

The Best Piecewise Linearization of Nonlinear Functions

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Abstract

In this paper, we propose a method for finding the best piecewise linearization of nonlinear functions. For this aim, we try to obtain the best approximation of a nonlinear function as a piecewise linear function. Our method is based on an optimization problem. The optimal solution of this optimization problem is the best piecewise linear approximation of nonlinear function. Finally, we examine our method to some examples.

Keywords

Nonlinear Systems, Piecewise Linearization, Optimization Problem, Linear Programming

1. Introduction

The linearization of nonlinear systems is an efficient tool for finding approximate solutions and treatment analysis of these systems, especially in application [1]-[3]. Some researchers have used some methods based on the optimization problem [4]. But in many applications for nonlinear and nonsmooth functions, we are faced to some problems. In fact, piecewise linearization is a more efficient tool for finding approximate solutions. Some researchers have used piecewise linearization in applications [5] [6]. Also, some researchers have used piecewise linearization to solve ODEs and PDEs [7].

First, we consider a nonlinear function. Let $F : A \subseteq R^n \rightarrow R$ be a nonlinear function. We suppose that $x = (x_1, x_2, \dots, x_n)$ varies in a subset of R^n as A and this subset is compact. Our aim is to approximate the nonlinear function F by a piecewise linear function as follows:

$$f(x) = \sum_{i=1}^N (a_{i0} + a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n) \chi_{A_i}(x), \quad a_{ij} \in R; \quad i = 1, 2, \dots, N \quad (1)$$

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where A_i is i th subset in partitioning of A as $P_N = \{A_1, A_2, \dots, A_N\}$. As we know, this partitioning has below properties:

- 1) $\forall i, j = 1, 2, \dots, N; A_i \cap A_j = \emptyset, A_i \in R^n$
- 2) $A = \bigcup_{i=1}^N A_i$

Also, $\chi_{A_i}(x)$ is a characteristic function on $\{A_1, A_2, \dots, A_N\}$ such that:

$$\chi_{A_i}(x) = \begin{cases} 1, & x \in A_i \\ 0, & x \notin A_i \end{cases} \tag{2}$$

Now, let $L_1(A) = \{f : A \rightarrow R \mid \int_A |f| dx < \infty\}$. As we know $L_1(A)$ is a Hilbert space of A with the following inner product:

$$\langle f, g \rangle = \int_A f(x)g(x) dx, \quad f, g \in L(A) \tag{3}$$

and

$$\|f\|_1 = \int_A |f| dx \tag{4}$$

Definition 1. We define $S_N(A)$ ($N \in \mathbb{N}$) be the set of all $f \in L_1(A)$ of the form (1).

Definition 2. If $F : R^n \rightarrow R$ is a nonlinear function and $f \in S_N(A)$, we define $\|F - f\|_{L_1}$ as follows:

$$\|F - f\|_{L_1} = \int_A |F - f| dx \tag{5}$$

Theorem. The subset $S_N(A)$ is dens on $L_1(A)$.

Proof. Suppose that F be a nonlinear function that $F : A \subseteq R^n \rightarrow R$.

$$\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N}, \quad \exists f(\cdot) \in S_N(A) \Rightarrow \|F - f\|_{L_1} < \varepsilon$$

Definition 3. We call $f^* \in S_N(A)$ the best piecewise linear approximation of F if for any $f \in S_N(A)$ we have

$$\|F - f^*\|_{L_1} \leq \|F - f\|_{L_1}.$$

In fact, by above definition f^* is optimal solution of the following optimization problem:

$$\begin{aligned} & \text{Min} \|F - f\|_{L_1} \\ & f \in S_N(A) \end{aligned} \tag{6}$$

Obviously, because $0 \leq \|F - f\|_{L_1}$, the optimization problem (6) has optimal solution.

2. Approach

At first, we consider a nonlinear function $F : R \rightarrow R$. Secondly, we explain this approach for a nonlinear function $F : R^2 \rightarrow R$. Then, we explain this approach for a nonlinear function $F : R^n \rightarrow R$.

1) Let to consider the bellow optimization problem

$$\begin{aligned} & \text{Min} \|F - f\|_{L_1} \\ & f \in S_N(A) \end{aligned}$$

where, $F : A \subseteq R \rightarrow R$ is a nonlinear function and $A = [a, b]$. As we know $[a, b]$ can be replaced by $[0, 1]$.

Now, we decompose interval $[0, 1]$ to N subintervals $\left[\frac{i-1}{N}, \frac{i}{N}\right]; i = 1, 2, \dots, N$ (See **Figure 1**).

