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Amenability and the Extension Property

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Abstract

Let *G* be a locally compact group, *H* a closed amenable subgroup and *u* an element of the Herz Figà-Talamanca algebra of *H* with compact support, we prove the existence of an extension of *u* to *G*, with a good control of the norm and of the support of the extension.

Keywords

Convolution Operators, Locally Compact Groups, Abstract Harmonic Analysis, Amenable Groups

1. Introduction

Let G be a locally compact group and H a closed subgroup, this paper is concerned with the problem of extending coefficients of the regular representation of H to G. Suppose H normal in G. In 1973 [1] C. Herz proved that for $u \in A_p(H)$ with compact support, for every $\varepsilon > 0$ and for every U neighborhood of $\operatorname{supp} u$ in G there is $v \in A_p(G)$ with $\operatorname{Res}_H v = u$, $\|v\| < \|u\| + \varepsilon$ and $\operatorname{supp} v \subset U$. In this work we want to treat the case of non normal subgroups. We succeed assuming that the subgroup H is amenable (Theorem 5). C. Fiorillo obtained [2] already this result assuming however the unimodularity of G and of H. But the AN part of the Iwasawa decomposition of $SL_2(\mathbb{R})$ was out of reach. Even for G amenable our result is new: the case of the non-normal copy of $\mathbb{R}$ in the $ax+b$-group was also out of reach.

Without control of norm and support of the extension, the theorem has been obtained in 1972 by McMullen [3]. With control of the norm, but not considering the supports, the statement is due Herz [1] (see also [4]).

2. A Property of Amenable Subgroups

We denote by $C_{00}(G)$ the set of all complex valued continuous functions on G with compact support. We choose a positive continuous function q on G such that $q(xh) = q(x)\Delta_H(h)\Delta_G(h^{-1})$, left invariant measures on G and H and a measure $d_q\dot{x}$ on G/H as in Chapter 8 of [5]. The following Lemma will be used in the proof of our main result. See below the steps $1)_3$ and $1)_4$ of the proof of Lemma 2.

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Lemma 1 *Let G be a locally compact group, H a closed amenable subgroup, K a compact subset of G, U a neighborhood of e in G and $\varepsilon>0$. Then there is $f\in C_{00}^{+}(G)$ such that $N_1(f)=1$, $\operatorname{supp} f\subset U$ and*

$$\int_K\left|\int_H f(hx)\Delta_G\left(h^{-1}\right)\mathrm{d}h-\int_H f(xh)\mathrm{d}h\right|\mathrm{d}x<\varepsilon$$

Proof. Let U_0 be a compact neighborhood of e in G with $U_0\subset U$, $K_1=\left(U_0K^{-1}\cup K^{-1}U_0\right)\cap H$ and $\delta=\max_{h\in K_1}\Delta_G\left(h^{-1}\right)$ By the Proposition 2.1 of [6] (p. 463), there is $f\in C_{00}^{+}(G)$ such that $N_1(f)=1$, $\operatorname{supp} f\subset U_0$ and such that $N_1\left(f_h\Delta_G(h)-_h f\right)<\varepsilon/2\delta m_H(K_1)$ for every $h\in K_1$. For every $x\in K$ we have

$$\left|\int_H f(hx)\Delta_G\left(h^{-1}\right)\mathrm{d}h-\int_H f(xh)\mathrm{d}h\right|\le\int_H 1_{K_H}(h)\left|f(hx)\Delta_G\left(h^{-1}\right)-f(xh)\right|\mathrm{d}h$$

where $K_H=\left(\operatorname{supp} fK^{-1}\cup K^{-1}\operatorname{supp} f\right)\cap H$ Consequently

$$\begin{aligned}\int_K\left|\int_H f(hx)\Delta_G\left(h^{-1}\right)\mathrm{d}h-\int_H f(xh)\mathrm{d}h\right|\mathrm{d}x&\le\int_G 1_K(x)\left(\int_H 1_{K_H}(h)\left|f(hx)\Delta_G\left(h^{-1}\right)-f(xh)\right|\mathrm{d}h\right)\mathrm{d}x\\&\le\int_H 1_{K_H}(h)\left(\int_G\left|f(hx)\Delta_G\left(h^{-1}\right)-f(xh)\right|\mathrm{d}x\right)\mathrm{d}h.\ \square\end{aligned}$$

