $k := \dfrac{A+B+E}{1-C-E}$, then $0 \le k < 1$ by i) and

$$d(x_n, x_{n+1}) \le kd(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}.$$

Repeating this process,

$$d(x_n, x_{n+1}) \le k^n d(x_0, x_1), \quad \forall n \in \mathbb{N}.$$

Let $m > n \ge 1$, then from the above,

$$d(x_n, x_m) \le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \le \sum_{i=n}^{m-1} k^i d(x_0, x_1) \le \left(\frac{k^n}{1-k}\right) d(x_0, x_1) := a_n.$$

Obviously, $a_n \in P_0$ and $a_n \to 0$ as $n \to \infty$ since $0 \le k < 1$ So $d(x_n, x_m) \le a_n$ for all $m > n \ge 1$, hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence by Lemma 2 and there exists $x^* \in X$ such that $x_n \to x^*$ by the completeness of *X*. Since *f* is continuous and $x_n = fx_{n+1}$, so $x^* = fx^*$, *i.e.*, x^* is a fixed point of *f*.

If x^* and y^* are all fixed points of *f* and suppose that $x^* \sqsubseteq y^*$, then by i),

$$\begin{aligned} d(x^*, y^*) &\le Ad(fx^*, fy^*) + Bd(x^*, fx^*) + Cd(y^*, fy^*) + Dd(x^*, fy^*) + Ed(fx^*, y^*) \\ &\le (A+D+E) d(x^*, y^*). \end{aligned}$$

Hence $x^* = y^*$ by (2) in Lemma 1, so x^* is the unique fixed point of *f*.

Another version of Theorem 1 is following:

Theorem 2 Let $(X, \sqsubseteq, d)$ be a complete partially ordered-cone metric space. Suppose that $f: X \to X$ is continuous and the following two assertions hold:

i) there exist $\alpha, \beta, \gamma \ge 0$ with $\alpha + 2\beta + 2\gamma < 1$ and for all $x, y \in X$ with $x \sqsubseteq y$,

$$d(x, y) \le \alpha d(fx, fy) + \beta\left[d(x, fx) + d(y, fy)\right] + \gamma\left[d(x, fy) + d(fx, y)\right];$$

ii) for each $x \in X$, there exists $y \in X$ such that $x = fy$ and $x \sqsubseteq y$.

Then *f* has a fixed point $x^* \in X$. Furthermore, if *x* and *y* is comparative for all $x, y \in Fix(f)$, then *f* has an unique fixed point in *X*.

Proof Take $A = \alpha$, $B = C = \beta$ and $D = E = \gamma$, then the conclusion is true by Theorem 1.

From now, we give common fixed point theorems for a pare of maps.

Theorem 3 Let $(X, \sqsubseteq, d)$ be a complete partially ordered-cone metric space. If $f, g: X \to X$ are two maps such that *f* or *g* is continuous and the following two assertions hold:

i) there exist *A*, *B*, *C*, *D*, *E* ≥ 0 with $B + D < 1$, $C + E < 1$ and $\dfrac{A+B+E}{1-C-E}\dfrac{A+C+D}{1-B-D} < 1$ such that for all comparative $x, y \in X d(x, y) \le Ad(fx, gy) + Bd(x, fx) + Cd(y, gy) + Dd(x, gy) + Ed(fx, y)$;

ii) for each $x \in X$, there exist $y_1, y_2 \in X$ such that $x = fy_1$, $x = gy_2$ and $x \sqsubseteq y_1$, $x \sqsubseteq y_2$.

Then *f* and *g* have a common fixed point $x^* \in X$. Furthermore, if *x* and *y* in $Fix(f) \cap Fix(g)$ are comparative and $A+D+E<1$, then $Fix(f)\cap Fix(g)$ is singleton.

Proof Take any element $x_0\in X$, then using ii), we can construct a sequence $\{x_n\}$ satisfying the following condition $x_{2n}=fx_{2n+1}$, $x_{2n+1}=gx_{2n+2}$ for all $n=0,1,\cdots$, and $x_0\sqsubseteq x_1\sqsubseteq x_2\sqsubseteq\cdots\sqsubseteq x_n\sqsubseteq\cdots$.

