Scientific
Research

Application of Interpolation Inequalities to the Study of Operators with Linear Fractional Endpoint Singularities in Weighted Hölder Spaces

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Abstract

In this paper we consider operators with endpoint singularities generated by linear fractional Carleman shift in weighted Hölder spaces. Such operators play an important role in the study of algebras generated by the operators of singular integration and multiplication by function. For the considered operators, we obtained more precise relations between norms of integral operators with local singularities in weighted Lebesgue spaces and norms in weighted Hölder spaces, making use of previously obtained general results. We prove the boundedness of operators with linear fractional singularities.

Keywords

Endpoint Singularities, Weighted Holder Space, Weighted Lebesgue Spaces, Relation between Norms, Boundedness

1. Introduction

The solvability theory of singular integral operators has developed independently in Hölder and Lebesgue spaces [1]-[7], as norms in these spaces differ widely in their structure.

The norm in weighted Hölder spaces is defined in the following way. A function $\varphi(x)$ that satisfies the following condition on contour $J = [0,1]$,

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$$\left|\varphi\left(x_1\right)-\varphi\left(x_2\right)\right| \le C\left|x_1-x_2\right|^{\mu}, \quad x_1 \in J, \ x_2 \in J, \ 0<\mu<1,$$

is called Hölder function with exponent μ and constant C on contour J.

Let J be a power function which has zeros at the endpoints $x=0, \ x=1$:

$$h\left(x\right)=\left(x-0\right)^{\mu_0}\left(1-x\right)^{\mu_1}, \quad \mu<\mu_0<1+\mu, \ \mu<\mu_1<1+\mu.$$

The functions that become Hölder functions and turn into zero at the endpoints, after being multiplied by $h\left(x\right)$, form a Banach space of Hölder functions with weight h:

$$H_{\mu}^{0}\left(J, \ h\right), \quad J=\left[0,1\right].$$

The norm in space $H_{\mu}^{0}\left(J, \ h\right)$ is defined by

$$\left\|f\right\|_{H_{\mu}^{0}(J, \ h)} = \left\|hf\right\|_{H_{\mu}(J)}$$

where

$$\left\|hf\right\|_{H_{\mu}(J)} = \left\|hf\right\|_{C} = \left\|hf\right\|_{\mu}$$

and

$$\left\|hf\right\|_{C} = \max_{x \in J}\left|h\left(x\right)f\left(x\right)\right|,$$

$$\left\|hf\right\|_{\mu} = \sup_{x_1,x_2 \in J, \ x_1 \neq x_2}\left|h\left(x\right)f\left(x\right)\right|_{\mu},$$

specifying that

$$\left|h\left(x\right)f\left(x\right)\right|_{\mu} = \frac{\left|h\left(x_1\right)f\left(x_1\right)-h\left(x_2\right)f\left(x_2\right)\right|}{\left|x_1-x_2\right|^{\mu}}.$$

We denote by $\Omega\left(X\right)$ the set of all bounded linear operators mapping the Banach space X into X.
The norm of an operator $A \in \Omega$ will be denoted by $\left\|A\right\|_{\Omega(V)}$.

We denote a class of continuous functions on the segment $\left[0,1\right]$ by $C\left(\left[0,1\right]\right)$, also denote a class of differentiable functions on interval $\left(0,1\right)$ by $C^1\left(\left(0,1\right)\right)$, and we denote by $C^{01}\left(\left(0,1\right),h\right)$ a class of functions

$$\left\{\varphi \big| h_{\varphi} \in C^1\left(\left(0,1\right)\right), \ \lim_{t \to j} h\left(t\right)\varphi\left(t\right)=0, \ j=0,1\right\}$$

Let us introduce the following notation:

$$\left(Dv\right)\left(x\right)=x\left(1-x\right)\frac{\mathrm{d}}{\mathrm{d}x}v\left(x\right),$$

D^0 is the identity operator I, $D^1=D$, $D^n=D^{n-1}D$; $\chi_x\left(t\right)$ is the characteristic function of segment $\left[0,x\right]$, $x \in \left(0,1\right)$.

Let ρ be a power function which has zeros at the endpoints $x=0, \ x=1$:

$$\rho=\left(x-0\right)^{r}\left(1-x\right)^{s}, \quad -1<r<1-p, \ -1<s<p-1, \ p>1$$

Let $L_p\left(J,\rho\right)$ denote the space of functions on J which are integrable in the p-th power after multiplication by the weight-function ρ.