3. Approximation Theorem for Convolution Operators Supported by Subgroups

We refer to [7] for $A_p(G)$, $CV_p(G)$, $PM_p(G)$ and the canonical map i of $CV_p(H)$ into $CV_p(G)$ (Section 7.1 p. 101). We denote by $\mathcal{L}\left(L^p(G)\right)$ the Banach space of all bounded operators of $L^p(G)$.

We define a family of linear maps $\Lambda_{k,l}^{q}$ of $\mathcal{L}\left(L^p(G)\right)$ into $\mathcal{L}\left(L^p(H)\right)$ where H is an arbitrary closed subgroup of G. We precise that τ_p is the involution of $L^p(G)$ $\tau_p(f)(x)=f\left(x^{-1}\right)\Delta_G\left(x^{-1}\right)^{1/p}$ and that for $k\in C_{00}(G)$, $\varphi\in C_{00}(H)$ and $x\in G$ we have $(k*_H\varphi)(x)=\int_H k(xh)\varphi\left(h^{-1}\right)\mathrm{d}h$.

Definition 1. *Let G be a locally compact group, H an arbitrary closed subgroup, $1<p<\infty$ and $k,\ l\in C_{00}(G)$. For $T\in\mathcal{L}\left(L^p(G)\right)$ we set for $\varphi,\ \psi\in C_{00}(H)$*

$$\left\langle\Lambda_{k,l}^{q}(T)[\varphi],[\psi]\right\rangle=\left\langle T\left[\tau_p\left(q^{1/p}\left(k*_H\tau_p\varphi\right)\right)\right],\left[\tau_{p'}\left(q^{1/p'}\left(k*_H\tau_{p'}\psi\right)\right)\right]\right\rangle.$$

Then $\Lambda_{k,l}^{q}(T)\in\mathcal{L}\left(L^p(H)\right)$ and $\left\|\Lambda_{k,l}^{q}(T)\right\|\le\|T\|N_p\left(T_H|k|\right)N_{p'}\left(T_H|l|\right)$ where $T_Hk(\dot{x})=\int_H k(xh)\mathrm{d}h$. If $T\in CV_p(G)$ then $\Lambda_{k,l}^{q}(T)\in CV_p(H)$ and $\operatorname{supp}\Lambda_{k,l}^{q}(T)$ is contained in $(\operatorname{supp}k)^{-1}\operatorname{supp}T(\operatorname{supp}l)$ [8].

Lemma 2. *Let G be a locally compact group, H a closed amenable subgroup, $p>1$, $\varphi_1,\cdots,\varphi_m$, $\psi_1,\cdots,\psi_m\in C_{00}(H)$, $\varepsilon>0$ and U an open neighborhood of e in G. Then there is $k,\ l\in C_{00}^{+}(G)$ with $\operatorname{supp}k\subset U$, $\operatorname{supp}l\subset U$, $N_p(T_Hk)N_{p'}(T_Hl)<1+\varepsilon$ and such that*

$$\left|\left\langle\Lambda_{k,l}^{q}(i(S))\left[\varphi_j\right],\left[\psi_j\right]\right\rangle-\left\langle S\left[\varphi_j\right],\left[\psi_j\right]\right\rangle\right|\le\varepsilon\|S\|$$

for every $1\le j\le m$ and every $S\in CV_p(H)$.

