

Embeddings of Almost Hermitian Manifold in Almost Hyper Hermitian Manifold and Complex (Hypercomplex) Numbers in Riemannian Geometry

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Abstract

Tubular neighborhoods play an important role in differential topology. We have applied these constructions to geometry of almost Hermitian manifolds. At first, we consider deformations of tensor structures on a normal tubular neighborhood of a submanifold in a Riemannian manifold. Further, an almost hyper Hermitian structure has been constructed on the tangent bundle TM with help of the Riemannian connection of an almost Hermitian structure on a manifold M then, we consider an embedding of the almost Hermitian manifold M in the corresponding normal tubular neighborhood of the null section in the tangent bundle TM equipped with the deformed almost hyper Hermitian structure of the special form. As a result, we have obtained that any Riemannian manifold M of dimension n can be embedded as a totally geodesic submanifold in a Kaehlerian manifold of dimension $2n$ (Theorem 6) and in a hyper Kaehlerian manifold of dimension $4n$ (Theorem 7). Such embeddings are “good” from the point of view of Riemannian geometry. They allow solving problems of Riemannian geometry by methods of Kaehlerian geometry (see Section 5 as an example). We can find similar situation in mathematical analysis (real and complex).

Keywords

Riemannian Manifolds, Almost Hermitian and Almost Hyper Hermitian Structures, Tangent Bundle

1. Deformations of Tensor Structures on a Normal Tubular Neighborhood of a Submanifold

1°. Let (M', g') be a k -dimensional Riemannian manifold isometrically embedded in a n -dimensional Rie-

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mannian manifold (M, g) . The restriction of g to M' coincides with g' and for any $p \in M'$.

$$T_p(M) = T_p(M') \oplus T_p(M')^\perp.$$

So, we obtain a vector bundle $M' \rightarrow T(M')^\perp : p \rightarrow T_p(M')^\perp$ over the submanifold M' . There exists a neighborhood \tilde{U}_0 of the null section $O_{M'}$ in $T(M')^\perp$ such that the mapping

$$\pi \times \exp : v \rightarrow (\pi(v), \exp_{\pi(v)} v), v \in \tilde{U}_0,$$

is a diffeomorphism of \tilde{U}_0 onto an open subset $\tilde{U} \subset M$. The subset \tilde{U} is called a *tubular neighborhood* of the submanifold M' in M .

For any point $p \in M'$ we can consider a set $\{\delta(p)\}$ of positive numbers such that the mapping $\exp_{U(\delta(p))}$ is defined and injective on $U(\delta(p)) \subset T_p(M)$. Let $\bar{\varepsilon}(p) = \sup\{\delta(p)\}$.

Lemma [1]. The mapping $M \rightarrow R_+ : p \rightarrow \bar{\varepsilon}(p)$ is continuous on M .

If we take the restriction of the function $\bar{\varepsilon}(p)$ on \tilde{U} then it is clear that there exists a continuous positive function $\varepsilon(p)$ on M' such that for any $p \in M'$ open geodesic balls $B\left(p; \frac{\varepsilon(p)}{2}\right) \subset B(p; \varepsilon(p)) \subset \tilde{U}$. For

compact manifolds we can choose a constant function $\varepsilon(p) = \varepsilon > 0$. We denote $\tilde{U}_p = \exp(\tilde{U}_0 \cap T_p(M')^\perp)$,

$$D\left(p; \frac{\varepsilon(p)}{2}\right) = B\left(p; \frac{\varepsilon(p)}{2}\right) \cap \tilde{U}_p, \quad D(p; \varepsilon(p)) = B(p; \varepsilon(p)) \cap \tilde{U}_p. \text{ It is obvious that}$$

$\dim \tilde{U}_p = \dim D(p; \varepsilon(p)) = n - k$. For any point $o \in M'$ we can consider such an orthonormal frame $(X_{i_0}, \dots, X_{n_0})$ that $T_0(M') = L[X_{i_0}, \dots, X_{k_0}]$ and $T_0(M')^\perp = L[X_{k+1_0}, \dots, X_{n_0}]$. There exist coordinates

x_1, \dots, x_k in some neighborhood $\tilde{V}_0 \subset M'$ of the point o that $\frac{\partial}{\partial x_{i_0}} = X_{i_0}, i = \overline{1, k}$. We consider orthonormal

vector fields X_{k+1}, \dots, X_n which are cross-sections of the vector bundle $p \rightarrow T_p(M')^\perp$ over \tilde{V}_0 and the neighborhood $\tilde{W}_0 = \bigcup_{p \in \tilde{V}_0} \tilde{U}_p$. The basis $\{X_{k+1_p}, \dots, X_{n_p}\}$ defines the normal coordinates x_{k+1}, \dots, x_n on \tilde{U}_p

[2]. For any point $x \in \tilde{W}_0$ there exists such unique point $p \in \tilde{V}_0$ that $x = \exp_p(t\xi), \|\xi\| = 1, \xi \in T_p(M')^\perp$. A point $x \in \tilde{W}_0$ has the coordinates $x_1, \dots, x_k, x_{k+1}, \dots, x_n$ where x_1, \dots, x_k are coordinates of the point p in \tilde{V}_0

and x_{k+1}, \dots, x_n are normal coordinates of x in \tilde{U}_p . We denote $X_i = \frac{\partial}{\partial x_i}, i = \overline{1, n}$, on \tilde{W}_0 . Thus, we can con-

sider *tubular neighborhoods* $Tb\left(M'; \frac{\varepsilon(p)}{2}\right) = \bigcup_{p \in M'} D\left(p; \frac{\varepsilon(p)}{2}\right)$ and $Tb(M'; \varepsilon(p)) = \bigcup_{p \in M'} D(p; \varepsilon(p))$ of the

submanifold M' .