For any $n\in\mathbb{N}$, by i), we have

$$\begin{aligned}d(x_{2n+1},x_{2n+2})&\le Ad(fx_{2n+1},gx_{2n+2})+Bd(x_{2n+1},fx_{2n+1})+Cd(x_{2n+2},gx_{2n+2})\\&\quad+Dd(x_{2n+1},gx_{2n+2})+Ed(fx_{2n+1},x_{2n+2})\\&=Ad(x_{2n},x_{2n+1})+Bd(x_{2n+1},x_{2n})+Cd(x_{2n+2},x_{2n+1})+Ed(x_{2n},x_{2n+2})\\&\le(A+B+E)d(x_{2n},x_{2n+1})+(C+E)d(x_{2n+1},x_{2n+2}),\end{aligned}$$

hence

$$d(x_{2n+1},x_{2n+2})\le K_1d(x_{2n},x_{2n+1}),\quad\forall n\in\mathbb{N},$$

where $K_1=\dfrac{A+B+E}{1-C-E}$. And

$$\begin{aligned}d(x_{2n+3},x_{2n+2})&\le Ad(fx_{2n+3},gx_{2n+2})+Bd(x_{2n+3},fx_{2n+3})+Cd(x_{2n+2},gx_{2n+2})\\&\quad+Dd(x_{2n+3},gx_{2n+2})+Ed(fx_{2n+3},x_{2n+2})+Ad(x_{2n+2},x_{2n+1})\\&\quad+Bd(x_{2n+3},x_{2n+2})+Cd(x_{2n+2},x_{2n+1})+Dd(x_{2n+3},x_{2n+1})\\&\le(A+C+D)d(x_{2n+2},x_{2n+1})+(B+D)d(x_{2n+3},x_{2n+2}),\end{aligned}$$

hence

$$d(x_{2n+3},x_{2n+2})\le K_2d(x_{2n+2},x_{2n+1}),\quad\forall n\in\mathbb{N},$$

where $K_2=\dfrac{A+C+D}{1-B-D}$.

Let $K=K_1K_2$, then $0<K<1$ by i), and by induction, for any $n=0,1,2,\cdots$

$$\begin{aligned}d(x_{2n+1},x_{2n+2})&\le K_1d(x_{2n},x_{2n+1})\le K_1K_2d(x_{2n-1},x_{2n})=Kd(x_{2n-1},x_{2n})\\&\le\cdots\le K^nd(x_1,x_2)K_1K^nd(x_0,x_1);\end{aligned}$$

$$d(x_{2n+2},x_{2n+3})\le K_2d(x_{2n+1},x_{2n+2})\le K_1\left[K_1K^nd(x_0,x_1)\right]=K^{n+1}d(x_0,x_1).$$

For any $p,\ q\in\mathbb{N}$ with $p<q$,

$$\begin{aligned}d(x_{2p+1},x_{2q+1})&\le\sum_{i=2p+1}^{2q}d(x_i,x_{i+1})\le\left(K_1\sum_{i=p}^{q-1}K^i+\sum_{i=p+1}^{q}K^i\right)d(x_0,x_1)\le\frac{K^p(K_1+K)}{1-K}d(x_0,x_1)\\&\le\frac{K^p(K_1+1)}{1-K}d(x_0,x_1)\le MK^pd(x_0,x_1),\end{aligned}$$

where $M=\dfrac{2}{1-k}\max\{K_1,1\}$. Similarly,

$$\begin{aligned}d(x_{2p},x_{2q+1})&\le\sum_{i=2p}^{2q}d(x_i,x_{i+1})\le\left(\sum_{i=p}^{q}K^i+K_1\sum_{i=p}^{q-1}K^i\right)d(x_0,x_1)\\&\le\frac{K^p(1+K_1)}{1-K}d(x_0,x_1)\le MK^pd(x_0,x_1);\end{aligned}$$

$$\begin{aligned}d(x_{2p},x_{2q})&\le\sum_{i=2p}^{2q-1}d(x_i,x_{i+1})\le\left(\sum_{i=p}^{q-1}K^i+K_1\sum_{i=p}^{q-1}K^i\right)d(x_0,x_1)\\&\le\frac{K^p(1+K_1)}{1-K}d(x_0,x_1)\le MK^pd(x_0,x_1);\end{aligned}$$

$$d\left(x_{2p+1},x_{2q}\right)\le\sum_{i=2p+1}^{2q-1}d\left(x_i,x_{i+1}\right)\le\left(K_1\sum_{i=p}^{q-1}K^i+\sum_{i=p+1}^{q-1}K^i\right)d\left(x_0,x_1\right)$$
$$\le\frac{K^p\left(K_1+K\right)}{1-K}d\left(x_0,x_1\right)\le MK^p d\left(x_0,x_1\right).$$

So for any $m, n\in\mathbb{N}$ with $m > n > 0$, there exists $\alpha(n)\in\mathbb{N}$ with $\frac{n-1}{2}\le\alpha(n)\le\frac{n}{2}$, that is, $\alpha(n)=\left[\frac{n}{2}\right]$ such that

$$d\left(x_m,x_n\right)\le MK^{\alpha(n)}d\left(x_0,x_1\right):=a_n.$$

Obviously, $a_n\in P_0$ and $a_n\to 0$ as $n\to\infty$ since $K<1$. So $d\left(x_m,x_n\right)\le a_n$ for all $m>n\ge 1$, hence $\{x_n\}$ is Cauchy by Lemma 2 and there exists $x^*\in X$ such that $x_n\to x^*$.