Proof. Let $0<\eta<1$ with $\eta<2^{-1}\varepsilon\left(4+N_p\left(\varphi_j\right)+N_{p'}\left(\psi_j\right)\right)^{-1}$ for every $1\le j\le m$. There is U_0 a compact symmetric neighborhood of e in G with $U_0\subset U$ and such that $\Delta_H(h)>(1+\eta)^{-1}$ for every $h\in U_0^2\cap H$. There is V open neighborhood of e in H such that $N_p\left(\varphi_j-\left(\varphi_j\right)_{h^{-1}}\Delta_H\left(h^{-1}\right)\right)$ and $N_{p'}\left(\psi_j-\left(\psi_j\right)_{h^{-1}}\Delta_H\left(h^{-1}\right)\right)$ are both smaller than $\eta/2$ for every $1\le j\le m$ and for every $h\in V$. We can choose $k'\in C_{00}^{+}(G)$ with $\operatorname{supp}k'\subset U_0$, $\int_H k'(h)\mathrm{d}h=1$ $\int_H k'(xh)\mathrm{d}h\le1$ $\int_H k'(hx)\Delta_H\left(h^{-1}\right)\mathrm{d}h\le1$ for every $x\in G$ and such that $\operatorname{supp}k'\cap H\subset V$.

Let U_1 be a symmetric compact neighborhood of e in G contained in U_0 with

$$(1+\eta)^{-1}<\Delta_G(x)<1+\eta$$

for every $x \in U_1$ and such that

$$N_p\left(\mathrm{Res}_H\left(\varphi_j *_H k'\right) - \left(\varphi_j *_H k'\right)_{x,H}\right) < \eta/2,$$

$$N_{p'}\left(\mathrm{Res}_H\left(\psi_j *_H k'\right) - \left(\psi_j *_H k'\right)_{x,H}\right) < \eta/2$$

for every $1 \le j \le m$ and for every $x \in U_1$ (for $f : G \to \mathbb{C}$ and $x \in G$ we denote by $f_{x,H}$ the function defined on H by $h \mapsto f(xh)$).

We put $K = \bigcup_{j=1}^m \mathrm{supp}\,\varphi_j \cup \mathrm{supp}\,\psi_j$, $A = \max_{x\in U_1}\left(T_H q^{-1/p}\tau_p k'\right)(\omega(x))$ and $B = \max_{x\in U_1}\left(T_H q^{-1/p'}\tau_{p'} k'\right)(\omega(x))$ where ω is the canonical map of G onto G/H.

By the preceding Lemma there is $f \in C_{00}^+(G)$ with $\mathrm{supp}\, f \subset U_1$ $N_1(f) = 1$ and such that

$$\int_{(KU_0)^{-1}} \left| \int_H f(hx)\Delta_G\left(h^{-1}\right)\mathrm{d}h - \int_H f(xh)\,\mathrm{d}h \right| \mathrm{d}x$$

is smaller than

$$\frac{\eta^p}{2^p\left(1 + \left\|\varphi_j *_H k'\right\|_\infty\right)^p \max_{x\in KU_0}\Delta_G(x)\left(1 + N_{p'}\left(\psi_j\right)\max_{x\in U_1} q(x)^{1/p'} B\right)^p}$$

and also smaller than

$$\frac{\eta^{p'}}{2^{p'}\left(1 + \left\|\psi_j *_H k'\right\|_\infty\right)^{p'}\left(1 + N_p\left(\varphi_j\right)\right)^{p'} \max_{x\in KU_0}\Delta_G(x)^{p'/p} \max_{x\in U_1} q(x)^{p'/p} A^{p'}}$$

for every $1 \le j \le m$. We finally put $F = (T_H f)\circ\omega$ $L = \left(T_H q^{-1}\breve{f}\right)\circ\omega$, $k'' = q^{-1/p}F^{1/p}\tau_p k'$ and $l'' = q^{-1/p'}F^{1/p'}\tau_{p'} k'$.

