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Embeddings of Almost Hermitian Manifold in Almost Hyper Hermitian Manifold and Complex (Hypercomplex) Numbers in Riemannian Geometry

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Abstract

Tubular neighborhoods play an important role in differential topology. We have applied these constructions to geometry of almost Hermitian manifolds. At first, we consider deformations of tensor structures on a normal tubular neighborhood of a submanifold in a Riemannian manifold. Further, an almost hyper Hermitian structure has been constructed on the tangent bundle *TM* with help of the Riemannian connection of an almost Hermitian structure on a manifold *M* then, we consider an embedding of the almost Hermitian manifold *M* in the corresponding normal tubular neighborhood of the null section in the tangent bundle *TM* equipped with the deformed almost hyper Hermitian structure of the special form. As a result, we have obtained that any Riemannian manifold *M* of dimension *n* can be embedded as a totally geodesic submanifold in a Kaehlerian manifold of dimension 2*n* (Theorem 6) and in a hyper Kaehlerian manifold of dimension 4*n* (Theorem 7). Such embeddings are "good" from the point of view of Riemannian geometry. They allow solving problems of Riemannian geometry by methods of Kaehlerian geometry (see Section 5 as an example). We can find similar situation in mathematical analysis (real and complex).

Keywords

Riemannian Manifolds, Almost Hermitian and Almost Hyper Hermitian Structures, Tangent Bundle

1. Deformations of Tensor Structures on a Normal Tubular Neighborhood of a Submanifold

1°. Let (M', g') be a k-dimensional Riemannian manifold isometrically embedded in a n-dimensional Rie-

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mannian manifold (M,g). The restriction of g to M' coincides with g' and for any $p\in M'$.

$$T_p(M)=T_p(M')\oplus T_p(M')^{\perp}.$$

So, we obtain a vector bundle $M'\to T(M')^{\perp}:p\to T_p(M')^{\perp}$ over the submanifold M'. There exists a neighborhood $\tilde{U}_0$ of the null section $O_{M'}$ in $T(M')^{\perp}$ such that the mapping

$$\pi\times\exp:v\to\left(\pi(v),\exp_{\pi(v)}v\right),\ v\in\tilde{U}_0,$$

is a diffeomorphism of $\tilde{U}_0$ onto an open subset $\tilde{U}\subset M$. The subset $\tilde{U}$ is called *a tubular neighborhood of the submanifold* M' *in* M.

For any point $p\in M$ we can consider a set $\{\delta(p)\}$ of positive numbers such that the mapping $\exp_{U(\delta(p))}$ is defined and injective on $U(\delta(p))\subset T_p(M)$. Let $\bar{\varepsilon}(p)=\sup\{\delta(p)\}$.

Lemma [1]. *The mapping* $M\to R_+:p\to\bar{\varepsilon}(p)$ *is continuous on M.*

If we take the restriction of the function $\bar{\varepsilon}(p)$ on $\tilde{U}$ then it is clear that there exists a continuous positive function $\varepsilon(p)$ on M' such that for any $p\in M'$ open geodesic balls $B\left(p;\frac{\varepsilon(p)}{2}\right)\subset B(p;\varepsilon(p))\subset\tilde{U}$. For compact manifolds we can choose a constant function $\varepsilon(p)=\varepsilon>0$. We denote $\tilde{U}_p=\exp\left(\tilde{U}_0\cap T_p(M')^{\perp}\right)$, $D\left(p;\frac{\varepsilon(p)}{2}\right)=B\left(p;\frac{\varepsilon(p)}{2}\right)\cap\tilde{U}_p$, $D(p;\varepsilon(p))=B(p;\varepsilon(p))\cap\tilde{U}_p$. It is obvious that $\dim\tilde{U}_p=\dim D(p;\varepsilon(p))=n-k$. For any point $o\in M'$ we can consider such an orthonormal frame $\left(X_{1_0},\cdots,X_{n_0}\right)$ that $T_0(M')=L\left[X_{1_0},\cdots,X_{k_0}\right]$ and $T_0(M')^{\perp}=L\left[X_{k+1_0},\cdots,X_{n_0}\right]$. There exist coordinates $x_1,\cdots,x_k$ in some neighborhood $\tilde{V}_0\subset M'$ of the point o that $\frac{\partial}{\partial x_i|_0}=X_{i_0},\ i=\overline{1,k}$. We consider orthonormal vector fields $X_{k+1},\cdots,X_n$ which are cross-sections of the vector bundle $p\to T_p(M')^{\perp}$ over $\tilde{V}_0$ and the neighborhood $\tilde{W}_0=\bigcup_{p\in\tilde{V}_0}\tilde{U}_p$. The basis $\left\{X_{k+1_p},\cdots,X_{n_p}\right\}$ defines the normal coordinates $x_{k+1},\cdots,x_n$ on $\tilde{U}_p$ [2]. For any point $x\in\tilde{W}_0$ there exists such unique point $p\in\tilde{V}_0$ that $x=\exp_p(t\xi)$, $\|\xi\|=1$, $\xi\in T_p(M')^{\perp}$. A point $x\in\tilde{W}_0$ has the coordinates $x_1,\cdots,x_k,x_{k+1},\cdots,x_n$ where $x_1,\cdots,x_k$ are coordinates of the point p in $\tilde{V}_0$ and $x_{k+1},\cdots,x_n$ are normal coordinates of x in $\tilde{U}_p$. We denote $X_i=\frac{\partial}{\partial x_i}$, $i=\overline{1,n}$, on $\tilde{W}_0$. Thus, we can consider *tubular neighborhoods* $Tb\left(M';\frac{\varepsilon(p)}{2}\right)=\bigcup_{p\in M'}D\left(p;\frac{\varepsilon(p)}{2}\right)$ and $Tb(M';\varepsilon(p))=\bigcup_{p\in M'}D(p;\varepsilon(p))$ of the submanifold M'.