Suppose that f is continuous, then $x^*=fx^*$ since $x_{2n}=fx_{2n+1}$. For x^* there exists $y'\in X$ such that $x^*=gy'$ and $x^*\sqsubseteq y'$ by ii). By i),

$$d\left(x^*,y'\right)\le Ad\left(fx^*,gy'\right)+Bd\left(x^*,fx^*\right)+Cd\left(y',gy'\right)+Dd\left(x^*,gy'\right)+Ed\left(fx^*,y'\right)$$
$$=\left[C+E\right]d\left(x^*,y'\right).$$

So $x^*=y'$ by (2) in Lemma 1, hence $fx^*=x^*=gy'=gx^*$. Therefore $x^*\in Fix(f)\cap Fix(g)$. Similarly, we can give the same result for the case of g being continuous.

If $x^*, y^*\in Fix(f)\cap Fix(g)$ then x^* and y^* are comparative, hence by i),

$$d\left(x^*,y^*\right)\le Ad\left(fx^*,gy^*\right)+Bd\left(x^*,fx^*\right)+Cd\left(y^*,gy^*\right)+Dd\left(x^*,gy^*\right)+Ed\left(fx^*,y^*\right)$$
$$\le\left(A+D+E\right)d\left(x^*,y^*\right)$$

so $x^*=y^*$ by (2) in Lemma 1. Hence $Fix(f)\cap Fix(g)=\{x^*\}$.

Modifying the idea of Zhang [16], we obtain next three corollaries.

Corollary 1 The conditions of *A*, *B*, *C*, *D*, *E* in i) of Theorem 3 can be replaced by the following:

i') there exist *A*, *B*, *C*, *D*, $E\ge 0$ and $\epsilon>0$ such that $A+B+C+D+E=1+\epsilon$, $A+D+E<1$, $C+E<1$, $B+D<1$, $(C-B)(D-E)>2\epsilon$.

Proof Since $(C-B)(D-E)>2\epsilon\Leftrightarrow A+CD+BE-\epsilon>A+\epsilon+BD+CE$ so

$$A+CD+BE-\epsilon>A+\epsilon+BD+CE,$$

hence $(1-B-C-D-E)+CD+BE>A(A+B+C+D+E)+BD+CE$ therefore

$$K=\frac{A+B+E}{1-C-E}\frac{A+C+D}{1-B-D}<1.$$

Corollary 2 The conditions of *A*, *B*, *C*, *D*, *E* in i) of Theorem 3 can be replaced by the following:

i'') there exist *A*, *B*, *C*, *D*, $E\ge 0$ such that $A+B+C+D+E=1$, $C>B$ and $D>E$ or $C<B$ and $D<E$.

Proof Take $\epsilon>0$ such that $(C-B)(D-E)>2\epsilon$ and $A+D+E+\epsilon<1$, and let $A'=A+\epsilon$. Then the following holds: for all comparative elements $x, y\in X$,

$$d(x,y)\le A'd(fx,gy)+Bd(x,fx)+Cd(y,gy)+Dd(x,y)+Ed(gx,y).$$

Obviously, A', *B*, *C*, *D*, *E* satisfy i') in Corollary 1.

Corollary 3 The conditions of *A*, *B*, *C*, *D*, *E* in 1) of Theorem 3 can be replaced by the following:

i''') there exist *A*, *B*, *C*, *D*, $E\ge 0$ such that $A+B+C+D+E<1$ and $B=C$ or $D=E$.

Proof Since *A*, *B*, *C*, *D*, $E<1$, so $A^2+A(1-A)<1-(B+C+D+E)$, hence

$$A^2+A(B+C+D+E)+(C+E)(B+D)<1-(B+C+D+E)+(C+E)(B+D),$$

or

$$A^2+A(B+C+D+E)+(C+D)(B+E)<1-(B+C+D+E)+(C+D)(B+E),$$

which implies that

$$\frac{A+C+E}{1-C-E}\frac{A+B+D}{1-B-D}<1,$$

or

$$\frac{A+C+D}{1-C-D}\frac{A+B+E}{1-B-E}<1.$$

If $B=D$ or $D=E$, then the above two relations reduce

$$K=\frac{A+B+E}{1-C-E}\frac{A+C+D}{1-B-D}<1.$$

The following is a non-continuous version of Theorem 3.

