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On Two Problems for Matrix Polytopes

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Abstract

We consider two problems from stability theory of matrix polytopes: the existence of common quadratic Lyapunov functions and the existence of a stable member. We show the applicability of the gradient algorithm and give a new sufficient condition for the second problem. A number of examples are considered.

Keywords

Stable Matrix, Matrix Family, Common Quadratic Lyapunov Functions, Switched System, Gradient Method

1. Introduction

Consider the switched system

$$\dot{x}(t) = Ax(t),\ \ A \in \{A_1, A_2, \cdots, A_N\} \tag{1}$$

where $\dot{x}(t) \in \mathbb{R}^n$, $t \geq 0$. In Equation (1), the matrix A switches among N matrices $A_1, A_2, \cdots, A_N$.

Switching signal $\sigma(t)$ is piecewise continuous from the right function $\sigma : [0, \infty) \to \{1, 2, \cdots, N\}$ and the switching times are arbitrary. For the switched system (1) with initial condition $x(0) = x_0$ and with switching signal $\sigma(t)$ denotes the solution by $x(t, x_0, \sigma(\cdot))$.

Definition 1. The origin is uniformly asymptotically stable (UAS) for the system (1) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every signal $\sigma(t)$ and initial state x_0 with $\|x_0\| < \delta$, the inequality $\|x(t, x_0, \sigma(\cdot))\| < \varepsilon$ is satisfied for all $t > 0$ and uniformly on $\sigma(\cdot)$

$$\lim_{t \to \infty} x(t, x_0, \sigma(\cdot)) = 0.$$

If all systems in (1) share a common quadratic Lyapunov function (CQLF) $V(x) = x^{\mathrm{T}} P x$ then the switched

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system is UAS (T denotes the transpose).

In this case there exists a common $P > 0$ such that

$$A_i^{\mathrm{T}} P + P A_i < 0 \quad (i = 1, 2, \cdots, N) \tag{2}$$

and P is called a common solution to the set of Lyapunov matrix inequalities (2).

The problem of existence of common positive definite solution P of (2) has been studied in a lot of works (see [1]-[9] and references therein). Numerical solution for common P via nondifferentiable convex optimization has been discussed in [10].

In the first part of the paper, the problem of existence of CQLF is investigated by Kelley's method. This method is applied when CQLF problem is treated as a convex optimization problem.

Second part of the paper is devoted to the following question:

Let $B \subset \mathbb{R}^l$ be a compact, for $q \in B$ the matrix $A(q)$ is a real $n \times n$ matrix. Is there a Hurwitz stable member (all eigenvalues lie in the open left half plane) in the family

$$\{A(q) : q \in B\}$$

or equivalently is there $q^* \in B$ such that $A(q^*)$ is stable? This problem is one of the hard and important problems in control theory (see [11]). Numerical solution of this problem is considered in [12]. In this paper we reduce this problem to a non-convex optimization problem.

2. Common Quadratic Lyapunov Function

For the switched system

$$\dot{x} = \{A_1, A_2, \cdots, A_N\} x$$

consider the problem of determination of CQLF $V(x) = x^{\mathrm{T}} P x$ where $P > 0$. We are going to investigate it by Kelley's cutting-plane method. This method gives new sufficient condition (Theorem 2) and new algorithm (Algorithm 1) which is more effective in comparison with the algorithm from [10].

Consider the problem of existence of a common $P > 0$ such that

$$A_i^{\mathrm{T}} P + P A_i < 0 \quad (i = 1, 2, \cdots, N). \tag{3}$$

Let $x \in \mathbb{R}^r$ be $x^{\mathrm{T}} = (x_1, x_2, \cdots, x_r)$ and P be an $n \times n$ symmetric matrix defined as

$$P = P(x) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_{n+1} & \cdots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n-1} & \cdots & x_r \end{pmatrix} \qquad \left(r = \frac{n(n+1)}{2} \right)$$

Define

$$\phi(x) = \max_{1 \le i \le N} \lambda_{\max} \left(A_i^{\mathrm{T}} P + P A_i \right) = \max_{1 \le i \le N, \|u\|=1} u^{\mathrm{T}} \left(A_i^{\mathrm{T}} P + P A_i \right) u. \tag{4}$$

If there exists x_* such that $P(x_*) > 0$ and $\phi(x_*) < 0$ then the matrix $P(x_*)$ is required solution. This problem can be reduced to the minimization of a convex function under convex constraints.

