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A Study on B-Spline Wavelets and Wavelet Packets

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Abstract

In this paper, we discuss the B-spline wavelets introduced by Chui and Wang in [1]. The definition for B-spline wavelet packets is proposed along with the corresponding dual wavelet packets. The properties of B-spline wavelet packets are also investigated.

Keywords

B-Splines, Spline Wavelets, Wavelet Packets

1. Introduction

Spline wavelet is one of the most important wavelets in the wavelet family. In both applications and wavelet theory, the spline wavelets are especially interesting because of their simple structure. All spline wavelets are linear combination of B-splines. Thus, they inherit most of the properties of these basis functions. The simplest example of an orthonormal spline wavelet basis is the Haar basis. The orthonormal cardinal spline wavelets in $L^2(\mathbb{R})$ were first constructed by Battle [2] and Lemarié [3]. Chui and Wang [4] found the compactly supported spline wavelet bases of $L^2(\mathbb{R})$ and developed the duality principle for the construction of dual wavelet bases [1] [5].

Wavelets are a fairly simple mathematical tool with a variety of possible applications. If $2^{j/2}\psi(2^j x-k)$, $k\in\mathbb{Z}$ is an orthonormal basis of $L^2(\mathbb{R})$, then ψ is called a wavelet. Usually a wavelet is derived from a given multiresolution analysis of $L^2(\mathbb{R})$. The construction of wavelets has been discussed in a great number of papers. Now, considerable attention has been given to wavelet packet analysis as an important generalization of wavelet analysis. Wavelet packet functions consist of a rich family of building block functions and are localized in time, but offer more flexibility than wavelets in representing different kinds of signals. The main feature of the wavelet transform is to decompose general functions into a set of approximation functions with different scales. Wavelet packet transform is an extension of the wavelet transform. In wavelet transformation signal de-

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composes into approximation coefficients and detailed coefficients, in which further decomposition takes place only at approximation coefficients whereas in wavelet packet transformation, detailed coefficients are decomposed as well which gives more wavelet coefficients for further analysis.

For a given multiresolution analysis and the corresponding orthonormal wavelet basis of $L^2(\mathbb{R})$, wavelet packets were constructed by Coifman, Meyer and Wickerhauser [6] [7]. This construction is an important generalization of wavelets in the sense that wavelet packets are used to further decompose the wavelet components. There are many orthonormal bases in the wavelet packets. Efficient algorithms for finding the best possible basis do exist. Chui and Li [8] generalized the concept of orthogonal wavelet packets to the case of nonorthogonal wavelet packets. Yang [9] constructed a scale orthogonal multiwavelet packets which were more flexible in applications. Xia and Suter [10] introduced the notion of vector valued wavelets and showed that multiwavelets can be generated from the component functions in vector valued wavelets. In [11], Chen and Cheng studied compactly supported orthogonal vector valued wavelets and wavelet packets. Other notable generalizations are biorthogonal wavelet packets [12], non-orthogonal wavelet packets with r-scaling functions [13].

The outline of the paper is as follows. In §2, we introduce some notations and recall the concept of B-splines and wavelets. In §3, we discuss the B-spline wavelet packets and the corresponding dual wavelet packets.

2. Preliminaries

In this Section, we introduce B-spline wavelets (or simply B-wavelets) and some notions used in this paper.

Every *m*th order cardinal spline wavelet is a linear combination of the functions $N_{2m}^{(m)}(2x-j)$. Here the function N_m is the *m*th order cardinal B-spline. Each wavelet is constructed by spline multiresolution analysis. Let *m* be any positive integer and let N_m denotes the *m*th order B-spline with knots at the set $\mathbb{Z}$ of integers such that

$$\operatorname{supp}(N_m) = [0, m].$$

The cardinal B-splines N_m are defined recursively by the equations

$$N_1(x) = \chi_{[0,1]}(x),$$

$$N_m(x) = (N_{m-1} * N_1)(x) = \int_0^1 N_{m-1}(x-t)\,\mathrm{d}t, \quad m = 2, 3, \cdots.$$

We use the following convention for the Fourier transform,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-i\omega t}\,\mathrm{d}t.$$

The Fourier transform of the scaling function N_m is given by

$$\hat{N}_m(\omega) = \left(\frac{1-\mathrm{e}^{-i\omega}}{i\omega}\right)^m. \tag{1}$$

For each $j, k \in \mathbb{Z}$, we set $N_{m;j,k} = N_m(2^j x - k)$, and for each $j \in \mathbb{Z}$, let V_j^m denotes the L^2-closure of the algebraic span of $\{N_{m;j,k}\}$. Then N_m is said to generate spline multiresolution analysis if the following conditions are satisfied.

1) $\cdots \subset V_{-1}^m \subset V_0^m \subset V_1^m \subset \cdots$

2) $\operatorname{clos}_{L^2(\mathbb{R})}\left(\bigcup_{j\in\mathbb{Z}} V_j^m\right) = L^2(\mathbb{R})$;

3) $\bigcap_{j\in\mathbb{Z}} V_j^m = \{0\}$,

4) for each j, $\{N_{m;j,k}\}$ is a Riesz basis of V_j^m.