Theorem 4 Let $(X,\sqsubseteq,d)$ be a complete partially ordered-cone metric space. If $f,g:X\to X$ are maps such that i) and ii) in Theorem 3 hold and iii) or iv) holds

iii) if an increasing sequence $\{x_n\}$ converges to $x\in X$, then $x_n\sqsubseteq x$ and $x_n\sqsubseteq fx$ for all $n\in\mathbb{N}$ and $f^2=1_X$;

iv) if an increasing sequence $\{x_n\}$ converges to $x\in X$, then $x_n\sqsubseteq x$ and $x_n\sqsubseteq gx$ for all $n\in\mathbb{N}$ and $g^2=1_X$.

Then *f* and *g* have a common fixed point $x^*\in X$. Furthermore, if *x* and *y* in $Fix(f)\cap Fix(g)$ are comparative and $A+D+E<1$, then $Fix(f)\cap Fix(g)$ is singleton.

Proof By i) and ii) in Theorem 3, we construct a sequence $\{x_n\}$ such that $x_{2n}=fx_{2n+1}$, $x_{2n+1}=gx_{2n+2}$, for all $n=0,1,\cdots$, and $x_0\sqsubseteq x_1\sqsubseteq x_2\sqsubseteq\cdots\sqsubseteq x_n\sqsubseteq\cdots$ and $x_n\to x^*$.

Case I: Suppose iv) holds, then $x_n\sqsubseteq x^*$ and $x_n\sqsubseteq gx^*$ for all $n\in\mathbb{N}$ and $g^2=1_X$. By i),

$$\begin{aligned}
d\left(x^*,gx^*\right)&\le d\left(x_{2n+1},x^*\right)+d\left(x_{2n+1},gx^*\right)\\
&\le d\left(x_{2n+1},x^*\right)+Ad\left(fx_{2n+1},ggx^*\right)+Bd\left(x_{2n+1},fx_{2n+1}\right)\\
&\quad+Cd\left(gx^*,ggx^*\right)+Dd\left(x_{2n+1},ggx^*\right)+Ed\left(fx_{2n+1},gx^*\right)\\
&=d\left(x_{2n+1},x^*\right)+Ad\left(x_{2n},x^*\right)+Bd\left(x_{2n+1},x_{2n}\right)\\
&\quad+Cd\left(x^*,gx^*\right)+Dd\left(x_{2n+1},x^*\right)+Ed\left(x_{2n},gx^*\right)\\
&\le\left(x_{2n+1},x^*\right)+Ad\left(x_{2n},x^*\right)+B\left[d\left(x_{2n+1},x^*\right)+d\left(x^*,x_{2n}\right)\right]\\
&\quad+Cd\left(x^*,gx^*\right)+Dd\left(x_{2n+1},x^*\right)+E\left[d\left(x_{2n},x^*\right)+d\left(x^*,gx^*\right)\right],
\end{aligned}$$

so we obtain

$$d\left(x^*,gx^*\right)\le L_1d\left(x_{2n},x^*\right)+L_2d\left(x_{2n+1},x^*\right),$$

where $L_1=\dfrac{A+B+E}{1-C-E}$ and $L_2=\dfrac{1+B+D}{1-C-E}$ Since $x_n\to x^*$, for any $c\in\mathrm{Int}P_0$ there exist enough large $N\in\mathbb{N}$ such that $d\left(x_{2n},x^*\right)\ll\dfrac{c}{2L_1}$ and $d\left(x_{2n+1},x^*\right)\ll\dfrac{c}{2L_2}$ for all $n\ge N$, hence

$$d\left(x^*,gx^*\right)\le K_1d\left(x_{2n},x^*\right)+K_2d\left(x_{2n+1},x^*\right)\ll\frac{c}{2}+\frac{c}{2}=c,\quad\forall n\ge N.$$

So $d\left(x^*,gx^*\right)=0$ by (1) in Lemma 1, hence $gx^*=x^*$.