2°. Let K be a smooth tensor field of type (r, s) on the manifold M and for $x\in\tilde{W}_0$, let

$$K_x=\sum_{i_1,\cdots,i_r,j_1,\cdots,j_s}k^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}(x)X_{i_{1_x}}\otimes\cdots\otimes X_{i_{r_x}}\otimes X^{j_1}_x\otimes\cdots\otimes X^{j_s}_x,$$

where $\left\{X^1_x,\cdots,X^n_x\right\}$ is the dual basis of $T^*_x(M)$, $x=\exp_p(t\xi)$, $\|\xi\|=1,\xi\in T_p(M')^{\perp}$. We define a tensor field $\bar{K}$ on M in the following way.

a) $x\in D\left(p;\frac{\varepsilon(p)}{2}\right)$, then

$$\bar{K}_x=\sum_{i_1,\cdots,i_r,j_1,\cdots,j_s}k^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}(p)X_{i_{1_x}}\otimes\cdots\otimes X_{i_{r_x}}\otimes X^{j_1}_x\otimes\cdots\otimes X^{j_s}_x;$$

b) $x\in D(p;\varepsilon(p))\setminus D\left(p;\frac{\varepsilon(p)}{2}\right)$, then

$$\bar{K}_x = \sum_{i_1,\cdots,i_r,j_1,\cdots,j_s} k^{i_1,\cdots,i_r}_{j_1,\cdots,j_s}\left(\exp_p\left(\left(2t-\varepsilon(p)\right)\xi\right)\right) X_{i_{1_x}} \otimes\cdots\otimes X_{i_{r_x}} \otimes X_x^{j_1} \otimes\cdots\otimes X_x^{j_s};$$

c) $x \in M \setminus \bigcup_{M'} D\left(p;\varepsilon(p)\right)$, then

$$\bar{K}_x = K_x .$$

It is easy to see the independence of the tensor field $\bar{K}$ on a choice of coordinates in $\tilde{W}_0$ for every point $o \in M'$.

Definition 1. *The tensor field* $\bar{K}$ *is called a deformation of the tensor field K on the normal tubular neighborhood of a submanifold* M'.

Remark. *The obtained tensor field* $\bar{K}$ *is continuous but is not smooth on the boundaries of the normal tubular neighborhoods* $Tb\left(M';\frac{\varepsilon(p)}{2}\right)$ *and* $Tb\left(M';\varepsilon(p)\right)$; $\bar{K}$ *is smooth in other points of the manifold M.*

3°. We consider a deformation $\bar{g}$ of the Riemannian metric *g* on the normal tubular neighborhood $Tb\left(M';\varepsilon(p)\right)$ of a submanifold M'. For $x \in \tilde{W}_0$, $x = \exp_p(t\xi)$, $\|\xi\| = 1$, $\xi \in T_p(M')$, we define the Riemannian metric $\bar{g}$ by the following way.

a) $\bar{g}_p = g_p$ for any $p \in M'$;

b) $\bar{g}_x\left(X_i, X_j\right) = \bar{g}_{ij}(x) = \bar{g}_{ij}(p)$, where $X_i = \frac{\partial}{\partial x_i}, i = \overline{1,n}, X_j = \frac{\partial}{\partial x_j}, \quad j = \overline{1,n},$ on $\tilde{W}_0$, $x \in D\left(p;\frac{\varepsilon(p)}{2}\right)$;

c) $\bar{g}_x\left(X_i, X_j\right) = \bar{g}_{ij}(x) = \bar{g}_{ij}\left(\exp_p\left(\left(2t-\varepsilon(p)\right)\xi\right)\right)$, for any $x \in D\left(p;\varepsilon(p)\right) / D\left(p;\frac{\varepsilon(p)}{2}\right)$;

d) $\bar{g}_x = g_x$ for each point $x \in M \setminus \bigcup_{p\in M'} D\left(p;\varepsilon(p)\right)$.

The independence of $\bar{g}$ on a choice of local coordinates follows and the correctly defined Riemannian metric $\bar{g}$ on *M* has been obtained.

It is known from [3] that every autoparallel submanifold of *M* is a totally geodesic submanifold and a submanifold M' is autoparallel if and only if $\nabla_X Y \in T(M')$ for any $X, Y \in \chi(M')$, where ∇ is the Riemannian connection of *g*.

Theorem 1. *Let* M' *be a submanifold of a Riemannian manifold (M, g) and* $\bar{g}$ *be the deformation of g on the normal tubular neighborhood* $Tb\left(M';\varepsilon(p)\right)$ *of* M' *constructed above. Then* M' *is a totally geodesic submanifold of* $\left(Tb\left(M';\frac{\varepsilon(p)}{2}\right),\bar{g}\right)$.

Proof. For any point $x \in D\left(p;\frac{\varepsilon(p)}{2}\right) \subset \tilde{W}_0$ the functions $\bar{g}_{ij}(x) = g_{ij}(p)$ and $\frac{\partial \bar{g}_{ij}}{\partial x_l} = 0, l = \overline{k+1,n}$ on $D\left(p;\frac{\varepsilon(p)}{2}\right)$ because the vector fields $X_l = \frac{\partial}{\partial x_l}$ are tangent to $D\left(p;\frac{\varepsilon(p)}{2}\right)$. By the formula of the Riemannian connection $\bar{\nabla}$ of the Riemannian metric $\bar{g}$, [2], we obtain for $i, j = \overline{1,k}, l = \overline{k+1,n}$

$$\begin{aligned} 2\bar{g}_p\left(\bar{\nabla}_{X_i} X_j, X_l\right) &= X_{i_p}\bar{g}\left(X_j, X_l\right) + X_{j_p}\bar{g}\left(X_i, X_l\right) - X_{l_p}\bar{g}\left(X_i, X_j\right) + \bar{g}_p\left(\left[X_i, X_j\right], X_l\right) \\ &+ \bar{g}_p\left(\left[X_l, X_i\right], X_j\right) + \bar{g}_p\left(X_i, \left[X_l, X_j\right]\right) = -\frac{\partial \bar{g}_{ij}}{\partial x_l} = 0. \end{aligned} \tag{1.1}$$

Here we use the fact that $\left[X_i, X_j\right] = \left[X_l, X_i\right] = \left[X_l, X_j\right] = 0$ and that $\bar{g}\left(X_j, X_l\right) = \bar{g}\left(X_i, X_l\right) = 0$ because $X_l \in T(M')^{\perp}$.

Thus, $\bar{\nabla}_{X_i} X_j \in T(M')$ and from the remarks above the theorem follows.

QED.

Corollary 1.1. *Let* $\bar{R}$ *be the Riemannian curvature tensor field of* $\bar{\nabla}$. *Then* $\bar{R}$ *vanishes on every*

$D\left(p;\frac{\varepsilon(p)}{2}\right)$ *for* $p \in M'$.

Proof. From the formula (1.1) it is clear that $\bar{\nabla}_{X_l} X_m = 0$ for $l, m = \overline{k+1, n}$. The rest is obvious.

QED.

2. Almost Hyper Hermitian Structures (ahHs) on Tangent Bundles

0°. We follow especially close to [4].

Let (M, g) be a n-dimensional Riemannian manifold and TM be its tangent bundle. For a Riemannian connection ∇ we consider the connection map K of ∇ [5], [1], defined by the formula

$$\nabla_X Z = KZ_* X, \tag{2.1}$$

where Z is considered as a map from M into TM and the right side means a vector field on M assigning to $p \in M$ the vector $KZ_* X_p \in M_p$.