Following Mallat [14], we consider the orthogonal complementary subspaces $W_{-1}^m, W_0^m, W_1^m, \cdots$ that is;

5) $V_{j+1}^m = V_j^m \oplus W_j^m, \quad \forall j \in \mathbb{Z}$.

6) $W_j^m \perp W_k^m, \quad \forall k \neq j$.

7) $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j^m$.

These subspaces W_j^m, $j \in \mathbb{Z}$, are called the wavelet subspaces of $L^2(\mathbb{R})$ relative to the B-spline N_m. Since $N_m(x) \in V_j^m$ and $V_j^m \subset V_{j+1}^m$, we have

$$N_m(x) = \sum_{k \in \mathbb{Z}} p_k N_m(2x-k), \tag{2}$$

where $\{p_k\}$ is some sequence in ℓ^2. Taking the Fourier transform on both sides of (2), we obtain

$$\hat{N}_m(\omega) = \frac{1}{2}\sum_{k \in \mathbb{Z}} p_k \mathrm{e}^{-ik\omega/2} \hat{N}_m\left(\frac{\omega}{2}\right). \tag{3}$$

Substituting the value of $\hat{N}_m(\omega)$ from (1) into (3), we have

$$\frac{1}{2}\sum_{k \in \mathbb{Z}} p_k \mathrm{e}^{-ik\omega/2} = \left(\frac{1-\mathrm{e}^{-i\omega}}{i\omega}\right)^m \left(\frac{i\omega/2}{1-\mathrm{e}^{-i\omega/2}}\right)^m = \left(\frac{1+\mathrm{e}^{-i\omega/2}}{2}\right)^m = 2^{-m}\sum_{k=0}^{m}\binom{m}{k}\mathrm{e}^{-ik\omega/2}.$$

This gives

$$p_k = \begin{cases} 2^{-m+1}\binom{m}{k} & \text{for } 0 \le k \le m \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

So, (2) can be written as

$$N_m(x) = \sum_{k=0}^{m} 2^{-m+1}\binom{m}{k} N_m(2x-k),$$

which is called the two scale relation for cardinal B-splines of order m.

Chui and Wang [1], introduced the following *m*th order compactly supported spline wavelet or B-wavelet

$$\psi_m(x) = \frac{1}{2^{m-1}}\sum_{j=0}^{2m-2}(-1)^j N_{2m}(j+1) N_{2m}^{(m)}(2x-j), \tag{5}$$

with support $[0, 2m-1]$ that generates W_0^m and consequently all the wavelet spaces W_j^m, $j \in \mathbb{Z}$. To verify that ψ_m is in V_1^m, we need the spline identity

$$N_{2m}^{(m)}(x) = \sum_{j=0}^{m}(-1)^j\binom{m}{j} N_m(x-j). \tag{6}$$

So, substituting (6) into (5), we have the two scale relation

$$\psi_m(x) = \sum_{k=0}^{3m-2} q_k N_m(2x-k), \tag{7}$$

where,

$$q_k = \frac{(-1)^k}{2^{m-1}}\sum_{j=0}^{m}\binom{m}{j} N_{2m}(k-j+1). \tag{8}$$

Let

$$\tilde{\psi}_m(x) = \sum_{k} \tilde{h}_{-k} \tilde{N}_m(2x-k), \tag{9}$$

with the corresponding two scale sequence $\{\tilde{h}_{-k}\}$. If ψ_m is a wavelet, then there exists another $\tilde{\psi}_m$ called the dual wavelet of ψ_m such that

$$\langle \psi_m(.-j), \tilde{\psi}_m(.-l)\rangle = \delta_{j,l}, \quad \forall j,\ l \in \mathbb{Z}. \tag{10}$$

For the scaling function N_m, we define its dual $\tilde{N}_m$ by

$$\tilde{N}_m(x) = \sum_{k \in \mathbb{Z}} \tilde{g}_{-k} \tilde{N}_m(2x-k), \tag{11}$$

such that

$$\left\langle N_m(\cdot - j), \tilde{N}_m(\cdot - l)\right\rangle = \delta_{j,l}, \quad \forall j,\ l \in \mathbb{Z}. \tag{12}$$

Now, we have

$$\begin{aligned} N_m(x) &= \sum_{k=0}^{m} p_k N_m(2x-k), \\ \tilde{N}_m(x) &= \sum_{k\in\mathbb{Z}} \tilde{g}_{-k} \tilde{N}_m(2x-k). \end{aligned} \tag{13}$$

Taking the Fourier transform of (13), we have

$$\begin{aligned} \hat{N}_m(\omega) &= \mathcal{P}\left(\frac{\omega}{2}\right)\hat{N}_m\left(\frac{\omega}{2}\right), \\ \hat{\tilde{N}}_m(\omega) &= \mathcal{G}\left(\frac{\omega}{2}\right)\hat{\tilde{N}}_m\left(\frac{\omega}{2}\right). \end{aligned} \tag{14}$$

where,

$$\begin{aligned} \mathcal{P}(\omega) &= \frac{1}{2}\sum_{k=0}^{m} p_k \mathrm{e}^{-i\omega k}, \\ \mathcal{G}(\omega) &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{g}_{-k} \mathrm{e}^{-i\omega k}. \end{aligned} \tag{15}$$