The norm in space $L_p\left(J,\rho\right)$ is defined by

$$\left\|f\right\|_{L_p(J,\rho)} = \left(\int_J\left|f\left(t\right)\right|^{p}\rho\left(t\right)\mathrm{d}t\right)^{\frac{1}{p}}.$$

As we can see, the norms in spaces $H_\mu^0(J, h)$ and $L_p(J, \rho)$ are different in their character, and the presence of a direct connection should not be expected. However, in this work, we describe a class of operators with local singularities for which we were able to find inequalities that connect the norms in weighted Lebesgue spaces with the norms in weighted Hölder spaces. Operators with fixed singularities perform an essential role in the study of singular integral operators with shift [8]-[10], in particular in the construction of regularizations.

By way of representatives of such types of operators we may consider the following operators with local singularities:

$$\frac{1}{\pi}\int_0^1 \frac{f(t)\,dt}{t+x-2tx}, \quad x \in (0,1),$$

$$\frac{1}{\pi}\int_0^1 \frac{f(t)\,dt}{t+kx}, \quad k>0, \ x \in (0,1)$$

Such operators can be used in the study of boundedness, of belonging of some operators to Banach algebras and of the solvability of operators in weighted Hölder spaces, on the basis of known results for operators in weighted Lebesgue spaces.

2. Inequality Which Connects the Norms in Lebesque and Hölder Weighted Spaces

It is useful to avoid two variables in the second term of the definition of the norm in Hölder spaces, for which we make use of

Lemma 1.

Let

$$h(x)f(x) \in C([0,1]) \cap C^1((0,1)),$$

then

$$\|hf\|_{H_\mu(J)} \le C_1 \left(\sup_{o<x<1} \frac{|h(x)f(x)|}{x^\mu(1-x)^\mu} + \sup_{o<x<1} \frac{|h(x)Df(x)|}{x^\mu(1-x)^\mu} \right),$$

where C_1 is a constant which does not depend on $f(x)$.

On the basis of Lemma 1 the following theorem can be proved [11].

Theorem 1.

Let the following conditions hold for some operator B :

1) Operators $D^n B$; $n = 0;1;2$ are bounded in spaces

$$L_{q_j}(J,\rho_j), \quad 1 < q_j < +\infty, \quad \rho_j(x) = x^{r_j}(1-x)^{s_j}, \quad -1 < r_j < q_j - 1, \ -1 < s_j < q_j -1, \ j = 1,2 \ ;$$

2) For any fixed $x \in (0;1)$ and for any function φ from space $H_\mu^0(J, h)$,

$$h(x) = x^{\mu+r}(1-x)^{\mu+s}; \quad 0 < \mu < 1; \ 0 < r < 1; \ 0 < s < 1$$

the following properties are fulfilled:

$$\begin{aligned}
\left(D^n B\chi_x\varphi\right)(t) &\in C^{01}((0,1),h), \\
\left(D^n B(1-\chi_x)\varphi\right)(t) &\in C^{01}((0,1),h), \quad n = 0,1.
\end{aligned} \tag{1}$$

Moreover, inequalities

$$\frac{1+r_2}{q_2} < r < \frac{1+r_1}{q_1}, \quad \frac{1+s_1}{q_1} < s < \frac{1+s_2}{q_2} \tag{2}$$

are correct.

It follows that operator B is bounded in space $H_\mu^0(J, h)$ and for its norm the following estimation is ful-

filled

$$\|B\|_{\Omega\left(H_{\mu}^{0}(J,h)\right)} \leq C_2 \sum_{n=0}^{2}\left(\left\|D^n B\right\|_{\Omega\left(L_{q_1}(J,\rho_1)\right)} + \left\|D^n B\right\|_{\Omega\left(L_{q_2}(J,\rho_2)\right)}\right), \tag{3}$$

where C_2 is a certain positive constant.

These results can be used in the study of operators in weighted Hölder spaces, on the basis of known results for operators in weighted Lebesgue spaces. In particular, operators with local endpoint singularities can be used in the construction of the left and the right regularizers in the study of Fredholmness of operators in weighted Hölder spaces.

3. Operators with Linear Fractional Endpoint Singularities

We formulate a useful assertion which follows directly from Theorem 1.