If $U \in TM$, we denote by H_U the kernel of $K_{|TM_U}$ and this n-dimensional subspace of TM_U is called the horizontal subspace of TM_U.

Let π denote the natural projection of TM onto M, then π_* is a C^∞-map of TTM onto TM. If $U \in TM$, we denote by V_U the kernel of $\pi_{*|TM_U}$ and this n-dimension subspace of TM_U is called the vertical subspace of TM_U $\left(\dim TM_U = 2\dim M = 2n\right)$. The following maps are isomorphisms of corresponding vector spaces $\left(p = \pi(U)\right)$

$$\pi_{*|TM_U} : H_U \to M_p,\ K_{|TM_U} : V_U \to M_p$$

and we have

$$TM_U = H_U \oplus V_U$$

If $X \in \chi(M)$, then there exists exactly one vector field on TM called the "horizontal lift" (resp. "vertical lift") of X and denoted by $\bar{X}^h\left(\bar{X}^v\right)$, such that for all $U \in TM$:

$$\pi_* \bar{X}_U^h = X_{\pi(U)},\quad K\bar{X}_U^h = 0_{\pi(U)}, \tag{2.2}$$

$$\pi_* \bar{X}_U^v = 0_{\pi(U)},\quad K\bar{X}_U^v = X_{\pi(U)}, \tag{2.3}$$

Let R be the curvature tensor field of ∇, then following [5] we write

$$\left[\bar{X}^v, \bar{Y}^v\right] = 0, \tag{2.4}$$

$$\left[\bar{X}^h, \bar{Y}^v\right] = \left(\overline{\nabla_X Y}\right)^v \tag{2.5}$$

$$\pi_*\left(\left[\bar{X}^h, \bar{Y}^h\right]_U\right) = [X, Y], \tag{2.6}$$

$$K\left(\left[\bar{X}^h, \bar{Y}^h\right]_U\right) = R(X, Y)U. \tag{2.7}$$

For vector fields $\bar{X} = \bar{X}^h \oplus \bar{X}^v$ and $\bar{Y} = \bar{Y}^h \oplus \bar{Y}^v$ on TM the natural Riemannian metric $\hat{g} = \langle , \rangle$ is defined on TM by the formula

$$\left\langle \bar{X}, \bar{Y} \right\rangle = g\left(\pi_* \bar{X}, \pi_* \bar{Y}\right) + g\left(K\bar{X}, K\bar{Y}\right). \tag{2.8}$$

It is clear that the subspaces H_U and V_U are orthogonal with respect to $\langle , \rangle$.

It is easy to verify that $\bar{X}_1^h, \bar{X}_2^h, \cdots, \bar{X}_n^h, \bar{X}_1^v, \bar{X}_2^v, \cdots, \bar{X}_n^v$ are orthonormal vector fields on TM if $X_1, X_2, \cdots, X_n$ are those on M *i.e.* $g\left(X_i, X_j\right) = \delta_j^i$.

1°. We define a tensor field J_1 on TM by the equalities

$$J_1 \bar{X}^h = \bar{X}^v,\ J_1 \bar{X}^v = -\bar{X}^h,\ X \in \chi(M). \tag{2.9}$$

For $X \in \chi(M)$ we get

$$J_1^2\bar{X}=J_1\left(J_1\left(\bar{X}^h\oplus\bar{X}^v\right)\right)=J_1\left(-\bar{X}^h\oplus\bar{X}^v\right)=-\left(\bar{X}^h\oplus\bar{X}^v\right)=-I\bar{X}$$

and

$$J_1^2=-I\ .$$

For $X,Y\in\chi(M)$ we obtain

$$\left\langle J_1\bar{X},J_1\bar{Y}\right\rangle=\left\langle-\bar{X}^h\oplus\bar{X}^v,-\bar{Y}^h\oplus\bar{Y}^v\right\rangle=\left\langle-\bar{X}^h,-\bar{Y}^v\right\rangle+\left\langle\bar{X}^v,\bar{Y}^v\right\rangle,$$
$$\left\langle\bar{X},\bar{Y}\right\rangle=\left\langle\bar{X}^h\oplus\bar{X}^v,\bar{Y}^h\oplus\bar{Y}^v\right\rangle=\left\langle\bar{X}^h,\bar{Y}^h\right\rangle+\left\langle\bar{X}^v,\bar{Y}^v\right\rangle$$

and it follows that $\left\langle J_1\bar{X},J_1\bar{Y}\right\rangle=\left\langle\bar{X},\bar{Y}\right\rangle$, $\left(TM,J_1,\langle,\rangle\right)$ is an almost Hermitian manifold.

Further, we want to analyze the second fundamental tensor field h^1 of the pair $\left(J_1,\langle,\rangle\right)$ where h^1 is defined by (2.11), [6].

The Riemannian connection $\hat{\nabla}$ of the metric $\hat{g}=\langle,\rangle$ on *TM* is defined by the formula (see [1])

$$\begin{aligned}\left\langle\hat{\nabla}_{\bar{X}}\bar{Y},\bar{Z}\right\rangle=&\frac{1}{2}\Big(\bar{X}\left\langle\bar{Y},\bar{Z}\right\rangle+\bar{Y}\left\langle\bar{Z},\bar{X}\right\rangle-\bar{Z}\left\langle\bar{X},\bar{Y}\right\rangle+\left\langle\bar{Z},[\bar{X},\bar{Y}]\right\rangle\\&+\left\langle\bar{Y},\left[\bar{Z},\bar{X}\right]\right\rangle+\left\langle\bar{X},\left[\bar{Z},\bar{Y}\right]\right\rangle\Big),\ X,Y,Z\in\chi(M).\end{aligned}\tag{2.10}$$

For orthonormal vector fields $\bar{X},\bar{Y},\bar{Z}$ on *TM* we obtain

$$\begin{aligned}h^1_{\overline{XYZ}}&=\left\langle h^1_{\bar{X}}\bar{Y},\bar{Z}\right\rangle=\frac{1}{2}\left\langle\hat{\nabla}_{\bar{X}}\bar{Y}+J_1\hat{\nabla}_{\bar{X}}J_1\bar{Y},\bar{Z}\right\rangle\\&=\frac{1}{2}\left(\left\langle\hat{\nabla}_{\bar{X}}\bar{Y},\bar{Z}\right\rangle-\left\langle\hat{\nabla}_{\bar{X}}J_1\bar{Y},J_1\bar{Z}\right\rangle\right)\\&=\frac{1}{4}\Big(\left\langle\left[\bar{X},\bar{Y}\right],\bar{Z}\right\rangle+\left\langle\left[\bar{Z},\bar{X}\right],\bar{Y}\right\rangle+\left\langle\left[\bar{Z},\bar{Y}\right],\bar{X}\right\rangle-\left\langle\left[\bar{X},J_1\bar{Y}\right],J_1\bar{Z}\right\rangle\\&\quad-\left\langle\left[J_1\bar{Z},\bar{X}\right],J_1\bar{Y}\right\rangle-\left\langle\left[J_1\bar{Z},J_1\bar{Y}\right],\bar{X}\right\rangle\Big).\end{aligned}\tag{2.11}$$