1) For every $S \in CV_p(H)$ and every $1 \le j \le m$ we have

$$\left|\left\langle \Lambda_{k'',l''}^q(i(S))\left[\varphi_j\right], \left[\psi_j\right]\right\rangle - \left\langle i(S)\left[L^{1/p} q^{1/p}\left(\varphi_j *_H k'\right)\right], \left[L^{1/p'} q^{1/p'}\left(\psi_j *_H k'\right)\right]\right\rangle\right| \le \eta\|S\|.$$

$1)_1$ We show at first that

$$\begin{aligned}&\left|\left\langle \Lambda_{k'',l''}^q(i(S))\left[\varphi_j\right], \left[\psi_j\right]\right\rangle - \left\langle i(S)\left[L^{1/p} q^{1/p}\left(\varphi_j *_H k'\right)\right], \left[L^{1/p'} q^{1/p'}\left(\psi_j *_H k'\right)\right]\right\rangle\right| \\ &\le \|S\| N_p\left(\left(\varphi_j *_H k'\right)\left(\breve{F}^{1/p} - L^{1/p} q^{1/p}\right)\right) N_{p'}\left(\breve{F}^{1/p'}\left(\psi_j *_H k'\right)\right) \\ &\quad + \|S\| N_p\left(q^{1/p}\breve{L}^{1/p'}\left(\varphi_j *_H k'\right)\right) N_{p'}\left(\left(\psi_j *_H k'\right)\left(\breve{F}^{1/p'} - L^{1/p'} q^{1/p'}\right)\right).\end{aligned}$$

From

$$\tau_p\left(q^{1/p}\left(k'' *_H \tau_p\varphi_j\right)\right) = \varphi_j *_H \left(\breve{F}^{1/p} k'\right) = \breve{F}^{1/p}\left(\varphi_j *_H k'\right)$$

we obtain indeed

$$\left\langle \Lambda_{k'',l''}^q(i(S))\left[\varphi_j\right], \left[\psi_j\right]\right\rangle = \left\langle i(S)\left[\breve{F}^{1/p}\left(\varphi_j *_H k'\right)\right], \left[\breve{F}^{1/p'}\left(\psi_j *_H k'\right)\right]\right\rangle.$$

$1)_2$ For every $1 \le j \le m$ we have

$$N_{p'}\left(\breve{F}^{1/p'}\left(\psi_j *_H k'\right)\right) \le N_{p'}\left(\psi_j\right)\max_{x\in U_1} q(x)^{1/p'} B.$$

We have

$$N_{p'}\left(\breve{F}^{1/p'}\left(\psi_j *_H k'\right)\right)^p = \int_{G/H} T_H f(\dot{x})\left(\int_H \left|\tau_{p'}\left(\psi_j *_H k'\right)(xh)\right|^{p'} q(xh)^{-1}\,\mathrm{d}h\right)\mathrm{d}_q\dot{x}.$$

But for every $x \in G$

$$\frac{\left|\tau_{p'}\left(\psi_j *_H k'\right)(xh)\right|^{p'}}{q(xh)} = \left|\left(\tau_{p'}\left(\breve{q}^{-1/p'}k'\right) *_H \tau_{p'}\psi_j\right)(xh)\right|^{p'}$$

and therefore

$$\int_H \frac{\left|\tau_{p'}\left(\psi_j *_H k'\right)(xh)\right|^{p'}}{q(xh)} \mathrm{d}h = N_{p'}\left(\left(\tau_{p'}\left(\breve{q}^{-1/p'}k'\right)\right)_{x,H} * \tau_{p'}\psi_j\right)^{p'}$$

consequently

$$N_{p'}\left(\breve{F}^{1/p'}\left(\psi_j *_H k'\right)\right)^{p'} \le N_{p'}\left(\psi_j\right)^{p'} \int_{G/H} T_H f(\dot{x}) N_1\left(\tau_{p'}\left(\breve{q}^{-1/p'}k'\right)_{x,H}\right)^{p'} \mathrm{d}_q\dot{x}$$

$$= N_{p'}\left(\psi_j\right)^{p'} \int_G q(x) f(x)\left(T_H\left(q^{-1/p'}\tau_{p'}k'\right)(\omega(x))\right)^{p'} \mathrm{d}x.$$