A necessary and sufficient condition for the duality relationship (12) is that $\mathcal{P}(\omega)$ and $\mathcal{G}(\omega)$ are dual two scale symbols in the sense that

$$\mathcal{P}(\omega)\mathcal{G}(\omega) + \mathcal{P}(-\omega)\mathcal{G}(-\omega) = 1, \quad \omega \in \mathbb{R}. \tag{16}$$

A proof of this statement is given in ([15], Theorem 5.22). Also from (7) and (9), we have

$$\begin{aligned} \hat{\psi}_m(\omega) &= \mathcal{Q}\left(\frac{\omega}{2}\right)\hat{N}_m\left(\frac{\omega}{2}\right), \\ \hat{\tilde{\psi}}_m(\omega) &= \mathcal{H}\left(\frac{\omega}{2}\right)\hat{\tilde{N}}_m\left(\frac{\omega}{2}\right). \end{aligned} \tag{17}$$

where,

$$\begin{aligned} \mathcal{Q}(\omega) &= \frac{1}{2}\sum_{k=0}^{3m-2} q_k \mathrm{e}^{-i\omega k}, \\ \mathcal{H}(\omega) &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{h}_{-k} \mathrm{e}^{-i\omega k}. \end{aligned} \tag{18}$$

We observe that

$$\begin{aligned} &\mathcal{P}(\omega)\mathcal{G}(\omega) + \mathcal{P}(-\omega)\mathcal{G}(-\omega) = 1 \\ &\mathcal{Q}(\omega)\mathcal{H}(\omega) + \mathcal{Q}(-\omega)\mathcal{H}(-\omega) = 1 \\ &\mathcal{P}(\omega)\mathcal{H}(\omega) + \mathcal{P}(-\omega)\mathcal{H}(-\omega) = 0 \\ &\mathcal{Q}(\omega)\mathcal{G}(\omega) + \mathcal{Q}(-\omega)\mathcal{G}(-\omega) = 0, \quad \omega \in \mathbb{R}. \end{aligned} \tag{19}$$

See ([15], Section 5.3).

If $N_m(x)$ is an orthogonal scaling function, then

$$\left\langle N_m(\cdot), N_m(\cdot - l)\right\rangle = \delta_{0,l}, \quad \forall l \in \mathbb{Z}. \tag{20}$$

We say that $\psi_m(x)$ is orthogonal (o.n) B-wavelet function associated with orthogonal scaling function $N_m(x)$ if

$$\langle N_m(.),\psi_m(.-l)\rangle = 0,\quad \forall l\in\mathbb{Z}, \tag{21}$$

and $\psi_m(x-l),\ l\in\mathbb{Z}$ is an orthonormal basis of W_0^m, so we have

$$\langle \psi_m(.),\psi_m(.-l)\rangle = \delta_{0,l},\quad \forall l\in\mathbb{Z}. \tag{22}$$

Lemma 1 *Let* $N_m(x)\in L^2(\mathbb{R})$. *Then* $N_m(x)$ *is an orthonormal family if and only if*

$$\sum_l \hat{N}_m(\omega+2\pi l)\overline{\hat{N}_m(\omega+2\pi l)} = 1,\quad \omega\in\mathbb{R}. \tag{23}$$

Proof *See* ([15], page no. 75].

Theorem 1 *Let* $N_m(x)$ *defined by* (13) *is an orthonormal scaling function. Assume that* $\psi_m(x)\in L^2(\mathbb{R})$ *whereas* $\mathcal{P}(\omega)$ *and* $\mathcal{Q}(\omega)$ *are defined by* (15) *and* (18) *respectively. Then* $\psi_m(x)$ *is an orthonormal wavelet function associated with* $N_m(x)$ *if and only if*

$$\begin{aligned}&\mathcal{P}(\omega)\overline{\mathcal{Q}(\omega)}+\mathcal{P}(\omega+\pi)\overline{\mathcal{Q}(\omega+\pi)} = 0,\quad \omega\in\mathbb{R},\\ &\mathcal{Q}(\omega)\overline{\mathcal{Q}(\omega)}+\mathcal{Q}(\omega+\pi)\overline{\mathcal{Q}(\omega+\pi)} = 1,\quad \omega\in\mathbb{R}.\end{aligned} \tag{24}$$