Using (2.4)-(2.7) and (2.11) we consider the following cases for the tensor field h^1 assuming all the vector fields to be orthonormal.

$$\begin{aligned}h^1_{\bar{X}^h\bar{Y}^h\bar{Z}^h}&=\frac{1}{4}\Big(\left\langle\left[\bar{X}^h,\bar{Y}^h\right],\bar{Z}^h\right\rangle+\left\langle\left[\bar{Z}^h,\bar{X}^h\right],\bar{Y}^h\right\rangle+\left\langle\left[\bar{Z}^h,\bar{Y}^h\right],\bar{X}^h\right\rangle\\&\quad-\left\langle\left[\bar{X}^h,J_1\bar{Y}^h\right],J_1\bar{Z}^h\right\rangle-\left\langle\left[J_1\bar{Z}^h,\bar{X}^h\right],J_1\bar{Y}^h\right\rangle-\left\langle\left[J_1\bar{Z}^h,J_1\bar{Y}^h\right],\bar{X}^h\right\rangle\Big)\\&=\frac{1}{4}\Big(g([X,Y],Z)+g([Z,X],Y)+g([Z,Y],X)-\left\langle\left[\bar{X}^h,\bar{Y}^v\right],\bar{Z}^v\right\rangle\\&\quad-\left\langle\left[\bar{Z}^v,\bar{X}^h\right],\bar{Y}^v\right\rangle-\left\langle\left[\bar{Z}^v,\bar{Y}^v\right],\bar{X}^h\right\rangle\Big)\\&=\frac{1}{2}g(\nabla_XY,Z)-\frac{1}{4}\left(g(\nabla_XY,Z)-g(\nabla_XZ,Y)\right)\\&=\frac{1}{2}\left(g(\nabla_XY,Z)-g(\nabla_XY,Z)\right)=0.\end{aligned}\tag{1.1°}$$

$$\begin{aligned}h^1_{\bar{X}^h\bar{Y}^h\bar{Z}^v}&=\frac{1}{4}\Big(\left\langle\left[\bar{X}^h,\bar{Y}^h\right],\bar{Z}^v\right\rangle+\left\langle\left[\bar{Z}^v,\bar{X}^h\right],\bar{Y}^h\right\rangle+\left\langle\left[\bar{Z}^v,\bar{Y}^h\right],\bar{X}^h\right\rangle\\&\quad-\left\langle\left[\bar{X}^h,J_1\bar{Y}^h\right],J_1\bar{Z}^v\right\rangle-\left\langle\left[J_1\bar{Z}^v,\bar{X}^h\right],J_1\bar{Y}^h\right\rangle-\left\langle\left[J_1\bar{Z}^v,J_1\bar{Y}^h\right],\bar{X}^h\right\rangle\Big)\\&=\frac{1}{4}\left(g\left(R(X,Y)U,Z\right)+\left\langle\left[\bar{Z}^h,\bar{X}^h\right],\bar{Y}^v\right\rangle\right)\\&=\frac{1}{4}\left(g\left(R(X,Y)U,Z\right)+g\left(R(Z,X)U,Y\right)\right)\\&=-\frac{1}{4}\left(g\left(R(X,Y)Z,U\right)+g\left(R(Z,X)Y,U\right)\right).\end{aligned}\tag{2.1°}$$

By similar arguments we obtain

$$h^1_{\bar{X}^h\bar{Y}^v\bar{Z}^h} = -\frac{1}{4}\Big(g\big(R(Z,X)Y,U\big)+g\big(R(X,Y)Z,U\big)\Big). \quad (3.1^\circ)$$

$$h^1_{\bar{X}^v\bar{Y}^h\bar{Z}^h} = -\frac{1}{4}\Big(g\big(R(Z,Y)X,U\big)\Big). \quad (4.1^\circ)$$

$$h^1_{\bar{X}^v\bar{Y}^v\bar{Z}^v} = \frac{1}{4}\Big(g\big(R(Z,Y)X,U\big)\Big). \quad (5.1^\circ)$$

$$h^1_{\bar{X}^v\bar{Y}^v\bar{Z}^h} = 0. \quad (6.1^\circ)$$

$$h^1_{\bar{X}^v\bar{Y}^h\bar{Z}^v} = 0. \quad (7.1^\circ)$$

$$h^1_{\bar{X}^h\bar{Y}^v\bar{Z}^v} = 0. \quad (8.1^\circ)$$

It is obvious that $(J_1,\hat{g})$ is a Kaehlerian structure if and only if $h^1=0$.

2°. Now assume additionally that we have an almost Hermitian structure J on (M, g). We define a tensor field J_2 on TM by the equalities

$$J_2\bar{X}^h = \left(\overline{JX}\right)^h, \quad J_2\bar{X}^v = -\left(\overline{JX}\right)^v, \quad X \in \chi(M). \quad (2.12)$$

For $X \in \chi(M)$ we get

$$J_2^2\bar{X} = J_2\left(J_2\left(\bar{X}^h \oplus \bar{X}^v\right)\right) = J_2\left(\left(\overline{JX}\right)^h \oplus -\left(\overline{JX}\right)^v\right) = -\left(\bar{X}^h \oplus \bar{X}^v\right) - I\bar{X}$$

and

$$J_2^2 = -I.$$

For $X,Y \in \chi(M)$ we obtain

$$\begin{aligned}\left\langle J_2\bar{X}, J_2\bar{Y}\right\rangle &= \left\langle \left(\overline{JX}\right)^h \oplus -\left(\overline{JX}\right)^v, \left(\overline{JY}\right)^h \oplus -\left(\overline{JY}\right)^v\right\rangle = \left\langle \left(\overline{JX}\right)^h, \left(\overline{JY}\right)^h\right\rangle + \left\langle \left(\overline{JX}\right)^v, \left(\overline{JY}\right)^v\right\rangle \\ &= g(JX,JY) + g(JX,JY) = g(X,Y) + g(X,Y) \\ &= \left\langle \bar{X}^h, \bar{Y}^h\right\rangle + \left\langle \bar{X}^v, \bar{Y}^v\right\rangle = \left\langle \bar{X}^h \oplus \bar{X}^v, \bar{Y}^h \oplus \bar{Y}^v\right\rangle = \left\langle \bar{X}, \bar{Y}\right\rangle.\end{aligned}$$