$1)_3$ For every $1 \le j \le m$ we have

$$N_p\left(\left(\varphi_j *_H k'\right)\left(\breve{F}^{1/p} - L^{1/p}q^{1/p}\right)\right) \le \left\|\varphi_j *_H k'\right\|_\infty \max_{x\in KU_0} \Delta_G(x)^{1/p}$$

$$\times\left(\int_{(KU_0)^{-1}} \left|\int_H f(hx)\Delta_G\left(h^{-1}\right)\mathrm{d}h - \int_H f(xh)\mathrm{d}h\right| \mathrm{d}x\right)^{1/p}.$$

As above

$$N_p\left(\left(\varphi_j *_H k'\right)\left(\breve{F}^{1/p} - L^{1/p}q^{1/p}\right)\right)^p \le \int_G \left|\left(\varphi_j *_H k'\right)(x)\right|^p \left|\breve{F}(x) - q(x)L(x)\right| \mathrm{d}x$$

taking in account that $q(x)L(x) = \int_H f\left(hx^{-1}\right)\Delta_G\left(h^{-1}\right)\mathrm{d}h$ we obtain

$$N_p\left(\left(\varphi_j *_H k'\right)\left(\breve{F}^{1/p} - L^{1/p}q^{1/p}\right)\right)^p \le \int_G \left|\left(\varphi_j *_H k'\right)\left(x^{-1}\right)\right|^p \Delta_G\left(x^{-1}\right)\left|\int_H f(hx)\Delta_G\left(h^{-1}\right)\mathrm{d}h - \int_H f(xh)\mathrm{d}h\right| \mathrm{d}x.$$

$1)_4$ Proof of 1). Using $1)_3$ and $1)_2$ one obtains an estimate for $N_p\left(q^{1/p}L^{1/p}\left(\varphi_j *_H k'\right)\right)$. We finish then the proof of 1) using $1)_1$.

2) For every $S \in CV_p(H)$ and every $1 \le j \le m$ we have

$$\left|\left\langle i(S)\left[L^{1/p}q^{1/p}\left(\varphi_j *_H k'\right)\right],\left[L^{1/p'}q^{1/p'}\left(\psi_j *_H k'\right)\right]\right\rangle - \int_G \breve{f}(x)\mathrm{d}x \left\langle S\left[\varphi_j\right],\left[\psi_j\right]\right\rangle\right|$$

$$\le \eta\|S\|\int_G \breve{f}(x)\mathrm{d}x\left(1 + N_p\left(\varphi_j\right) + N_{p'}\left(\psi_j\right)\right).$$

By the Corollary 6 of section 7.2 p.112 of [7]

$$\left\langle i(S)\left[L^{1/p}q^{1/p}\left(\varphi_j *_H k'\right)\right],\left[L^{1/p'}q^{1/p'}\left(\psi_j *_H k'\right)\right]\right\rangle$$

$$= \int_G \left(T_{H,q}\breve{f}\right)(\omega(x))\beta(x)q(x)\left\langle S\left[\left(\varphi_j *_H k'\right)_{x,H}\right],\left[\left(\psi_j *_H k'\right)_{x,H}\right]\right\rangle \mathrm{d}x.$$

Consequently

$$\left|\left\langle i(S)\left[L^{1/p}q^{1/p}\left(\varphi_j *_H k'\right)\right],\left[L^{1/p'}q^{1/p'}\left(\psi_j *_H k'\right)\right]\right\rangle - \int_G \breve{f}(x)\mathrm{d}x \left\langle S\left[\varphi_j\right],\left[\psi_j\right]\right\rangle\right|$$

$$\le \int_G \left(T_{H,q}\breve{f}\right)(\omega(x))\beta(x)q(x)1_{U_1H}(x)\left|\left\langle S\left[\left(\varphi_j *_H k'\right)_{x,H}\right],\left[\left(\psi_j *_H k'\right)_{x,H}\right]\right\rangle - \left\langle S\left[\varphi_j\right],\left[\psi_j\right]\right\rangle\right| \mathrm{d}x.$$