Consider the following convex minimization problem

$$\begin{aligned} &\phi(x) \to \text{minimize.} \\ &\min_{\|v\|=1} v^{\mathrm{T}} P(x) v > 0 \end{aligned} \tag{5}$$

Let $X \subset \mathbb{R}^n$ be a convex set and $F : X \to \mathbb{R}$ be convex function. The vector $g \in \mathbb{R}^n$ is said to be a subgradient of $F(x)$ at $x_* \in X$ if for all $x \in X$

$$F(x) \ge F(x_*) + g^{\mathrm{T}} (x - x_*).$$

The set of all subgradients of $F(x)$ at $x = x_*$ is denoted by $\partial F(x_*)$. If x_* is an interior point of X then the set $\partial F(x_*)$ is nonempty and convex. The following proposition follows from nondifferentiable optimization theory.

Proposition 1. Let $\phi(x)$ be defined as

$$\phi(x) = \max_{y \in Y} f(x, y) \tag{6}$$

where Y is compact, $f(x, y)$ is continuous and differentiable in x. Then

$$\partial\phi(x) = \operatorname{conv}\left\{\frac{\partial f(x, y)}{\partial x} : y \in Y(x)\right\}$$

where $Y(x)$ is the set of all maximizing elements y in (6), *i.e.*

$$Y(x) = \{y \in Y : f(x, y) = \phi(x)\}.$$

If for a given x the maximizing element is unique, *i.e.* $Y(x) = \{y(x)\}$ then $\phi(x)$ is differentiable at x and its gradient is

$$\nabla\phi(x) = \frac{\partial f(x, y)}{\partial x}.$$

In the case of the Function (4)

$$\partial\phi(x) = \operatorname{conv}\left\{\frac{\partial}{\partial x}\left(u^{\mathrm{T}}\left(A_i^{\mathrm{T}}P + PA_i\right)u\right) : i \text{ maximizes} \lambda_{\max}\left(A_i^{\mathrm{T}}P + PA_i\right), \right.$$
$$\left. u \text{ is a corresponding unit eigenvector}\right\}.$$

If for the given x the maximizing i is unique and $\lambda_{\max}\left(A_i^{\mathrm{T}}P + PA_i\right)$ is a simple eigenvalues, the differentiability of ϕ at the point x is guaranteed [13].

We investigate problem (5) by Kelley's cutting-plane method.

This method converts the problem (5) to the problem

$$\begin{aligned} &c^{\mathrm{T}}z \to \min \\ &c_1(z) \geq 0, c_2(z) \geq 0, \ -1 \leq x_i \leq 1 \ (i = 1, 2, \cdots, r) \end{aligned} \tag{7}$$

where $z = (x_1, x_2, \cdots, x_r, L)^{\mathrm{T}}$, $c = (0, 0, \cdots, 0, 1)^{\mathrm{T}}$, $c_1(z) = L - \phi(x)$, $c_2(z) = \min_{\|v\|=1} v^{\mathrm{T}}Pv$.

Let z^0 be a starting point and $z^0, z^1, \cdots, z^k$ be $k+1$ distinct points.