Since, $f \in S_N(A)$, we have

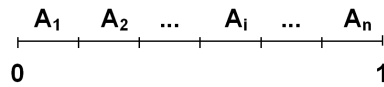


Figure 1. Partitioning A to subintervals A_i .

$$\text{Min} \int_0^1 \left| F(x) - \sum_{i=1}^N (a_i + b_i x) \chi_{\left[\frac{i-1}{N}, \frac{i}{N}\right]}(x) \right| dx \tag{7}$$

Our objective function is a functional. Now, we reduce this functional to a summation as follows:

$$\int_0^1 \left| F(x) - \sum_{i=1}^N (a_i + b_i x) \chi_{\left[\frac{i-1}{N}, \frac{i}{N}\right]}(x) \right| dx = \frac{1}{N} \sum_{i=1}^N |F(x_i) - (a_i + b_i x_i)| \tag{8}$$

So, the optimization problem (8) is as follows:

$$\begin{aligned} & \text{Min} \sum_{i=1}^N |r_i - s_i| \\ & \text{s.t} \\ & r_i - s_i = F(x_i) - (a_i + b_i x_i) \\ & 0 \leq r_i, s_i; \quad i = 1, 2, \dots, N. \end{aligned} \tag{9}$$

But, the optimization problem (9) is a nonlinear programming problem. We reduce this problem to a linear programming problem by relation $|r_i - s_i| = r_i + s_i$ such that $r_i \cdot s_i = 0$. So, our optimization problem will be as follows:

$$\begin{aligned} & \text{Min} \frac{1}{N} \sum_{i=1}^N r_i + s_i \\ & \text{s.t} \\ & r_i - s_i = F(x_i) - (a_i + b_i x_i) \\ & r_i \cdot s_i = 0 \\ & 0 \leq r_i, s_i; \quad I = 1, 2, \dots, N. \end{aligned} \tag{10}$$

2) Second, we consider a nonlinear function $F : A \subseteq R^2 \rightarrow R$. So, we have the optimization problem as follows:

$$\text{Min} \int_A \left| F(x) - \sum_{i=1}^N (a_i + b_i x + c_i y) \chi_{A_i}(x) \right| dx \tag{11}$$

where A_i is the i th partition in partitioning of $A = [a, b] \times [c, d]$ and $dx = dx dy$. Also we can replace A by $[0, 1] \times [0, 1]$. As, we explained in 1) the optimization problem (11) will be reduced to a linear programming problem as follows:

$$\begin{aligned} & \text{Min} \sum_{i=1}^N r_i + s_i \\ & \text{s.t} \\ & r_i - s_i = F(x_i, y_i) - (a_i + b_i x_i + c_i y_i) \\ & r_i \cdot s_i = 0 \\ & 0 \leq r_i, s_i \\ & i = 1, 2, \dots, N \\ & N = m \times n. \end{aligned} \tag{12}$$

where m and n are numbers of subintervals on axes x and y , respectively (See Figure 2).

3) Third, we consider a nonlinear function $F : A \subseteq R^n \rightarrow R$. So, we have the optimization problem as fol-

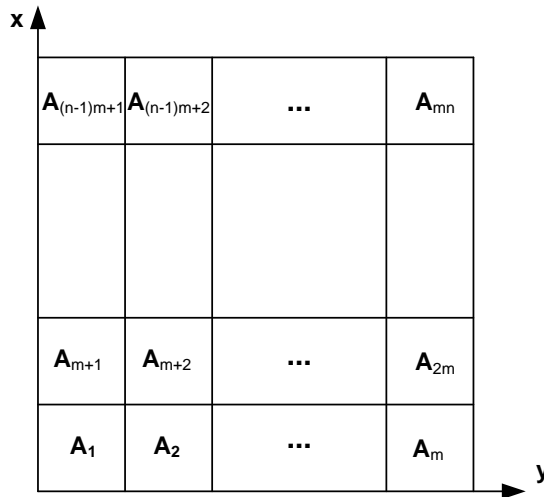


Figure 2. Partitioning A to subintervals A_i .

lows:

$$\text{Min} \int_A \left| F(x_1, x_2, \dots, x_n) - \sum_{i=1}^N (a_{i0} + a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n) \chi_{A_i}(x) \right| dx \tag{13}$$

As, we explained in sections 1) and 2) this optimization problem will be reduced to a linear programming problem as follows:

$$\begin{aligned} &\text{Min} \sum_{i=1}^N r_i + s_i \\ &\text{s.t} \\ &r_i - s_i = F(x_1, x_2, \dots, x_n) - (a_{i0} + a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n) \\ &r_i \cdot s_i = 0 \\ &0 \leq r_i, s_i; \quad i = 1, 2, \dots, N \\ &N = m_1 \times m_2 \times \dots \times m_n. \end{aligned} \tag{14}$$

where m_1, m_2, \dots, m_n are numbers of subintervals on axes x_1, x_2, \dots, x_n respectively.