But by definition of U_1 for every $x \in U_1H$ we have

$$\left|\left\langle S\left[\left(\varphi_j *_H k'\right)_{x,H}\right],\left[\left(\psi_j *_H k'\right)_{x,H}\right]\right\rangle - \left\langle S\left[\varphi_j\right],\left[\psi_j\right]\right\rangle\right| \le \eta\|S\|\left(1 + N_p\left(\varphi_j\right) + N_{p'}\left(\psi_j\right)\right).$$

3) End of the proof of Lemma 2. We are now able to define the functions k and l of the Lemma $k = \left(\int_G \breve{f}(x)\mathrm{d}x\right)^{-1/p} k''$ and $l = \left(\int_G \breve{f}(x)\mathrm{d}x\right)^{-1/p'} l''$. Using 1) and 2) we get

$$\left|\left\langle \Lambda_{k,l}^q(i(S))\left[\varphi_j\right],\left[\psi_j\right]\right\rangle - \left\langle S\left[\varphi_j\right],\left[\psi_j\right]\right\rangle\right| \le \left(\int_G \breve{f}(x)\mathrm{d}x\right)^{-1}\eta\|S\| + \eta\|S\|\left(1 + N_p\left(\varphi_j\right) + N_{p'}\left(\psi_j\right)\right)$$

$$\le (1+\eta)\eta\|S\| + \left(1 + N_p\left(\varphi_j\right) + N_{p'}\left(\psi_j\right)\right)\|S\|\eta$$

$$\le \eta\|S\|\left(3 + N_p\left(\varphi_j\right) + N_{p'}\left(\psi_j\right)\right) \le \varepsilon\|S\|.$$

Clearly $\operatorname{supp} k \subset U$ and $\operatorname{supp} l \subset U$. It remains to show that $N_p(T_H k) N_{p'}(T_H l) < 1+\varepsilon$. We have

$$N_p(T_H k'')^p = \int_G f(x) 1_{U_1}(x) \Delta_G(x^{-1}) \left(\left(\int_H k'(hx^{-1}) \Delta_H(h^{-1}) \right)^{1/p'} \mathrm{d}h \right)^p \mathrm{d}x.$$

But for $x \in U_1$

$$\int_H k'(hx^{-1}) \Delta_H(h^{-1})^{1/p'} \mathrm{d}h \le (1+\eta)^{1/p}$$

hence $N_p(T_H k) < (1+\eta)^{3/p}$ and similarly $N_{p'}(T_H l) < (1+\eta)^{3/p'}$, we finally get $N_p(T_H k) N_{p'}(T_H l) < 1+\varepsilon$. □

Theorem 3 *Let G be a locally compact group, H a closed amenable subgroup, $p > 1$, (r_n) a sequence of $\mathcal{L}^p(H)$, (s_n) a sequence of $\mathcal{L}^{p'}(H)$, $\varepsilon > 0$ and U an open neighborhood of e in G. Suppose that the series $\sum N_p(r_n) N_{p'}(s_n)$ converges. Then there is $k, l \in C_{00}^+(G)$ with* $\operatorname{supp} k \subset U$, $\operatorname{supp} l \subset U$, *$N_p(T_H k) N_{p'}(T_H l) < 1$ and such that*

$$\sum_{n=1}^{\infty} \left| \left\langle \Lambda_{k,l}^q(i(S))[r_n],[s_n] \right\rangle - \left\langle S[r_n],[s_n] \right\rangle \right| \le \varepsilon \|S\|$$

for every $S \in CV_p(H)$

Proof. We choose $0 < \eta < 1$ with $\eta < \varepsilon \left(1 + \sum N_p(r_n) N_{p'}(s_n)\right)^{-1}$.

1) There is $k', l' \in C_{00}^+(G)$ with $\operatorname{supp} k' \subset U$ $\operatorname{supp} l' \subset U$, $N_p(T_H k') N_{p'}(T_H l') < 1+\eta$ and such that

$$\sum_{n=1}^{\infty} \left| \left\langle \Lambda_{k',l'}^q(i(S))[r_n],[s_n] \right\rangle - \left\langle S[r_n],[s_n] \right\rangle \right| \le \eta \|S\|$$

for every $S \in CV_p(H)$.