Proof *Let us suppose that* $\psi_m(x)$ *is an orthonormal wavelet function associated with* $N_m(x)$. *By Lemma* 1 *and* (21), *we have*

$$\begin{aligned}0 &= \sum_{l\in\mathbb{Z}}\hat{N}_m(2\omega+2\pi l)\overline{\hat{\psi}_m(2\omega+2\pi l)}\\ &= \sum_{l\in\mathbb{Z}}\mathcal{P}(\omega+l\pi)\hat{N}_m(\omega+l\pi)\overline{\hat{N}_m(\omega+l\pi)}\,\overline{\mathcal{Q}(\omega+l\pi)}\\ &= \sum_{\rho=0}^{1}\mathcal{P}(\omega+\rho\pi)\sum_{\nu\in\mathbb{Z}}\hat{N}_m(\omega+\rho\pi+2\nu\pi)\overline{\hat{N}_m(\omega+\rho\pi+2\nu\pi)}\,\overline{\mathcal{Q}(\omega+\rho\pi)}\\ &= \mathcal{P}(\omega)\overline{\mathcal{Q}(\omega)}+\mathcal{P}(\omega+\pi)\overline{\mathcal{Q}(\omega+\pi)}.\end{aligned}$$

Again by Lemma 1 and (22), we have

$$\begin{aligned}1 &= \sum_{l\in\mathbb{Z}}\hat{\psi}_m(2\omega+2\pi l)\overline{\hat{\psi}_m(2\omega+2\pi l)}\\ &= \sum_{\eta=0}^{1}\mathcal{Q}(\omega+\eta\pi)\sum_{\nu\in\mathbb{Z}}\hat{N}_m(\omega+\eta\pi+2\nu\pi)\overline{\hat{N}_m(\omega+\eta\pi+2\nu\pi)}\,\overline{\mathcal{Q}(\omega+\eta\pi)}\\ &= \mathcal{Q}(\omega)\overline{\mathcal{Q}(\omega)}+\mathcal{Q}(\omega+\pi)\overline{\mathcal{Q}(\omega+\pi)}.\end{aligned}$$

On the other hand, let (24) holds.

Now,

$$\begin{aligned}\langle N_m(.),\psi_m(.-l)\rangle &= \frac{1}{2\pi}\left\langle \hat{N}_m(\omega), e^{-i\omega l}\hat{\psi}_m(\omega)\right\rangle\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{N}_m(\omega)\overline{\hat{\psi}_m(\omega)}e^{i\omega l}\,d\omega\\ &= \frac{1}{\pi}\sum_{\nu\in\mathbb{Z}}\int_{2\pi\nu}^{2\pi(\nu+1)}\hat{N}_m(2\omega)\overline{\hat{\psi}_m(2\omega)}e^{2i\omega l}\,d\omega\\ &= \frac{1}{\pi}\int_0^{\pi}\sum_{\nu\in\mathbb{Z}}\hat{N}_m(2\omega+2\nu\pi)\overline{\hat{\psi}_m(2\omega+2\nu\pi)}e^{2i\omega l}\,d\omega\\ &= 0.\end{aligned}$$

Also,

$$\langle \psi_m(.),\psi_m(.-l)\rangle = \frac{1}{\pi}\int_0^{\pi}\sum_{\nu\in\mathbb{Z}}\hat{\psi}_m(2\omega+2\nu\pi)\overline{\hat{\psi}_m(2\omega+2\nu\pi)}e^{2i\omega l}\,d\omega = \delta_{0,l}\quad [\text{by Lemma } 1].$$

Thus, $N_m(x)$ and $\psi_m(x)$ are orthogonal and $\psi_m(x)$ is an orthonormal wavelet function associated with

$N_m(x)$.

3. B-Spline Wavelet Packets and Their Duals

Following Coifman and Meyer [6] [7], we introduce two sequences of L^2 functions $\{\mathcal{W}_{n,m}\}$ and $\{\tilde{\mathcal{W}}_{n,m}\}$ defined by

$$\mathcal{W}_{2n+\lambda,m}(x)=\sum_{k\in\mathbb{Z}}\zeta_k^{(\lambda)}\mathcal{W}_{n,m}(2x-k),\quad \lambda=0,1 \tag{25}$$

$$\tilde{\mathcal{W}}_{2n+\lambda,m}(x)=\sum_{k\in\mathbb{Z}}\overline{\gamma_{-k}^{(\lambda)}}\tilde{\mathcal{W}}_{n,m}(2x-k),\quad \lambda=0,1 \tag{26}$$

where $n=0,1,\cdots$

When $\lambda=0$ and $n=0$, we have

$$\zeta_k^{(0)}=p_k,\quad \mathcal{W}_{0,m}=N_m$$
$$\overline{\gamma_{-k}^{(0)}}=\tilde{g}_{-k},\quad \tilde{\mathcal{W}}_{0,m}=\tilde{N}_m,$$

and for $\lambda=1$ and $n=0$, we have

$$\zeta_k^{(1)}=q_k,\quad \mathcal{W}_{1,m}=\psi_m,$$
$$\overline{\gamma_{-k}^{(1)}}=\tilde{h}_{-k},\quad \tilde{\mathcal{W}}_{1,m}=\tilde{\psi}_m.$$