3. Examples

In this section, we show efficiency of our approach by several examples. Also, we define the root mean squared error by follow relation:

$$\text{RMSE} = \sqrt{\frac{\sum_{i=1}^N e_i^2}{N}} \tag{15}$$

Example 1. We consider nonlinear nonsmooth function $F(x) = |x - .2|$ on interval $[0,1]$.

As we explained in section 1), the linear programming corresponding to this function is as follows:

$$\begin{aligned} &\text{Min} \frac{1}{N} \sum_{i=1}^N r_i + s_i \\ &\text{s.t} \\ &r_i - s_i = |x_i - .2| - (a_i + b_i x_i) \\ &r_i \cdot s_i = 0 \\ &0 \leq r_i, s_i; \quad I = 1, 2, \dots, N. \end{aligned} \tag{16}$$

The optimal solution of linear programming problem (16) is the best piecewise linearization of the function $F(x) = |x - .2|$ on $[0,1]$. We let $N = 50$ and $N = 100$, respectively (See **Figure 3**, **Figure 4**). In this example, we have $RMSE = 7.81 \times 10^{-14}$ for $N = 100$. As we can see the approximate piecewise linearization of this function is high accurate.

Example 2. We consider nonlinear function $F(x) = x^2$ on interval $[0,1]$. We have obtained the piecewise approximation of this nonlinear function using two other methods. These methods are Splines Piecewise Approximation (SPA) and Mixture of Polynomials (MOP). Then we have compared these with our method.

As we explained in section 1), the linear programming corresponding to this function is as follows:

$$\begin{aligned} & \text{Min } \frac{1}{N} \sum_{i=1}^N r_i + s_i \\ & \text{s.t} \\ & r_i - s_i = x_i^2 - (a_i + b_i x_i) \\ & r_i \cdot s_i = 0 \\ & 0 \leq r_i, s_i; \quad I = 1, 2, \dots, N. \end{aligned} \tag{17}$$

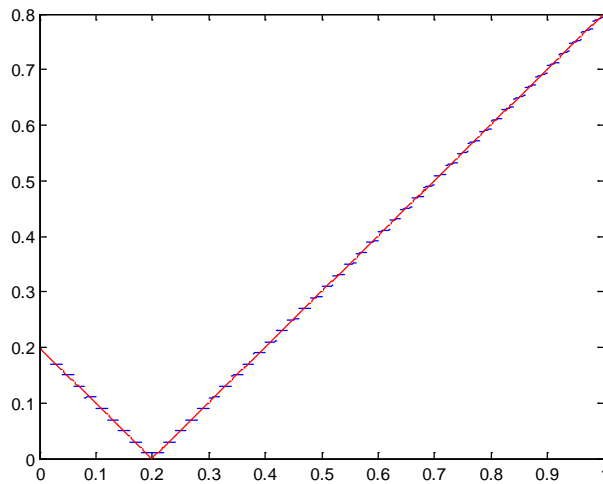


Figure 3. The figure of piecewise function approximation of nonlinear function $F(x) = |x - .2|$ for $N = 50$.

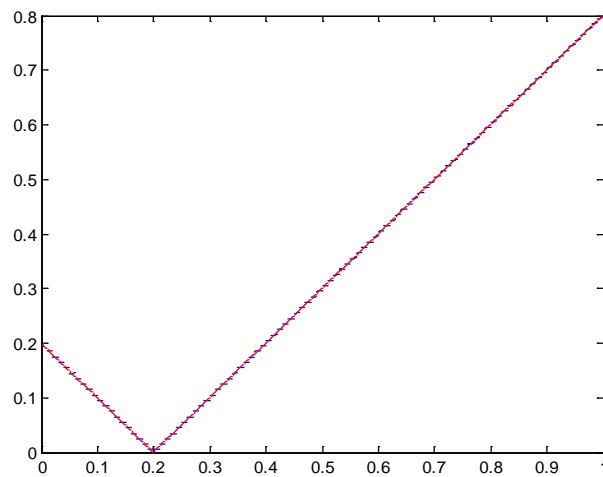


Figure 4. The figure of piecewise function approximation of nonlinear function $F(x) = |x - .2|$ for $N = 100$.

The optimal solution of linear programming problem (17) is the best piecewise linearization of the function $F(x) = x^2$ on $[0,1]$. We let $N = 100$ (See **Figure 5**). In this example, we have $RMSE = 7.52 \times 10^{-12}$ while $RMSE = 2.26 \times 10^{-5}$ for Splines Piecewise Approximation and $RMSE = 5.03 \times 10^{-7}$. As we can see the approximate piecewise linearization of this function using our method is more accurate in compared with two other methods.