At the $(k+1)$ th iteration, the cutting-plane algorithm solves the following LP problem

$$\begin{array}{ll} \text{minimize} & L \\ \text{subject to} & -h_1^{\mathrm{T}}(z^0)z \geq -h_1^{\mathrm{T}}(z^0)z^0 - c_1(z^0) \\ & -h_2^{\mathrm{T}}(z^0)z \geq -h_2^{\mathrm{T}}(z^0)z^0 - c_2(z^0) \\ & \vdots \\ & -h_1^{\mathrm{T}}(z^k)z \geq -h_1^{\mathrm{T}}(z^k)z^k - c_1(z^k) \\ & -h_2^{\mathrm{T}}(z^k)z \geq -h_2^{\mathrm{T}}(z^k)z^k - c_2(z^k) \\ & -1 \leq x_i \leq 1 \end{array} \tag{8}$$

where $h_j(z^i)$ denotes a subgradient of $-c_j(z)$ at z^i $(i = 1, 2)$.

Let z_*^k be the minimizer of the problem (8).

If z_*^k satisfies the inequality $\min\left\{c_1(z_*^k), c_2(z_*^k)\right\} \geq -\varepsilon$, where ε is a tolerance then z_*^k is an approximate solution of the problem (7).

Otherwise define j^* as the index for the most negative $c_j\left(z_*^k\right)$, update the constraints in (8) by including the linear constraint

$$c_{j^*}\left(z^{k+1}\right)-h_{j^*}^{\mathrm{T}}\left(z^{k+1}\right)\left(z-z^{k+1}\right)\geq 0$$

and repeat the procedure.

Recall that our aim is to find x_* such that $P\left(x_*\right)>0$ and $\phi\left(x_*\right)<0$, but not the solution of the minimization problem (5), (7).

Theorem 2. If there exists k such that

$$c_1\left(z_*^k\right)>L^k, c_2\left(z_*^k\right)>0$$

where $z_*^k=\left(x_*^k,L^k\right)$ is the minimizer of the problem (8), then the matrix $P=P\left(x_*^k\right)$ is a common solution to (3).

Proof:

$$\phi\left(x_*^k\right)=L^k-c_1\left(z_*^k\right)<0,$$

$$0<c_2\left(z_*^k\right)=\min_{\|v\|=1} v^{\mathrm{T}}P\left(x_*^k\right)v$$

and by (5), $P\left(x_*^k\right)>0$ is a common solution to (3).

For the problem (5), (7) Kelley's method gives the following

Algorithm 1.

Step 1. Take an initial point $z^0=\left(x^0,L^0\right)^{\mathrm{T}}$. Compute $\phi\left(x^0\right)$ and $c_2\left(z^0\right)$. If $\phi\left(x^0\right)<0$ and $c_2\left(z^0\right)>0$ stop; otherwise continue.

Step 2. Determine z_*^k by solving LP problem in (8). If $c_1\left(z_*^k\right)>L^k$ and $c_2\left(z_*^k\right)>0$ then stop; otherwise continue. Set $z^{k+1}=z_*^k$, update the constraints in (8) and repeat the procedure.

Example 1. Consider the switched system

$$\dot{x}\in\left\{A_1,A_2,A_3\right\}x$$

where

$$A_1=\begin{pmatrix}-2&5&-6\\0&-8&0\\-5&-2&-20\end{pmatrix},\quad A_2=\begin{pmatrix}-8&17&-27\\9&-44&27\\22&-41&-2\end{pmatrix}\text{ and }A_3=\begin{pmatrix}4&9&-2\\-6&-8&4\\1&-10&-6\end{pmatrix}$$

are Hurwitz stable matrices.