We call $\{\mathcal{W}_{n,m}\}$ the sequence of B-spline wavelet packets induced by the wavelet ψ_m and its corresponding scaling function N_m whereas $\{\tilde{\mathcal{W}}_{n,m}\}$ denotes the corresponding sequence of dual wavelet packets. By applying the Fourier transformation on both sides of (25), we have

$$\hat{\mathcal{W}}_{2n+\lambda,m}(\omega)=\mathcal{M}^{(\lambda)}\left(\frac{\omega}{2}\right)\hat{\mathcal{W}}_{n,m}\left(\frac{\omega}{2}\right) \tag{27}$$

where,

$$\mathcal{M}^{(\lambda)}(\omega)=\frac{1}{2}\sum_{k\in\mathbb{Z}}\zeta_k^{(\lambda)}\mathrm{e}^{-ik\omega} \tag{28}$$

$$\mathcal{M}^{(0)}(\omega)=\mathcal{P}(\omega),\quad \mathcal{M}^{(1)}(\omega)=\mathcal{Q}(\omega). \tag{29}$$

So, (24) can be written as

$$\begin{aligned}&\mathcal{M}^{(0)}(\omega)\overline{\mathcal{M}^{(1)}(\omega)}+\mathcal{M}^{(0)}(\omega+\pi)\overline{\mathcal{M}^{(1)}(\omega+\pi)}=0,\quad \omega\in\mathbb{R},\\&\mathcal{M}^{(1)}(\omega)\overline{\mathcal{M}^{(1)}(\omega)}+\mathcal{M}^{(1)}(\omega+\pi)\overline{\mathcal{M}^{(1)}(\omega+\pi)}=1,\quad \omega\in\mathbb{R}.\end{aligned} \tag{30}$$

Similarly, taking the Fourier transformation on both sides of (26), we have

$$\hat{\tilde{\mathcal{W}}}_{2n+\lambda,m}(\omega)=\mathcal{L}^{(\lambda)}\left(\frac{\omega}{2}\right)\hat{\tilde{\mathcal{W}}}_{n,m}\left(\frac{\omega}{2}\right), \tag{31}$$

where,

$$\mathcal{L}^{(\lambda)}(\omega)=\frac{1}{2}\sum_{k\in\mathbb{Z}}\overline{\gamma_{-k}^{(\lambda)}}\mathrm{e}^{-ik\omega} \tag{32}$$

$$\mathcal{L}^{(0)}(\omega)=\mathcal{G}(\omega),\quad \mathcal{L}^{(1)}(\omega)=\mathcal{H}(\omega). \tag{33}$$

Using these conditions we can write

$$\mathcal{M}^{(\lambda)}(\omega)\mathcal{L}^{(\mu)}(\omega)+\mathcal{M}^{(\lambda)}(-\omega)\mathcal{L}^{(\mu)}(-\omega)=\delta_{\lambda,\mu},\quad \omega\in\mathbb{R},\quad \lambda,\ \mu=0,1. \tag{34}$$

We are now in a position to investigate the properties of B-spline wavelet packets.

Theorem 2 *Let* $\mathcal{W}_{0,m}(x)$ *be any orthonormal scaling function and* $\{\mathcal{W}_{n,m}(x)\}$ *its corresponding family of*

B-spline wavelet packets. Then for each $n \in \mathbb{Z}_+$, *we have*

$$\left\langle \mathcal{W}_{n,m}(.), \mathcal{W}_{n,m}(.-k) \right\rangle = \delta_{0,k}, \quad k \in \mathbb{Z}. \tag{35}$$

Proof *Since* $\mathcal{W}_{0,m}(x) = N_m(x)$ *satisfies* (35) *for* $n = 0$. *We may proceed to prove* (35) *by induction. Suppose that* (35) *holds for all* n, *where* $0 \le n < 2^r$, r *a positive integer and* $2^r \le n < 2^{r+1}$. *We have* $2^{r-1} \le \left[\frac{n}{2}\right] < 2^r$, *where* $[x]$ *denote the largest integer not exceeding* x. *By induction hypothesis and Lemma* 1, *we have*

$$\left\langle \mathcal{W}_{[n/2],m}(.), \mathcal{W}_{[n/2],m}(.-k) \right\rangle = \delta_{0,k} \Leftrightarrow \sum_{\nu \in \mathbb{Z}} \hat{\mathcal{W}}_{[n/2],m}(\omega + 2\nu\pi) \overline{\hat{\mathcal{W}}_{[n/2],m}(\omega + 2\nu\pi)} = 1. \tag{36}$$

By using (27), (30) and (36), we obtain

$$\begin{aligned}
&\sum_{\nu \in \mathbb{Z}} \hat{\mathcal{W}}_{n,m}(2\omega + 2\nu\pi) \overline{\hat{\mathcal{W}}_{n,m}(2\omega + 2\nu\pi)} \\
&= \sum_{\nu \in \mathbb{Z}} \mathcal{M}^{(\lambda)}(\omega + \nu\pi) \hat{\mathcal{W}}_{[n/2],m}(\omega + \nu\pi) \cdot \overline{\hat{\mathcal{W}}_{[n/2],m}(\omega + \nu\pi)}\, \overline{\mathcal{M}^{(\lambda)}(\omega + \nu\pi)} \\
&= \sum_{\rho=0}^{1} \mathcal{M}^{(\lambda)}(\omega + \rho\pi) \sum_{k \in \mathbb{Z}} \hat{\mathcal{W}}_{[n/2],m}(\omega + \rho\pi + 2k\pi) \cdot \overline{\hat{\mathcal{W}}_{[n/2],m}(\omega + \rho\pi + 2k\pi)}\, \overline{\mathcal{M}^{(\lambda)}(\omega + \rho\pi)} \\
&= \mathcal{M}^{(\lambda)}(\omega) \overline{\mathcal{M}^{(\lambda)}(\omega)} + \mathcal{M}^{(\lambda)}(\omega + \pi) \overline{\mathcal{M}^{(\lambda)}(\omega + \pi)} = 1.
\end{aligned}$$

Hence, by Lemma 1, (35) follows.