There are (φ_n) and (ψ_n) sequences of $C_{00}(H)$ with

$$N_p(r_n - \varphi_n) < \frac{\eta}{3^2 2^{n+1}\left(1 + N_{p'}(s_n)\right)}$$

and

$$N_{p'}(s_n - \psi_n) < \frac{\eta}{3^2 2^{n+1}\left(1 + N_p(r_n)\right)}$$

for every $n \in \mathbb{N}$. From the convergence of $\sum N_p(\varphi_n) N_{p'}(\psi_n)$ follows the existence of $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} N_p(\varphi_n) N_{p'}(\psi_n) < \eta/9.$$

By Lemma 2 there is $k', l' \in C_{00}^+(G)$ with $\operatorname{supp} k' \subset U$, $\operatorname{supp} l' \subset U$,

$$N_p(T_H k') N_{p'}(T_H l') < 1 + \frac{\eta}{3N}$$

and such that

$$\left| \left\langle \Lambda_{k',l'}^q(i(S))[\varphi_j],[\psi_j] \right\rangle - \left\langle S[\varphi_j],[\psi_j] \right\rangle \right| \le \frac{\eta\|S\|}{3N}$$

for every $1 \le n \le N$ and every $S \in CV_p(H)$. Consequently

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \left\langle \Lambda_{k',l'}^q(i(S))[r_n],[s_n] \right\rangle - \left\langle S[r_n],[s_n] \right\rangle \right| &\le \frac{\eta\|S\|}{3} + \sum_{n=1}^{\infty} \left| \left\langle \Lambda_{k',l'}^q(i(S))[\varphi_j],[\psi_j] \right\rangle - \left\langle S[\varphi_j],[\psi_j] \right\rangle \right| \\ &\le \frac{\eta\|S\|}{3} + \frac{2\eta\|S\|}{3}. \end{aligned}$$

2) End of the proof of Theorem 3. It suffices to put $k = k'(1+\eta)^{-1/p}$ and $l = l'(1+\eta)^{-1/p'}$ to obtain

$N_p(T_H k) N_{p'}(T_H l) < 1$ and

$$\sum_{n=1}^{\infty} \left| \left\langle \Lambda_{k,l}^q (i(S))[r_n],[s_n] \right\rangle - \left\langle S[r_n],[s_n] \right\rangle \right| \le \frac{\eta}{1+\eta} \|S\| \left(1 + \sum_{n=1}^{\infty} N_p(r_n) N_{p'}(s_n) \right) \le \varepsilon \|S\|. \quad \square$$

4. The Main Result

Definition 2 *Let* G *be a locally compact group,* H *an arbitrary closed subgroup,* $1 < p < \infty$ *and* $k, l \in C_{00}(G)$ *For* $u \in A_p(H)$ *we put*

$$\Phi_{k,l}^q(u) = \sum_{n=1}^{\infty} \overline{\left(q^{1/p}(k *_H \varphi_n) \right)} *_G \left(q^{1/p'}(l *_H \psi_n) \right)^{\vee}$$

where (φ_n) and (ψ_n) are sequences of $C_{00}(H)$ such that $\sum N_p(\varphi_n) N_{p'}(\psi_n)$ converges and such that $u = \sum \overline{\varphi_n} * \breve{\psi}_n$.

Then $\Phi_{k,l}^q$ is a linear map of $A_p(H)$ into $A_p(G)$, for $u \in A_p(H)$ and $T \in PM_p(G)$ one has $\left\langle u, \Lambda_{k,l}^q(T) \right\rangle = \left\langle \Phi_{k,l}^q(u), T \right\rangle$, $\Lambda_{k,l}^q(T) \in PM_p(H)$ and $\operatorname{supp}\Phi_{k,l}^q(u) \subset \operatorname{supp}k \operatorname{supp}u (\operatorname{supp}l)^{-1}$ [8].