Further, we obtain

$$J_1\left(J_2\bar{X}\right) = J_1\left(\left(\overline{JX}\right)^h \oplus -\left(\overline{JX}\right)^v\right) = \left(\overline{JX}\right)^h \oplus \left(\overline{JX}\right)^v,$$

$$J_2\left(J_1\bar{X}\right) = J_2\left(-\bar{X}^h \oplus \bar{X}^v\right) = -\left(\overline{JX}\right)^h \oplus -\left(\overline{JX}\right)^v.$$

Thus, we get $J_1J_2 = -J_2J_1 = J_3$ and ahHs $(J_1, J_2, J_3, \langle,\rangle)$ on TM has been constructed.
For orthonormal vector fields $\bar{X},\bar{Y},\bar{Z}$ on TM we obtain

$$\begin{aligned}h^2_{\bar{X}\bar{Y}\bar{Z}} &= \left\langle h^2_{\bar{X}}\bar{Y},\bar{Z}\right\rangle = \frac{1}{2}\left\langle \hat{\nabla}_{\bar{X}}\bar{Y} + J_2\hat{\nabla}_{\bar{X}}J_2\bar{Y},\bar{Z}\right\rangle = \frac{1}{2}\left(\left\langle \hat{\nabla}_{\bar{X}}\bar{Y},\bar{Z}\right\rangle - \left\langle \hat{\nabla}_{\bar{X}}J_2\bar{Y},J_2\bar{Z}\right\rangle\right) \\ &= \frac{1}{4}\Big(\left\langle\left[\bar{X},\bar{Y}\right],\bar{Z}\right\rangle + \left\langle\left[\bar{Z},\bar{X}\right],\bar{Y}\right\rangle + \left\langle\left[\bar{Z},\bar{Y}\right],\bar{X}\right\rangle - \left\langle\left[\bar{X},J_2\bar{Y}\right],J_2\bar{Z}\right\rangle \\ &\quad -\left\langle\left[J_2\bar{Z},\bar{X}\right],J_2\bar{Y}\right\rangle - \left\langle\left[J_2\bar{Z},J_2\bar{Y}\right],\bar{X}\right\rangle\Big).\end{aligned} \quad (2.13)$$

Using (2.4)-(2.7) and (2.13) we consider the following cases for the tensor field h^2 assuming all the vector fields to be orthonormal.

$$h^2_{\bar{X}^h\bar{Y}^h\bar{Z}^h} = \frac{1}{4}\Big(\big\langle\big[\bar{X}^h,\bar{Y}^h\big],\bar{Z}^h\big\rangle + \big\langle\big[\bar{Z}^h,\bar{X}^h\big],\bar{Y}^h\big\rangle + \big\langle\big[\bar{Z}^h,\bar{Y}^h\big],\bar{X}^h\big\rangle$$
$$-\big\langle\big[\bar{X}^h,J_2\bar{Y}^h\big],J_2\bar{Z}^h\big\rangle - \big\langle\big[J_2\bar{Z}^h,\bar{X}^h\big],J_2\bar{Y}^h\big\rangle - \big\langle\big[J_2\bar{Z}^h,J_2\bar{Y}^h\big],\bar{X}^h\big\rangle\Big)$$
$$= \frac{1}{4}\big(g([X,Y],Z) + g([Z,X],Y) + g([Z,Y],X) \qquad (1.2^\circ)$$
$$-g([X,JY],JZ) - g([JZ,X],JY) - g([JZ,JY],X)\big)$$
$$= \frac{1}{2}\big(g(\nabla_X Y,Z) - g(\nabla_X JY,JZ)\big) = h_{XYZ}.$$

$$h^2_{\bar{X}^h\bar{Y}^h\bar{Z}^v} = \frac{1}{4}\Big(\big\langle\big[\bar{X}^h,\bar{Y}^h\big],\bar{Z}^v\big\rangle + \big\langle\big[\bar{Z}^v,\bar{X}^h\big],\bar{Y}^h\big\rangle + \big\langle\big[\bar{Z}^v,\bar{Y}^h\big],\bar{X}^h\big\rangle$$
$$-\big\langle\big[\bar{X}^h,J_2\bar{Y}^h\big],J_2\bar{Z}^v\big\rangle - \big\langle\big[J_2\bar{Z}^v,\bar{X}^h\big],J_2\bar{Y}^h\big\rangle - \big\langle\big[J_2\bar{Z}^v,J_2\bar{Y}^h\big],\bar{X}^h\big\rangle\Big) \qquad (2.2^\circ)$$
$$= \frac{1}{4}\big(g(R(X,Y)U,Z) + g(R(X,JY)U,JZ)\big)$$
$$= -\frac{1}{4}\big(g(R(X,Y)Z,U) + g(R(X,JY)JZ,U)\big).$$

By similar arguments we obtain

$$h^2_{\bar{X}^h\bar{Y}^v\bar{Z}^h} = -\frac{1}{4}\big(g(R(X,Z)Y,U) + g(R(X,JZ)JY,U)\big). \qquad (3.2^\circ)$$

$$h^2_{\bar{X}^v\bar{Y}^h\bar{Z}^h} = -\frac{1}{4}\big(g(R(Z,Y)X,U) + g(R(JZ,JY)X,U)\big). \qquad (4.2^\circ)$$

$$h^2_{\bar{X}^v\bar{Y}^v\bar{Z}^v} = 0. \qquad (5.2^\circ)$$

$$h^2_{\bar{X}^v\bar{Y}^v\bar{Z}^h} = 0. \qquad (6.2^\circ)$$

$$h^2_{\bar{X}^v\bar{Y}^h\bar{Z}^v} = 0. \qquad (7.2^\circ)$$

$$h^2_{\bar{X}^h\bar{Y}^v\bar{Z}^v} = \frac{1}{2}\big(g(\nabla_X Y,Z) - g(\nabla_X JY,JZ)\big) = h_{XYZ}. \qquad (8.2^\circ)$$

Here h is the second fundamental tensor field of the pair (J, g) on M.

3. Embeddings of Almost Hermitian Manifolds in Almost Hyper Hermitian Those

For an almost Hermitian manifold (M, J, g) we have constructed in Section 2 ahHs $(J_1,J_2,J_3,\hat{g})$ on TM. The manifold M can be considered as the null section O_M in TM $(p \leftrightarrow o_p \in O_M \subset TM)$ and it is clear from (2.8) that $\hat{g}_{|M} = g$. All the results of 1 can be applied to a submanifold M in $(TM,\hat{g})$, see [7]. So, we can consider the normal tubular neighborhoods $Tb\left(M,\frac{\varepsilon(p)}{2}\right) \subset Tb(M,\varepsilon(p)) \subset TM$ and the deformations $\bar{J}_1,\bar{J}_2,\bar{J}_3,\bar{g}$ of the tensor fields $J_1,J_2,J_3,\hat{g}$ respectively.