Theorem 3 *Let* $\{\mathcal{W}_{n,m}(x)\}$ *be a B-spline wavelet packet with respect to the orthonormal scaling function* $N_m(x) = \mathcal{W}_{0,m}(x)$. *Then for every* $n \in \mathbb{Z}_+$, *we have*

$$\left\langle \mathcal{W}_{2n,m}(.), \mathcal{W}_{2n+1,m}(.-k) \right\rangle = 0, \quad k \in \mathbb{Z}. \tag{37}$$

Proof *By* (27), (30) *and* (36), *for* $k \in \mathbb{Z}$ *we have*

$$\begin{aligned}
\left\langle \mathcal{W}_{2n,m}(.), \mathcal{W}_{2n+1,m}(.-k) \right\rangle &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mathcal{W}}_{2n,m}(\omega) \overline{\hat{\mathcal{W}}_{2n+1,m}(\omega)} \mathrm{e}^{i\omega k} \mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{M}^{(0)}\left(\frac{\omega}{2}\right) \hat{\mathcal{W}}_{n,m}\left(\frac{\omega}{2}\right) \overline{\mathcal{M}^{(1)}\left(\frac{\omega}{2}\right)}\, \overline{\hat{\mathcal{W}}_{n,m}\left(\frac{\omega}{2}\right)} \mathrm{e}^{i\omega k} \mathrm{d}\omega \\
&= \frac{1}{\pi} \int_{\mathbb{R}} \mathcal{M}^{(0)}(\omega) \hat{\mathcal{W}}_{n,m}(\omega) \overline{\hat{\mathcal{W}}_{n,m}(\omega)}\, \overline{\mathcal{M}^{(1)}(\omega)} \mathrm{e}^{2i\omega k} \mathrm{d}\omega \\
&= \frac{1}{\pi} \sum_{\nu \in \mathbb{Z}} \int_{2\pi\nu}^{2\pi(\nu+1)} \mathcal{M}^{(0)}(\omega) \left\{ \hat{\mathcal{W}}_{n,m}(\omega) \overline{\hat{\mathcal{W}}_{n,m}(\omega)} \right\} \overline{\mathcal{M}^{(1)}(\omega)} \mathrm{e}^{2i\omega k} \mathrm{d}\omega \\
&= \frac{1}{\pi} \int_{0}^{2\pi} \mathcal{M}^{(0)}(\omega) \left\{ \sum_{\nu \in \mathbb{Z}} \hat{\mathcal{W}}_{n,m}(\omega + 2\nu\pi) \overline{\hat{\mathcal{W}}_{n,m}(\omega + 2\nu\pi)} \right\} \overline{\mathcal{M}^{(1)}(\omega)} \mathrm{e}^{2i\omega k} \mathrm{d}\omega \\
&= \frac{1}{\pi} \int_{0}^{\pi} \left[\mathcal{M}^{(0)}(\omega) \overline{\mathcal{M}^{(1)}(\omega)} + \mathcal{M}^{(0)}(\omega + \pi) \overline{\mathcal{M}^{(1)}(\omega + \pi)} \right] \mathrm{e}^{2i\omega k} \mathrm{d}\omega = 0.
\end{aligned}$$

For the family of B-spline wavelet packets $\{\mathcal{W}_{n,m}(x)\}$ corresponding to some orthonormal scaling function $\mathcal{W}_{0,m} = N_m$, consider the family of subspaces

$$\tau_j^{n,m} := \operatorname{clos}_{L^2(\mathbb{R})} \left\langle 2^{j/2} W_{n,m}\left(2^j x - k\right) : k \in \mathbb{Z} \right\rangle, \quad j \in \mathbb{Z}, \; n \in \mathbb{Z}_+$$

generated by $\{\mathcal{W}_{n,m}\}$. We observe that

$$\tau_j^{0,m} = V_j^m, \quad j \in \mathbb{Z}$$
$$\tau_j^{1,m} = W_j^m, \quad j \in \mathbb{Z},$$

where $\{V_j^m\}$ is the MRA of $L^2(\mathbb{R})$ generated by $\mathcal{W}_{0,m} = N_m$, and $\{W_j^m\}$ is the sequence of orthogonal complementary (wavelet) subspaces generated by the wavelet $\mathcal{W}_{1,m}=\psi_m$. Then the orthogonal decomposition

$$V_{j+1}^m = V_j^m \oplus W_j^m, \quad \forall j \in \mathbb{Z}$$

may be written as

$$\tau_{j+1}^{0,m} = \tau_j^{0,m} \oplus \tau_j^{1,m}, \quad \forall j \in \mathbb{Z}.$$