Choose the initial point $z^0=\left(x_1^0,x_2^0,x_3^0,x_4^0,x_5^0,x_6^0,L^0\right)^{\mathrm{T}}=\left(1,0,0,1,0,1,1\right)^{\mathrm{T}}$, then

$$P\left(x^0\right)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},$$

$$c_1\left(z^0\right)=-7.5247,\quad c_2\left(z^0\right)=1\text{ and }\phi\left(x^0\right)=\max_{i\in\{1,2,3\}}\lambda_{\max}\left(A_i^{\mathrm{T}}P\left(x^0\right)+P\left(x_0\right)A_i\right)=8.5247>0.$$

We obtain $z^1=\left(-1,1,1,1,-1,1,-27.9933\right)^{\mathrm{T}}$ by solving LP problem in (8). Calculations give the following Table 1, and

$$z^{15}=\left(x^{15},L^{15}\right)^{\mathrm{T}}=\left(0.7811,0.6268,-0.1283,1,-0.1254,0.2383,-0.8206\right)^{\mathrm{T}}.$$

Since $L^{15}-c_1\left(z^{15}\right)=-0.0287<0$ and $c_2\left(z^{15}\right)=0.2075>0$,

Table 1. Kelley's algorithm for Example 1.

k	L^k	$c_1(z^k)$	$c_2(z^k)$
1	–27.9933	–209.7383	–1.9999
2	–24.4038	–127.1153	–2.3326
3	–14.2596	–106.2473	–1.8092
4	–10.0497	–63.4433	–1.8878
⋮	⋮	⋮	⋮
14	–0.8465	–1.1881	0.2694
15	–0.8206	–0.7919	0.2075

$$P = P(x^{15}) = \begin{pmatrix} 0.7811 & 0.6268 & -0.1283 \\ 0.6268 & 1 & -0.1254 \\ -0.1283 & -0.1254 & 0.2383 \end{pmatrix}$$

is a common positive definite solution for

$$A_i^{\mathrm{T}} P + P A_i < 0 \quad (i = 1, 2, 3).$$

3. Stable Member in a Polytope

This part is devoted to the following question: Given a matrix family $\{A(q) : q \in B\}$ where $B \subset \mathbb{R}^l$ is a compact, is there a stable matrix in this family?

In [12], a numerical algorithm has been proposed for a stable member in the affine matrix family $\{A(q) : q \in \mathbb{R}^l\}$. In this algorithm the uncertainty vector q varies in the whole space $\mathbb{R}^l$. On the other hand we consider the case where q varies in a box $B \subset R^l$ and use the gradient algorithm for minimization of the nonconvex maximum eigenvalue function. By choosing appropriate step-size, we obtain the convergence.

Let $Z_1, Z_2, \cdots, Z_r$ $\left(r = \frac{n(n+1)}{2} \right)$ be a basis for the subspace of $n \times n$ symmetric matrices and

$$Q_i(q) = (-Z_i) \oplus \left(A^{\mathrm{T}}(q) Z_i + Z_i A(q) \right),$$

$$\phi(x, q) = \lambda_{\max} \left(\sum_{i=1}^{r} x_i Q_i(q) \right)$$

where $x = (x_1, x_2, \cdots, x_r)^{\mathrm{T}}$, $q = (q_1, q_2, \cdots, q_k)^{\mathrm{T}}$.

Consider the problem

$$\phi(x, q) \to \text{minimize.}$$

$$\min_{\|x\|=1, q \in Q} v^{\mathrm{T}} P(x) v > 0$$

Theorem 3. There is a stable matrix in the family $A(q)$ if and only if $\phi^* = \min_{(x,q)} \phi(x, q) < 0$.

Proof:

$$\phi^* < 0 \Leftrightarrow \text{there exists } (x^*, q^*) \text{ such that } \sum_{i=1}^{r} x_i^* Q_i(q^*) < 0$$

$$\Leftrightarrow \left(-\sum_{i=1}^{r} x_i^* Z_i \right) \oplus \left[A^{\mathrm{T}}(q^*) \left(-\sum_{i=1}^{r} x_i^* Z_i \right) + \left(-\sum_{i=1}^{r} x_i^* Z_i \right) A(q^*) \right] < 0$$

$$\Leftrightarrow P\left(x^*\right)=\sum_{i=1}^{r} x_i^* Z_i > 0 \text{ and } A\left(q^*\right)^{\mathrm{T}} P\left(x^*\right)+P\left(x^*\right)A\left(q^*\right)<0.$$

By Lyapunov theorem, the matrix $A\left(q^*\right)$ is stable.