For $x^*\in X$ there exists $y'\in X$ such that $x^*=fy'$ and $x^*\sqsubseteq y'$ by ii). Hence by i),

$$\begin{aligned}
d\left(y',x^*\right)&\le Ad\left(fy',gx^*\right)+Bd\left(y'fy'\right)+Cd\left(x^*,gx^*\right)+Dd\left(y',gx^*\right)+Ed\left(fy',x^*\right)\\
&=\left[B+D\right]d\left(y',x^*\right),
\end{aligned}$$

so $y'=x^*$ by (2) in Lemma 1. Hence $gx^*=x^*=fy'=fx^*$, *i.e.*, $x^*\in Fix(f)\cap Fix(g)$.

Case II: Suppose iii) holds, then $x_n \sqsubseteq x^*$ and $x_n \sqsubseteq fx^*$ for all $n \in \mathbb{N}$ and $f^2 = 1_X$. By (1),

$$\begin{aligned} d\left(x^*, fx^*\right) &\le d\left(x^*, x_{2n+2}\right) + d\left(x_{2n+2}, fx^*\right) = d\left(x^*, x_{2n+2}\right) + d\left(fx^*, x_{2n+2}\right) \\ &\le d\left(x^*, x_{2n+2}\right) + Ad\left(ffx^*, gx_{2n+2}\right) + Bd\left(fx^*, ffx^*\right) \\ &\quad + Cd\left(x_{2n+2}, gx_{2n+2}\right) + Dd\left(fx^*, gx_{2n+2}\right) + Ed\left(ffx^*, x_{2n+2}\right) \\ &= d\left(x^*, x_{2n+2}\right) + Ad\left(x^*, x_{2n+1}\right) + Bd\left(fx^*, x^*\right) \\ &\quad + Cd\left(x_{2n+2}, x_{2n+1}\right) + Dd\left(fx^*, x_{2n+1}\right) + Ed\left(x^*, x_{2n+2}\right) \\ &\le d\left(x^*, x_{2n+2}\right) + Ad\left(x^*, x_{2n+1}\right) + Bd\left(fx^*, x^*\right) + C\left[d\left(x_{2n+2}, x^*\right) + d\left(x^*, x_{2n+1}\right)\right] \\ &\quad + D\left[d\left(fx^*, x^*\right) + d\left(x^*, x_{2n+1}\right)\right] + Ed\left(x^*, x_{2n+2}\right), \end{aligned}$$

so we obtain

$$d\left(x^*, fx^*\right) \le L_3 d\left(x_{2n+1}, x^*\right) + L_4 d\left(x_{2n+2}, x^*\right),$$

where $L_3 = \dfrac{A+C+D}{1-B-D}$ and $L_4 = \dfrac{1+C+E}{1-B-D}$ Since $x_n \to x^*$, for any $c \in \mathrm{Int}P_0$ there exist enough large $N \in \mathbb{N}$ such that $d\left(x_{2n+1}, x^*\right) \ll \dfrac{c}{2L_3}$ and $d\left(x_{2n+2}, x^*\right) \ll \dfrac{c}{2L_4}$ for all $n \ge N$, hence $d\left(x^*, fx^*\right) \le L_3 d\left(x_{2n}, x^*\right) + L_4 d\left(x_{2n+1}, x^*\right) \ll \dfrac{c}{2} + \dfrac{c}{2} = c$, $\forall n \ge N$. So $d\left(x^*, fx^*\right) = 0$ by (1) in Lemma 1, that is, $fx^* = x^*$.

For $x^* \in X$ there exists $y'' \in X$ such that $x^* = gy''$ and $x^* \sqsubseteq y''$ by ii). Hence by i),

$$\begin{aligned} d\left(x^*, y''\right) &\le Ad\left(fx^*, gy''\right) + Bd\left(x^*, fx^*\right) + Cd\left(y'', gy''\right) + Dd\left(x^*, gy''\right) + Ed\left(fx^*, y''\right) \\ &= [C+E]\, d\left(x^*, y''\right), \end{aligned}$$

so $y'' = x^*$ by (1) in Lemma 1. Hence $fx^* = x^* = gy'' = gx^*$, *i.e.*, $x^* \in Fix(f) \cap Fix(g)$.

So in any case, $x^* \in Fix(f) \cap Fix(g)$ The uniqueness is obvious.

Remark 1 We can also modify Corollary 1 - 3 to give the corresponding corollaries of Theorem 4, but we omit the part.

Remark 2 In this paper, we discuss the common fixed point problems for mappings with quasi-contractive type (*i.e.*, expansive type) on partially ordered cone metric spaces, but some authors in references discussed the same problems for contractive or Lipschitz type. So our results improve and generalize the corresponding conclusions.

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