Corollary 1. *Let properties* (1) *and* (2) *be correct for the operator $B = D_1 R_1$ and furthermore*

$$\left\|D^n D_1 R_1\right\|_{L_{q_j}(J,\rho_j)} \leq C_3 \left\|D_1 M R_1\right\|_{L_{q_j}(J,\rho_j)}; \quad j = 1,2 \tag{4}$$

Here M is an operator that may be not linear; C_3 is a positive constant; the operators D, M and R_1 are bounded in spaces $L_{q_j}(J,\rho_j)$, $j = 1,2$.

Then

$$\left\|D_1 R_1\right\|_{H_{\mu}^{0}(J,\rho)} \leq C_4 \sum_{j=1}^{2}\left\|D_1\right\|_{L_{q_j}(J,\rho_j)} \cdot \left\|R_1\right\|_{L_{q_j}(J,\rho_j)}$$

where

$$C_4 = 3C_2 C_3 \max_{j=1,2}\left(\left\|M\right\|_{L_{q_j}(J,\rho_j)}, 1\right)$$

We consider the operators

$$(Q_J\varphi)(x) = \frac{1}{2x-1}\varphi\left[\frac{x}{2x-1}\right],$$

$$(S_J\varphi)(x) = \frac{1}{\pi i}\int_{0}^{1}\frac{\varphi(t)\,dt}{x-t}$$

$$(Q_J S_J\varphi)(x) = \frac{1}{\pi i}\int_{0}^{1}\frac{\varphi(t)\,dt}{2xt-x-t}$$

and

$$(V_k\varphi)(x) = \frac{1}{\pi i}\int_{0}^{1}\frac{\varphi(t)\,dt}{t+kx}, \quad k > 0$$

We note that for operators $Q_J S_J$ and V_k conditions (1), (2), (4) of corollary 1 are fulfilled.

Moreover, the following estimations hold

$$\left|(D^n Q_J S_J\varphi)(t)\right| = \left|\frac{1}{\pi i}\right|\left|\int_{0}^{1}\left[\frac{x(1-x)(1-2t)}{(x+t-2tx)^2}(1-2x)^{n-1} - \left(2\cdot\frac{x^2(1-x)^2(1-2t)^2}{(x+t-2tx)^3}\right)(n-1)\right]\varphi(t)\,dt\right|$$

$$\leq C_5 \cdot \frac{1}{\pi}\int_{0}^{1}\left[\frac{|\varphi(t)\,dt|}{x+t-2tx}\right] = C_5 \cdot |Q_J S_J|\varphi\|; \quad n = 1,2. \tag{5}$$

where

$$C_5 = \max_{n=1,2} \sup_{x,t \in J} \left| \frac{x(1-x)(1-2t)}{x+t-2tx}(1-2x)^{n-1} - \left(2 \cdot \frac{x^2(1-x)^2(1-2t)^2}{(x+t-2tx)^2} \right)(n-1) \right|,$$

and

$$\left| (D^n V_k \varphi)(t) \right| = \left| \frac{1}{\pi i} \int_0^1 \left[\frac{-kx(1-x)}{(t+kx)^2}(1-2x)^{n-1} + \left(2 \cdot \frac{k^2 x^2 (1-x)^2}{(t+kx)^3} \right)(n-1) \right] \varphi(t)\,dt \right| \le C_6 \frac{1}{\pi} \int_0^1 \frac{|\varphi(t)|\,dt}{t+kx} = C_6 \cdot |V_k |\varphi\|$$

$$n = 1, 2,$$

where

$$C_6 = \max_{n=1,2} \sup_{x,t \in J} \left| \frac{kx(1-x)}{t+kx}(1-2x)^{n-1} + \left(2 \cdot \frac{k^2 x^2 (1-x)^2}{(t+kx)^2} \right)(n-1) \right|.$$