Example 3. We consider nonlinear non smooth function $F(x, y) = |x - y|$ on $[0,1] \times [0,1]$.

As we explained in Section 2), the linear programming corresponding to this function is as follows:

$$\begin{aligned}
 & \text{Min} \sum_{i=1}^N r_i + s_i \\
 & \text{s.t} \\
 & r_i - s_i = |x_i - y_i| - (a_i + b_i x_i + c_i y_i) \\
 & r_i \cdot s_i = 0 \\
 & 0 \leq r_i, s_i \\
 & i = 1, 2, \dots, N \\
 & N = m \times n.
 \end{aligned} \tag{18}$$

The optimal solution of linear programming problem (18) is the best piecewise linearization of the function $F(x, y) = |x - y|$ on $[0,1]$. We let $m = n = 10$ and $m = n = 40$ (See **Figure 6**, **Figure 7**). In this example,

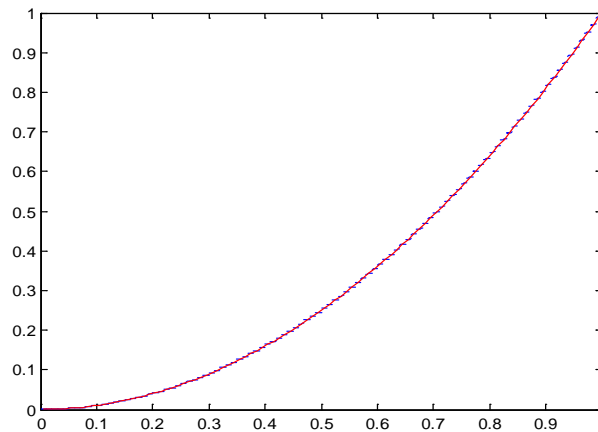


Figure 5. The figure of piecewise function approximation of nonlinear function $F(x) = x^2$ for $N = 100$.

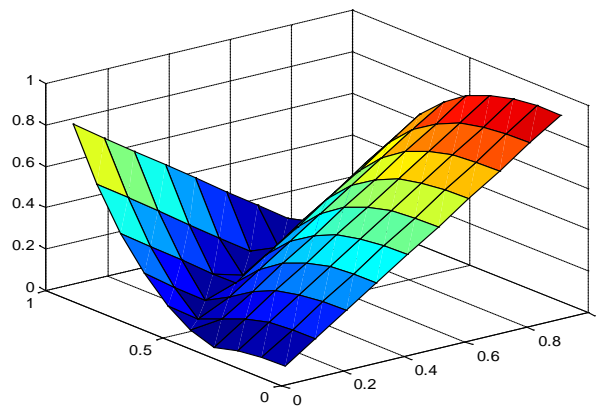


Figure 6. The figure of piecewise function approximation of nonlinear function $F(x, y) = |x - y|$ for $m = n = 10$.

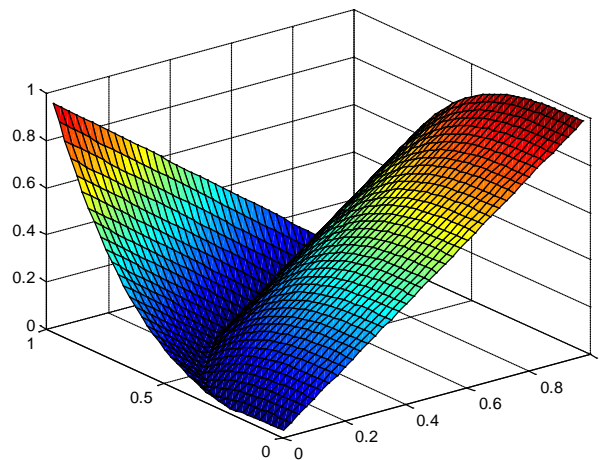


Figure 7. The figure of piecewise function approximation of nonlinear function $F(x, y) = |x - y|$ for $m = n = 40$.

we have $RMSE = 1.28 \times 10^{-14}$ and $RMSE = 7.97 \times 10^{-13}$, respectively. As we can see the approximate piecewise linearization of this function is high accurate.

4. Conclusion

Our method for piecewise linearization of nonlinear functions is extensible to $R = (-\infty, +\infty)$ by the function $:(-\infty, +\infty) \rightarrow [0, 1]$. As we can see, this approximation is high accurate in comparison of other methods and this method is very simple for achieving this optimal solution. Also, this piecewise linearization form of nonlinear functions is useful for many applications, especially for nonlinear nonsmooth optimization, nonlinear differential equations, fuzzy ODE and PDE differential equations and so on.

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