Example 2. Consider the family of matrices

$$A(q)=A_0+q_1A_1+q_2A_2+q_3A_3,\ \ q_1,q_2,q_3\in[-1,1]$$

where

$$A_0=\begin{pmatrix}1&0&2&0\\-2&0&-3&0\\-5&1&-1&0\\-3&-1&0&-2\end{pmatrix},A_1=\begin{pmatrix}-2&0&3&0\\-1&0&-3&2\\-3&3&-1&0\\-4&-1&0&-2\end{pmatrix},A_2=\begin{pmatrix}-1&0&2&0\\-3&-1&-3&0\\-3&2&-1&2\\-2&-1&0&-2\end{pmatrix},A_3=\begin{pmatrix}-1&0&0&1\\-1&-2&3&-2\\1&-2&0&-1\\0&-2&1&-5\end{pmatrix}.$$

For $q=(0,0,0)^{\mathrm{T}}$, $A(q)=A_0$ is unstable. We apply the gradient algorithm to find a stable member in the family.

Let $x^0=\left(\frac{1}{2},0,0,0,\frac{1}{2},0,0,\frac{1}{2},0,\frac{1}{2}\right)^{\mathrm{T}}$ and $q^0=(1,0,0)^{\mathrm{T}}$. So

$$a^0=\left(x^0,q^0\right)=\left(\frac{1}{2},0,0,0,\frac{1}{2},0,0,\frac{1}{2},0,\frac{1}{2},1,0,0\right)^{\mathrm{T}}.$$

Then

$$\sum_{i=1}^{10} x_i^0 Q_i\left(q^0\right)=\begin{pmatrix}-P\left(x^0\right)&0\\0&A\left(q^0\right)^{\mathrm{T}}P\left(x^0\right)+P\left(x^0\right)A\left(q^0\right)\end{pmatrix}$$

$$=\begin{pmatrix}-1/2&0&0&0&0&0&0&0\\0&-1/2&0&0&0&0&0&0\\0&0&-1/2&0&0&0&0&0\\0&0&0&-1/2&0&0&0&0\\0&0&0&0&-1&0&5&0\\0&0&0&0&-3&0&-6&2\\0&0&0&0&-8&4&-2&0\\0&0&0&0&-7&-2&0&-4\end{pmatrix}.$$

Maximum eigenvalue of this matrix and its corresponding unit eigenvector are

$$\lambda_{\max}=2.1866,\ v=(0,0,0,0,0.7644,-0.4480,-0.1668,-0.4324)^{\mathrm{T}}$$

respectively. Gradient of the function ϕ at a^0 is

$$\nabla\phi\big|_{a^0}=(-2.44,-1.86,-11.04,-2.78,1.93,7.50,4.30,2.52,7.46,2.35,0.28,0.50,-2.73)^{\mathrm{T}}.$$

The first tencomponent of the vector $a^1=a^0-t\cdot\nabla\phi\big|_{a^0}$ should be on the ten dimensional unit sphere. Therefore $t=0.01531$ and

$$a^1=(0.53,0.02,0.16,0.04,0.47,-0.11,-0.06,0.46,-0.11,0.46,0.99,-0.007,0.04)^{\mathrm{T}}.$$

After 4 steps, we get

$$a^4=\left(x^4,q^4\right)=(0.59,0.03,0.04,0.009,0.41,-0.05,-0.04,0.49,-0.15,0.45,0.98,-0.03,0.08)^{\mathrm{T}}$$

and $\phi\left(x^4, q^4\right) = -0.2585 < 0$. Therefore $A\left(q^4\right)$ is stable.

4. Conclusion

Two important problems from control theory are considered: the existence of common quadratic Lyapunov functions for switched linear systems and the existence of a stable member in a matrix polytope. We obtain new conditions which give new effective computational algorithms.

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