Theorem 2. *Let* (M, J, g) *be an almost Hermitian manifold and* $Tb(M,\varepsilon(p))$ *be the corresponding normal tubular neighborhood with respect to* $\hat{g} = \langle,\rangle$ *on TM. Then* $M(O_M)$ *is a totally geodesic submanifold of the almost hyper Hermitian manifold* $\left(Tb\left(M,\frac{\varepsilon(p)}{2}\right),\bar{J}_1,\bar{J}_2,\bar{J}_3,\bar{g}\right)$, *where the ahHs* $(\bar{J}_1,\bar{J}_2,\bar{J}_3,\bar{g})$ *is the deformation of the structure* $(\bar{J}_1,\bar{J}_2,\bar{J}_3,\hat{g})$ *obtained in* **2°**, Section 1. *The structure* $(\bar{J}_1,\bar{g})$ *is Kaehlerian one.*

Proof. It follows from Theorem 1 that M is a totally geodesic submanifold of the Riemannian manifold $\left(Tb\left(M,\frac{\varepsilon(p)}{2}\right),\bar{g}\right)$.

Let $\tilde{W}_0$ be a coordinate neighborhood in *TM* considered in **1**°, Section 1. A point $x \in \tilde{W}_0$ has the coordinates $x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}$ where $x_1, \cdots, x_n$ are coordinates of the point *p* in $\tilde{V}_0 \subset M$ and $x_{n+1}, \cdots, x_{2n}$ are normal coordinates of *x* in $D\left(p, \frac{\varepsilon(p)}{2}\right)$.

We denote $X_i = \frac{\partial}{\partial x_i}, i = \overline{1, 2n}$, $\hat{\nabla}_{X_i} X_j = \sum_k \hat{\Gamma}_{ij}^k X_k$, $\bar{\nabla}_{X_i} X_j = \sum_k \bar{\Gamma}_{ij}^k X_k$, $JX_j = \sum_k J_j^k X_k$, $\bar{J}X_j = \sum_k \bar{J}_j^k X_k$, $\hat{g}_{ij} = \hat{g}(X_i, X_j)$, $\bar{g}_{ij} = \bar{g}(X_i, X_j)$ where $\hat{\nabla}$ and $\bar{\nabla}$ are Riemannian connections of metrics $\hat{g}$ and $\bar{g}$, *J* is any tensor field from J_1, J_2, J_3.

Using the construction in **2**°, Section 1 we have $\bar{g}_{ij}(x) = \hat{g}_{ij}(p)$, $\bar{J}_j^i(x) = J_j^i(p)$ on $Tb\left(M, \frac{\varepsilon(p)}{2}\right) \cap \tilde{W}_0$.

According to [2] we can write

$$\sum_l \bar{g}_{lk} \bar{\Gamma}_{ij}^l = \frac{1}{2}\left(\frac{\partial \bar{g}_{kj}}{\partial x_i} + \frac{\partial \bar{g}_{ik}}{\partial x_j} - \frac{\partial \bar{g}_{ij}}{\partial x_k}\right) \tag{3.1}$$

It follows from (3.1) that $\bar{\Gamma}_{ij}^l(x) = \hat{\Gamma}_{ij}^l(p)$ and $\bar{\Gamma}_{ij}^l(x) = 0$ *i.e.* $\bar{\nabla}_{X_i} X_j = 0$ for $i = \overline{n+1, 2n}$. Further, we get

$$\begin{aligned}
\left(\bar{\nabla}_{X_i} \bar{J}\right) X_j &= \bar{\nabla}_{X_i} \bar{J} X_j - \bar{J} \bar{\nabla}_{X_i} X_j = \sum_k \bar{\nabla}_{X_i} \bar{J}_j^k X_k - \bar{J}\left(\sum_k \bar{\Gamma}_{ij}^k X_k\right) \\
&= \sum_k \left(\bar{J}_j^k \bar{\nabla}_{X_i} X_k + \left(X_i \bar{J}_j^k\right) X_k\right) - \sum_{k,l} \bar{\Gamma}_{ij}^l \bar{J}_l^k X_k \\
&= \sum_{k,l} \left(\bar{J}_j^l \bar{\Gamma}_{il}^k - \bar{\Gamma}_{ij}^l \bar{J}_l^k + X_i \bar{J}_j^k\right) X_k, \\
\left(\left(\bar{\nabla}_{X_i} \bar{J}\right) X_j\right)(x) &= \sum_{k,l} \left(\bar{J}_j^l \bar{\Gamma}_{il}^k - \bar{\Gamma}_{ij}^l \bar{J}_l^k + X_i \bar{J}_j^k\right)(x) X_{k|x} \\
&= \sum_{k,l} \left(\left(\bar{J}_j^l \bar{\Gamma}_{il}^k - \bar{\Gamma}_{ij}^l \bar{J}_l^k\right)(p) + \left(X_i \bar{J}_j^k\right)(x)\right) X_{k|x}.
\end{aligned}$$

It follows that $\bar{\nabla}_{X_i} \bar{J} = 0$ for $i = \overline{n+1, 2n}$.

For $i = \overline{1, n}$ $\left(X_i \bar{J}_j^k\right)(x) = \left(X_i J_j^k\right)(p)$ and we obtain

$$\left(\left(\bar{\nabla}_{X_i} \bar{J}\right) X_j\right)(x) = \sum_{k,l} \left(J_j^l \hat{\Gamma}_{il}^k - \hat{\Gamma}_{ij}^l J_l^k + X_i J_j^k\right)(p) X_{k|x}.$$

From the other side we can write

$$\left(\left(\hat{\nabla}_{X_i} \bar{J}\right) X_j\right)(p) = \sum_{k,l} \left(J_j^l \hat{\Gamma}_{il}^k - \hat{\Gamma}_{ij}^l J_l^k + X_i J_j^k\right)(p) X_{k|p}.$$

According to [6] we have $\left(\bar{\nabla}_{X_i} J\right) X_j = \left(2h_{X_i} JX_j\right)(p)$ where the second fundamental tensor field *h* is defined by (2.11). From (1.1°)-(8.1°) it follows that $h_p^1 = 0$ for any $p \in M$ $\left(U = o_p \in O_M\right)$. Thus, we have obtained $\bar{\nabla} J_1 = 0$ and the structure $\left(\bar{J}_1, \bar{g}\right)$ is Kaehlerian one on $Tb\left(M, \frac{\varepsilon(p)}{2}\right)$.

QED.