Theorem 2. *Let an operator* R_1 *be bounded in the space*

$$L_{q_j}(J,\rho_j), \quad 1 < q_j < \infty; \quad \rho_j(x) = x^{r_j}(1-x)^{s_j}; \quad -1 < r_j, \ s_j < q_j - 1; \ j = 1, 2,$$

and inequalities (2) *be true.*
  If

$$r + \mu > \frac{1+r_1}{q_1} \quad \text{and} \quad s + \mu > \frac{1+s_2}{q_2}, \tag{6}$$

then the operators $Q_J S_J R_1$ *and* $V_k R_1$ *are bounded in space* $H_\mu^0(J, h)$.
  Proof. Let a function $\varphi(t)$ belong to $H_\mu^0(J, \rho)$.
  We introduce functions

$$\varphi_1(t) = \chi_x(t)\varphi(t), \ x \in J$$

and

$$\varphi_2(t) = \varphi(t) - \varphi_1(t).$$

  From the fact that

$$(R_1\varphi_j)(t) \in L_{q_j}(J,\rho_j), \quad j = 1, 2$$

  It follows that the function

$$\Lambda(t) = \frac{\left| (R_1\varphi_j)(t) \right|}{t^{1-\frac{1}{2}\lambda_j} \cdot (1-t)^{1-\frac{1}{2}\lambda_j}}$$

is summable on segment J if

$$\frac{1+r_j}{q_j} + 1 - \frac{1}{2}\lambda_j < 1$$

and

$$\frac{1+s_j}{q_j} + 1 - \frac{1}{2}\gamma_j < 1; \quad j = 1, 2.$$

  Condition (6) of the theorem makes it possible to choose constants λ_j and γ_j from interval $(0,2)$ so that

$$\frac{1+r_j}{q_j} < \frac{1}{2}\lambda_j < r+\mu; \quad \frac{1+s_j}{q_j} < \frac{1}{2}\gamma_j < s+\mu.$$

Now, we carry out an estimation of the expression $\left|\rho\left(D^n Q_J S_J R_1 \varphi\right)\right|$.
In doing so, we will use inequalities (5),

$$\left|\rho\left(D^n Q_J S_J R_1\varphi\right)\right| \le C_5 \rho(t)\left|\sum_{j=1}^{2}\frac{1}{\pi i}\int_0^1 \frac{\left|(R_1\varphi_j)(\tau)\right|d\tau}{\tau+t-2t\tau}\right| \le C_1\rho(t)\left|\sum_{j=1}^{2}\frac{1}{\pi i}\int_0^1 \frac{\left|(R_1\varphi_j)(\tau)\right|d\tau}{t^{\frac{1}{2}\cdot\lambda_j}\cdot(1-t)^{\frac{1}{2}\cdot\lambda_j}\cdot\tau^{1-\frac{1}{2}\cdot\lambda_j}\cdot(1-\tau)^{1-\frac{1}{2}\cdot\lambda_j}}\right|$$

where

$$C_1 = C_5 \cdot \sup_{\substack{\tau,t\in J \\ \tau\ne t}}\left|\frac{(t+\tau)(t+\tau-2)}{(t+\tau-2t\tau)}\right|.$$

Here we have taken into account that

$$C_5 \cdot \frac{1}{(t+\tau-2t\tau)} \le \frac{C_1}{t^{\frac{1}{2}\cdot\lambda_j}\cdot\tau^{-\frac{1}{2}\lambda_j+1}\cdot(1-t)^{\frac{1}{2}\cdot\lambda_j}\cdot(1-\tau)^{1-\frac{1}{2}\lambda_j}}.$$

Since

$$\rho(t)\cdot t^{-\frac{1}{2}\cdot\lambda_j}\cdot(1-t)^{-\frac{1}{2}\lambda_j} = t^{r+\mu-\frac{1}{2}\lambda_j}\cdot(1-t)^{s+\mu-\frac{1}{2}\gamma_j} \to 0$$

when $t\to 0$; and since function $\Lambda(t)$ is summable, it follows that conditions (1) of Theorem 1 are fulfilled for the operator $R = Q_J S_J R_1$.
From properties (5), condition (4) follows:

$$\left\|D^n Q_J S_J R_1\right\|_{L_{q_j}(J,\rho_j)} \le C_5 \left\|Q_J S_J M R_1\right\|_{L_{q_j}(J,\rho_j)},$$

where $(M\varphi)(X) = |\varphi(x)|$, and as the operator $Q_J S_J$ is bounded in $L_{q_j}(J,\rho_j)$ it follows that all conditions of Corollary 1 are fulfilled and we can apply it. Therefore operator $Q_J S_J R_1$ is bounded in $H_\mu^0(J,\rho)$.
Since operator V_k is bounded in $L_{q_j}(J,\rho_j)$, the boundness of operator $V_k R_1$ in $H_\mu^0(J,\rho)$ may be proved analogously.

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