A generalization of the above result for other values of n can be written as

$$\tau_{j+1}^{n,m} = \tau_j^{2n,m} \oplus \tau_j^{2n+1,m}, \quad \forall j \in \mathbb{Z}.$$

Theorem 4 *For the B-spline wavelet packets, the following two scale relation*

$$\mathcal{W}_{n,m}\left(2^{j+1}x-l\right)=\frac{1}{2}\sum_k\left\{\overline{p}_{l-2k}\mathcal{W}_{2n,m}\left(2^jx-k\right)+\overline{q}_{l-2k}\mathcal{W}_{2n+1,m}\left(2^jx-k\right)\right\} \tag{38}$$

holds for all $l\in\mathbb{Z}$.

Proof *In order to prove the two scale relation, we need the following identity, see* ([15], Lemma 7.9)

$$\sum_k\left\{p_{r-2k}\overline{p}_{l-2k}+q_{r-2k}\overline{q}_{l-2k}\right\}=2\delta_{r,l}. \tag{39}$$

Taking the right-hand side of (38), and applying the identity (39), we have

$$\begin{aligned}\frac{1}{2}\sum_k\left\{\overline{p}_{l-2k}\mathcal{W}_{2n,m}\left(2^jx-k\right)+\overline{q}_{l-2k}\mathcal{W}_{2n+1,m}\left(2^jx-k\right)\right\} &= \frac{1}{2}\sum_k\sum_l\left\{p_l\overline{p}_{l-2k}+q_l\overline{q}_{l-2k}\right\}\mathcal{W}_{n,m}\left(2^{j+1}x-2k-l\right)\\ &=\frac{1}{2}\sum_k\sum_t\left\{p_{t-2k}\overline{p}_{l-2k}+q_{t-2k}\overline{q}_{l-2k}\right\}\mathcal{W}_{n,m}\left(2^{j+1}x-t\right)\\ &=\sum_t\left\{\frac{1}{2}\sum_k\left[p_{t-2k}\overline{p}_{l-2k}+q_{t-2k}\overline{q}_{l-2k}\right]\right\}\mathcal{W}_{n,m}\left(2^{j+1}x-t\right)\\ &=\sum_t\delta_{t,l}\mathcal{W}_{n,m}\left(2^{j+1}x-t\right).\end{aligned} \tag{40}$$

$t=l\Rightarrow\mathcal{W}_{n,m}\left(2^{j+1}x-l\right)$ This completes the proof of the theorem.

Next, we discuss the duality properties between the wavelet packets $\{W_{n,m}\}$ and $\{\tilde{W}_{n,m}\}$.

Lemma 2 *For all* $k,\ l\in\mathbb{Z}$ *and* $n\in\mathbb{Z}_+$,

$$\left\langle\mathcal{W}_{n,m}(.-k),\tilde{\mathcal{W}}_{n,m}(.-l)\right\rangle=\delta_{k,l},\quad k\in\mathbb{Z}. \tag{41}$$

Proof *We will prove* (41) *by induction on* n. *The case* $n=0$ *is the same as our assumption* (12) *on the dual scaling functions* $\mathcal{W}_{0,m}=N_m$ *and* $\tilde{\mathcal{W}}_{0,m}=\tilde{N}_m$. *Suppose that* (41) *holds for all* n *where* $0\le n<2^r$, *where* r *is a positive integer. Then for* $2^r\le n<2^{r+1}$, *we can write* $n=2\left[\frac{n}{2}\right]+\lambda$ *for some* $\lambda\in\{0,1\}$, *according to the proof of Theorem* 7.24 *in* [15]. *From the Fourier transform formulations of equations* (25) *and* (26) *and using* (34) *we have*

$$\begin{aligned}\left\langle\mathcal{W}_{n,m}(.-k),\tilde{\mathcal{W}}_{n,m}(.-l)\right\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\mathcal{W}}_{n,m}(\omega)\overline{\hat{\tilde{\mathcal{W}}}_{n,m}(\omega)}\mathrm{e}^{i(l-k)\omega}\mathrm{d}\omega\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{M}^{(\lambda)}\left(\frac{\omega}{2}\right)\mathcal{L}^{(\lambda)}\left(\frac{\omega}{2}\right)\hat{\mathcal{W}}_{\left[\frac{n}{2}\right],m}\left(\frac{\omega}{2}\right)\overline{\hat{\tilde{\mathcal{W}}}_{\left[\frac{n}{2}\right],m}\left(\frac{\omega}{2}\right)}\mathrm{e}^{i(l-k)\omega}\mathrm{d}\omega\\ &=\frac{1}{2\pi}\int_0^{4\pi}\mathrm{e}^{i(l-k)\omega}\mathcal{M}^{(\lambda)}\left(\frac{\omega}{2}\right)\mathcal{L}^{(\lambda)}\left(\frac{\omega}{2}\right)\sum_{j\in\mathbb{Z}}\hat{\mathcal{W}}_{\left[\frac{n}{2}\right],m}\left(\frac{\omega}{2}+2\pi j\right)\cdot\overline{\hat{\tilde{\mathcal{W}}}_{\left[\frac{n}{2}\right],m}\left(\frac{\omega}{2}+2\pi j\right)}\mathrm{d}\omega.\end{aligned}$$