As a corollary we have got the following:

Theorem 3 [8]. *Let* (*M*, *g*) *be a smooth Riemannian manifold and* $Tb(M, \varepsilon(p))$ *be the corresponding normal tubular neighborhood with respect to* $g = \langle , \rangle$ *on TM. Then M*(O_M) *is a totally geodesic submanifold of the Kaehlerian manifold* $\left(Tb\left(M, \frac{\varepsilon(p)}{2}\right), \bar{J}_1, \bar{g}\right)$.

The classification given in [9] can be rewritten in terms of the second fundamental tensor field *h* (**Table 1**),

Table 1. Classification of almost Hermitian structures.

Class	Defining condition
$\boldsymbol{K}$	$h = 0$
$\boldsymbol{U_1 = NK}$	$h_X X = 0$
$\boldsymbol{U_2 = AK}$	$\sigma h_{XYZ} = 0$
$\boldsymbol{U_3 = SK \cap H}$	$h_{XYZ} - h_{JXJYJZ} = \beta(Z) = 0$
$\boldsymbol{U_4}$	$h_{XYZ} = \frac{1}{2(n-1)}\left[\langle X,Y\rangle\beta(Z) - \langle X,Z\rangle\beta(Y) - \langle X,JY\rangle\beta(JZ) + \langle X,JZ\rangle\beta(JY)\right]$
$\boldsymbol{U_1 \oplus U_2 = QK}$	$h_{XYJZ} = h_{JXYZ}$
$\boldsymbol{U_3 \oplus U_4 = H}$	$N(J) = 0$ or $h_{XYJZ} = -h_{JXYZ}$
$\boldsymbol{U_1 \oplus U_3}$	$h_{XXY} - h_{JXJXY} = \beta(Z) = 0$
$\boldsymbol{U_2 \oplus U_4}$	$\sigma\left[h_{XYJZ} - \frac{1}{(n-1)}\langle JX,Y\rangle\beta(Z)\right] = 0$
$\boldsymbol{U_1 \oplus U_4}$	$h_{XXY} = -\frac{1}{2(n-1)}\left[\langle X,Y\rangle\beta(X) - \lVert X\rVert^2\beta(Y) - \langle X,JY\rangle\beta(JX)\right]$
$\boldsymbol{U_2 \oplus U_3}$	$\sigma\left[h_{XYJZ} + h_{JXYZ}\right] = \beta(Z) = 0$
$\boldsymbol{U_1 \oplus U_2 \oplus U_3 = SK}$	$\beta = 0$
$\boldsymbol{U_1 \oplus U_2 \oplus U_4}$	$h_{XYJZ} - h_{JXYZ} = \frac{1}{(n-1)}\left[\langle X,Y\rangle\beta(JZ) - \langle X,Z\rangle\beta(JY) + \langle X,JY\rangle\beta(Z) - \langle X,JZ\rangle\beta(Y)\right]$
$\boldsymbol{U_1 \oplus U_3 \oplus U_4}$	$h_{XJXY} + h_{JXXY} = 0$
$\boldsymbol{U_2 \oplus U_3 \oplus U_4}$	$\sigma\left[h_{XYJZ} + h_{JXYZ}\right] = 0$
$\boldsymbol{U}$	No condition

see chapter 5 of monograph [6].

Let $\dim M \geq 6$ and $2\beta(X) = \delta\Phi(JX)$, where $\Phi(X,Y) = g(JX,Y)$, then we have **Table 1**.

Proposition 4. *Let* (*J*, *g*) *be from some class from the* **Table 1**. *Then the structure* $\left(\bar{J}_2, \bar{g}\right)$ *has the analogous class on* $Tb\left(M, \frac{\varepsilon(p)}{2}\right)$.

Proof. From (1.2˚)-(8.2˚) it follows that $h^2_{XYZ} = 2h_{XYZ}$. The rest is obvious from the table.

QED.

4. Complex and Hypercomplex Numbers in Differential Geometry

For the manifold *M* we consider the products $M^2 = M \times M = \left\{(x;y) \middle| x;y \in M\right\}$, $M^4 = M^2 \times M^2 = \left\{(x;y;u;v) \middle| x;y,u;v \in M\right\}$ and the diagonals $\Delta\left(M^2\right) = \left\{(x;x) \in M^2\right\}$, $\Delta\left(M^4\right) = \left\{(x;x;x;x) \in M^4\right\}$. It is obvious that the manifold $\Delta\left(M^2\right)$ and $\Delta\left(M^4\right)$ are diffeomorphic to *M* $\left(\Delta\left(M^2\right) \cong \Delta\left(M^4\right) \cong M\right)$.

Theorem 5 [1]. *Let* (M, ∇) *be a manifold with a connection* ∇ *and* π: $TM \rightarrow M$ *be the canonical projection. Then there exists such a neighborhood* N_0 *of the null section* O_M *in TM that the mapping*

$$\varphi : \pi \times \exp : X \rightarrow \left(\pi(X), \exp_{\pi(X)} X\right)$$

is the diffeomorphic of N_0 *on a neighborhood* N_Δ *of the diagonal* $\Delta\left(M^2\right)$.

Further, ∇ is a Riemannian connection of the Riemannian metric *g*. Combining the Theorems 3 and 5 we have obtained the following.

Theorem 6. *The diffeomorphism* φ *induces the Kaehlerian structure* $\left(\bar{J}_1, \bar{g}\right)$ *on the neighborhood* N_Δ *of the diagonal* $\Delta\left(M^2\right)$ *and* $\Delta\left(M^2\right) \cong M$ *is a totally geodesic submanifold of the Kaehlerian manifold* $\left(N_\Delta, \bar{J}_1, \bar{g}\right)$.

Remark. *Generally speaking, the complex structure of the Kaehlerian manifold* $\left(N_\Delta, \bar{J}_1, \bar{g}\right)$ *is not compatible with the product structure of* M^2. *It means that if* $z_l, l=\overline{1,n}$ *are the complex coordinates of a point* $(x,y)\in N_\Delta$, *then, generally speaking, we can not find such real coordinates* $x_l, y_l, l=\overline{1,n}$ *of the points* $x, y\in M$ *respectively that* $z_l = x_l + iy_l$ *where* $i^2=-1$.

Combining the Theorems 2, 3, 4, 5 and 6 we have obtained the following.

Theorem 7. *There exists the hyper Kaehlerian structure* $\left(\bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{g}\right)$ *on a neighborhood* $\bar{N}_\Delta$ *of the diagonal* $\Delta\left(M^4\right)$ and $\Delta\left(M^4\right)\cong M$ *is a totally geodesic submanifold of the hyper Kaehlerian manifold* $\left(N_\Delta, \bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{g}\right)$.