Corollary 4 *Let* G *be a locally compact group,* H *a closed amenable subgroup,* $p > 1$, $u \in A_p(H) \cap C_{00}(H)$, $\varepsilon > 0$ *and* Ω *a neighborhood of* $\operatorname{supp}u$ *in* G. *Then there are* $k, l \in C_{00}^+(G)$ *with* $\left\| \Phi_{k,l}^q(u) \right\| \le \|u\|$, $\operatorname{supp}\Phi_{k,l}^q(u) \subset \Omega$ *and* $\left\| \operatorname{Res}_H \Phi_{k,l}^q(u) - u \right\| < \varepsilon$.

Proof. There are sequences (r_n), (s_n) of $C_{00}(H)$ such that $\sum N_p(r_n) N_{p'}(s_n)$ converges and such that $u = \sum \overline{r_n} * \breve{s}_n$. Let U be an open neighborhood of e in G such that $U \operatorname{supp}u U^{-1} \subset \Omega$. By Theorem 3 there is $k, l \in C_{00}^+(G)$ with $\operatorname{supp}k \subset U$, $\operatorname{supp}l \subset U$, $N_p(T_H k) N_{p'}(T_H l) < 1$ and such that

$$\sum_{n=1}^{\infty} \left| \left\langle \Lambda_{k,l}^q (i(S)) \left[\tau_p r_n \right], \left[\tau_{p'} s_n \right] \right\rangle - \left\langle S \left[\tau_p r \right]_n, \left[\tau_{p'} s_n \right] \right\rangle \right| \le \frac{\varepsilon}{2} \|S\|$$

for every $S \in CV_p(H)$.

Consider an arbitrary $S \in PM_p(H)$ with $\|S\| \le 1$. From

$$\langle u, S \rangle = \sum_{n=1}^{\infty} \overline{\left\langle S \left[\tau_p r_n \right], \left[\tau_{p'} s_n \right] \right\rangle}$$

and

$$\left\langle u, \Lambda_{k,l}^q(i(S)) \right\rangle = \sum_{n=1}^{\infty} \overline{\left\langle \Lambda_{k,l}^q(i(S)) \left[\tau_p r_n \right], \left[\tau_{p'} s_n \right] \right\rangle}$$

we get $\left| \left\langle u - \operatorname{Res}_H \Phi_{k,l}(u), S \right\rangle \right| \le \varepsilon/2$ and therefore $\left\| u - \operatorname{Res}_H \Phi_{k,l}(u) \right\| < \varepsilon$. $\square$

The following theorem is the main result of the paper.

Theorem 5 *Let* G *be a locally compact group,* H *a closed amenable subgroup,* $p > 1$, $u \in A_p(H) \cap C_{00}(H)$, $\varepsilon > 0$ *and* Ω *a neighborhood of* $\operatorname{supp}u$ *in* G. *Then there is* $v \in A_p(G) \cap C_{00}(G)$ *with* $\operatorname{Res}_H v = u$ $\|v\| < \|u\| + \varepsilon$ *and* $\operatorname{supp}v \subset \Omega$

Proof. This proof is identical with the one of Proposition 1 ii) p. 115 of [1]. Let Ω' be an open neighborhood of $\operatorname{supp}u$ in G such that the closure of Ω' in G is compact and contained in Ω. Using the Corollary 4 we show by induction the existence of a sequence (u_n) of $A_p(H) \cap C_{00}(H)$ and of a sequence (v_n) of $A_p(G) \cap C_{00}(G)$ such that $u_1 = u$, $\operatorname{supp}u_n \subset \Omega'$, $\operatorname{supp}v_n \subset \Omega'$, $\|v_n\| \le \|u_n\|$, $\left\| u_n - \operatorname{Res}_H v_n \right\| < 2^{-(n+1)} \varepsilon$ and $u_{n+1} = u_n - \operatorname{Res}_H v_n$ The function $\sum v_n$ satisfies all the requirements. $\square$

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