Since $\left[\frac{n}{2}\right]\le\frac{n}{2}<2^r$, it follows from the induction hypothesis that $\left\langle\mathcal{W}_{\left[\frac{n}{2}\right],m}(.-k),\tilde{\mathcal{W}}_{\left[\frac{n}{2}\right],m}(.-l)\right\rangle=\delta_{k,l}$ for all $k,\ l\in\mathbb{Z}$, and this is equivalent to

$$\sum_{j\in\mathbb{Z}}\hat{\mathcal{W}}_{\left[\frac{n}{2}\right],m}\left(\frac{\omega}{2}+2\pi j\right)\overline{\hat{\tilde{\mathcal{W}}}_{\left[\frac{n}{2}\right],m}\left(\frac{\omega}{2}+2\pi j\right)}=1\quad a.e.. \tag{42}$$

Thus, we have

$$\begin{aligned}\left\langle \mathcal{W}_{n,m}(.-k),\tilde{\mathcal{W}}_{n,m}(.-l)\right\rangle &= \frac{1}{2\pi}\int_0^{4\pi}\mathrm{e}^{i(l-k)\omega}\mathcal{M}^{(\lambda)}\left(\frac{\omega}{2}\right)\mathcal{L}^{(\lambda)}\left(\frac{\omega}{2}\right)\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{i(l-k)\omega}\left[\mathcal{M}^{(\lambda)}\left(\frac{\omega}{2}\right)\mathcal{L}^{(\lambda)}\left(\frac{\omega}{2}\right)+\mathcal{M}^{(\lambda)}\left(-\frac{\omega}{2}\right)\mathcal{L}^{(\lambda)}\left(-\frac{\omega}{2}\right)\right]\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{i(l-k)\omega}\mathrm{d}\omega=\delta_{k,l}.\end{aligned}$$

This shows that (41) also holds for $2^r \le n < 2^{r+1}$.

Lemma 3 *For all* $k,\ l\in\mathbb{Z}$ *and* $n\in\mathbb{Z}_+$, *and* $\lambda,\ \mu\in\{0,1\}$, *with* $\lambda\neq\mu$,

$$\left\langle \mathcal{W}_{2n+\lambda,m}(.-k),\tilde{\mathcal{W}}_{2n+\lambda,m}(.-l)\right\rangle=0,\quad k\in\mathbb{Z}. \tag{43}$$

Proof *By applying the Fourier transform formulations of Equations* (25) *and* (26) *and using* (42) *and* (34), *we have as in the proof of Lemma* 2 *that*

$$\begin{aligned}\left\langle \mathcal{W}_{2n+\lambda,m}(.-k),\tilde{\mathcal{W}}_{2n+\lambda,m}(.-l)\right\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\mathcal{W}}_{2n+\lambda,m}(\omega)\overline{\hat{\tilde{\mathcal{W}}}_{2n+\lambda,m}(\omega)}\mathrm{e}^{i(l-k)\omega}\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{M}^{(\lambda)}\left(\frac{\omega}{2}\right)\mathcal{L}^{(\lambda)}\left(\frac{\omega}{2}\right)\hat{\mathcal{W}}_{n,m}\left(\frac{\omega}{2}\right)\overline{\hat{\tilde{\mathcal{W}}}_{n,m}\left(\frac{\omega}{2}\right)}\mathrm{e}^{i(l-k)\omega}\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_0^{4\pi}\mathrm{e}^{i(l-k)\omega}\mathcal{M}^{(\lambda)}\left(\frac{\omega}{2}\right)\mathcal{L}^{(\lambda)}\left(\frac{\omega}{2}\right)\sum_{j\in\mathbb{Z}}\hat{\mathcal{W}}_{n,m}\left(\frac{\omega}{2}+2\pi j\right)\overline{\hat{\tilde{\mathcal{W}}}_{n,m}\left(\frac{\omega}{2}+2\pi j\right)}\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_0^{4\pi}\mathrm{e}^{i(l-k)\omega}\mathcal{M}^{(\lambda)}\left(\frac{\omega}{2}\right)\mathcal{L}^{(\lambda)}\left(\frac{\omega}{2}\right)\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{i(l-k)\omega}\left[\mathcal{M}^{(\lambda)}\left(\frac{\omega}{2}\right)\mathcal{L}^{(\lambda)}\left(\frac{\omega}{2}\right)+\mathcal{M}^{(\lambda)}\left(-\frac{\omega}{2}\right)\mathcal{L}^{(\lambda)}\left(-\frac{\omega}{2}\right)\right]\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{i(l-k)\omega}\delta_{\lambda,\mu}\mathrm{d}\omega=\delta_{\lambda,\mu}\delta_{k,l}=0,\quad \lambda\neq\mu,\quad k=l.\end{aligned}$$

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