Remark. *Generally speaking, the hypercomplex structure of the hyper Kaehlerian manifold* $\left(\bar{N}_\Delta, \bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{g}\right)$ *is not compatible with the product structure of* M^4. *It means that if* $q_l, l=\overline{1,n}$ *are the hypercomplex coordinates of a point* $(x;y;u;v)\in \bar{N}_\Delta$, *then, generally speaking we can not find such real coordinates* $x_l, y_l, u_l, v_l, l=\overline{1,n}$ *of the points x; y; u;* $v\in M$ *respectively that* $q_l = x_l + iy_l + ju_l + kv_l$ *where* $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$.

5. A Local Construction of Kaehlerian and Riemannian Metrics

1°. We consider a Riemannian manifold (M, g) as a totally geodesic subanifold of the Kaehlerian manifold $Tb\left(M, \frac{\varepsilon(p)}{2}, \bar{J}=J_1, \bar{g}\right)$ (see Theorem 3) then $\bar{g}_{|M} = g$.

Let $x_1,\cdots,x_n$ be coordinates in some coordinate neighborhood $U\subset M$ and $\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}$ be the corresponding vector fields. We can choose a neighborhood $\bar{U} = U\times D = \bigcup_{p\in U} D(p;\varepsilon)\subset Tb\left(M,\frac{\varepsilon(p)}{2}\right)$ where $\varepsilon \le \frac{\varepsilon(p)}{2}$ for every point $p\in U$. It is clear from **3°**, 1 that $U\times D$ is a Riemannian product with respect the metric $\bar{g}$. For every point $x\in\bar{U}$ where $\pi(x)=p$ we denote $Y_{jx} = \bar{J}\frac{\partial}{\partial x_{jx}}$, $j=\overline{1,n}$ and the vector fields Y_j define the coordinates $y_1,\cdots,y_n$ on $D_{(p;\varepsilon)}$ hence $Y_j = \frac{\partial}{\partial y_j}$ is tangent to $D_{(p;\varepsilon)}$ for $j=\overline{1,n}$.

So, $\bar{U}$ is an coordinate neighborhood of the Kaehlerian manifold $\left(Tb\left(M,\frac{\varepsilon(p)}{2}\right), \bar{J}\bar{g}\right)$, with complex coordinates $z_j = x_j + iy_j$, $j=\overline{1,n}$, $i^2=-1$, and the vector fields $\frac{\partial}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\right)$, $\frac{\partial}{\partial \bar{z}_\beta} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\right)$, $\alpha,\beta=\overline{1,n}$. It is known [3] that the Kaehlerian metric $\bar{g}^c$ has on $\bar{U}$ the following decomposition

$$\mathrm{d}s^2 = 2\sum_{\alpha,\beta}\bar{g}^c_{\alpha\bar{\beta}}\mathrm{d}z^\alpha \mathrm{d}\bar{z}^\beta,\quad \bar{g}^c_{\alpha\bar{\beta}} = \frac{\partial^2 u}{\mathrm{d}z_\alpha \mathrm{d}\bar{z}_\beta},$$

where u is a real-valued function on $\bar{U}$.

We have

$$\frac{\partial^2 u}{\partial z_\alpha \partial z_\beta} = \frac{1}{4}\left\{\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} - i\left(\frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta}\right)\right\} = 0,$$

$$\frac{\partial^2 u}{\partial \bar{z}_\alpha \partial \bar{z}_\beta} = \frac{1}{4}\left\{\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} + i\left(\frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta}\right)\right\} = 0.$$

It follows that

$$\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} = \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}, \quad \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} = -\frac{\partial^2 u}{\partial y_\alpha \partial x_\beta}.$$

Further, we obtain

$$\bar{g}^c_{\alpha\bar{\beta}} = \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{1}{4}\left\{\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} + i\left(\frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial x_\beta}\right)\right\} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + i\frac{\partial^2 u}{\partial x_\alpha \partial y_\beta}\right),$$

$$\bar{g}^c_{\bar{\alpha}\beta} = \frac{\partial^2 u}{\partial \bar{z}_\alpha \partial z_\beta} = \frac{1}{4}\left\{\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} - i\left(\frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial x_\beta}\right)\right\} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} - i\frac{\partial^2 u}{\partial x_\alpha \partial y_\beta}\right).$$

Finally, we get

$$\bar{g}\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \frac{1}{2} Re\bar{g}^c\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \frac{1}{2} Re\bar{g}^c\left(\frac{\partial}{\partial z_\alpha} + \frac{\partial}{\partial z_\beta}, \frac{\partial}{\partial \bar{z}_\beta} + \frac{\partial}{\partial \bar{z}_\beta}\right)$$

$$= Re\left(\bar{g}^c_{\alpha\beta} + \bar{g}^c_{\alpha\bar{\beta}} + + \bar{g}^c_{\alpha\bar{\beta}} + \bar{g}^c_{\bar{\alpha}\beta}\right) = Re\left(\bar{g}^c_{\alpha\bar{\beta}} + \bar{g}^c_{\bar{\alpha}\beta}\right) = \frac{\partial^2 u}{\partial x_\alpha \partial y \beta}.$$

We can consider the restriction of $\bar{g}$ and the function u on the neighborhood U. So, we have obtained.

Theorem 8. *Let* (M, g) *be a Riemannian manifold and* $x_1, \cdots, x_n$ *be coordinates is some coordinate neighborhood* $U \subset M$. *There exists a smooth function u*: $U \to \mathbf{R}$ *that* $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 u}{\partial x_i \partial x_j}$ *on U*.

2°. Let (M, J, g) be a Kaehlerian manifold $x_1, \cdots, x_n$, $y_1, \cdots, y_n$, be coordinates is some coordinate neighborhood $U \subset M$, where $\frac{\partial}{\partial y_\alpha} = J\frac{\partial}{\partial x_\alpha}, \alpha = \overline{1, n}$. We consider a function u: $U \to \mathbf{R}$ from Theorem 5. Then, we have the following conditions on this function.

$$\frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} = g\left(\frac{\partial}{\partial x_\alpha}, J\frac{\partial}{\partial x_\beta}\right) = -g\left(J\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = -\frac{\partial^2 u}{\partial y_\alpha \partial y_\beta};$$

$$\frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} = g\left(J\frac{\partial}{\partial x_\alpha}, J\frac{\partial}{\partial x_\beta}\right) = g\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}, \quad \alpha, \beta = \overline{1, n}.$$

6. Conclusion

We consider such mappings in the category of Riemannian manifolds that metrics are invariant with respect to them. It follows that only totally geodesic submanifolds are "naturally good". Theorems 6 and 7 allow considering any Riemannian manifold as a totally geodesic submanifold of a Kaehlerian (hyper Kaehlerian) one *i.e.* to apply the results of Kaehlerian (hyper Kaehlerian) geometry to Riemannian metrics. We remark that Whitnies embeddings are not suitable in this context.

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