2°. Let K be a smooth tensor field of type (r, s) on the manifold M and for $x \in \tilde{W}_0$, let

$$K_x = \sum_{i_1, \dots, i_r, j_1, \dots, j_s} k_{j_1, \dots, j_s}^{i_1, \dots, i_r}(x) X_{i_{i_1}} \otimes \dots \otimes X_{i_{i_r}} \otimes X_x^{j_1} \otimes \dots \otimes X_x^{j_s},$$

where $\{X_x^1, \dots, X_x^n\}$ is the dual basis of $T_x^*(M), x = \exp_p(t\xi), \|\xi\| = 1, \xi \in T_p(M')^\perp$. We define a tensor field \bar{K} on M in the following way.

a) $x \in D\left(p; \frac{\varepsilon(p)}{2}\right)$, then

$$\bar{K}_x = \sum_{i_1, \dots, i_r, j_1, \dots, j_s} k_{j_1, \dots, j_s}^{i_1, \dots, i_r}(p) X_{i_{i_1}} \otimes \dots \otimes X_{i_{i_r}} \otimes X_x^{j_1} \otimes \dots \otimes X_x^{j_s};$$

b) $x \in D(p; \varepsilon(p)) \setminus D\left(p; \frac{\varepsilon(p)}{2}\right)$, then

$$\bar{K}_x = \sum_{i_1, \dots, i_r, j_1, \dots, j_s} k_{j_1, \dots, j_s}^{i_1, \dots, i_r} \left(\exp_p \left((2t - \varepsilon(p)) \xi \right) \right) X_{i_{i_x}} \otimes \dots \otimes X_{i_{i_x}} \otimes X_{j_1} \otimes \dots \otimes X_{j_s};$$

c) $x \in M \setminus \bigcup_{M'} D(p; \varepsilon(p))$, then

$$\bar{K}_x = K_x.$$

It is easy to see the independence of the tensor field \bar{K} on a choice of coordinates in \tilde{W}_0 for every point $o \in M'$.

Definition 1. The tensor field \bar{K} is called a deformation of the tensor field K on the normal tubular neighborhood of a submanifold M' .

Remark. The obtained tensor field \bar{K} is continuous but is not smooth on the boundaries of the normal tubular neighborhoods $Tb\left(M'; \frac{\varepsilon(p)}{2}\right)$ and $Tb(M'; \varepsilon(p))$; \bar{K} is smooth in other points of the manifold M .

3°. We consider a deformation \bar{g} of the Riemannian metric g on the normal tubular neighborhood $Tb(M'; \varepsilon(p))$ of a submanifold M' . For $x \in \tilde{W}_0$, $x = \exp_p(t\xi)$, $\|\xi\| = 1$, $\xi \in T_p(M')$, we define the Riemannian metric \bar{g} by the following way.

a) $\bar{g}_p = g_p$ for any $p \in M'$;

b) $\bar{g}_x(X_i, X_j) = \bar{g}_{ij}(x) = \bar{g}_{ij}(p)$, where $X_i = \frac{\partial}{\partial x_i}$, $i = \overline{1, n}$, $X_j = \frac{\partial}{\partial x_j}$, $j = \overline{1, n}$, on \tilde{W}_0 , $x \in D\left(p; \frac{\varepsilon(p)}{2}\right)$;

c) $\bar{g}_x(X_i, X_j) = \bar{g}_{ij}(x) = \bar{g}_{ij}\left(\exp_p\left((2t - \varepsilon(p))\xi\right)\right)$, for any $x \in D(p; \varepsilon(p)) \setminus D\left(p; \frac{\varepsilon(p)}{2}\right)$;

d) $\bar{g}_x = g_x$ for each point $x \in M \setminus \bigcup_{p \in M'} D(p; \varepsilon(p))$.

The independence of \bar{g} on a choice of local coordinates follows and the correctly defined Riemannian metric \bar{g} on M has been obtained.

It is known from [3] that every autoparallel submanifold of M is a totally geodesic submanifold and a submanifold M' is autoparallel if and only if $\nabla_X Y \in T(M')$ for any $X, Y \in \chi(M')$, where ∇ is the Riemannian connection of g .

Theorem 1. Let M' be a submanifold of a Riemannian manifold (M, g) and \bar{g} be the deformation of g on the normal tubular neighborhood $Tb(M'; \varepsilon(p))$ of M' constructed above. Then M' is a totally geodesic submanifold of $\left(Tb\left(M'; \frac{\varepsilon(p)}{2}\right), \bar{g}\right)$.

Proof. For any point $x \in D\left(p; \frac{\varepsilon(p)}{2}\right) \subset \tilde{W}_0$ the functions $\bar{g}_{ij}(x) = g_{ij}(p)$ and $\frac{\partial \bar{g}_{ij}}{\partial x_l} = 0$, $l = \overline{k+1, n}$ on $D\left(p; \frac{\varepsilon(p)}{2}\right)$ because the vector fields $X_l = \frac{\partial}{\partial x_l}$ are tangent to $D\left(p; \frac{\varepsilon(p)}{2}\right)$. By the formula of the Riemannian connection $\bar{\nabla}$ of the Riemannian metric \bar{g} , [2], we obtain for $i, j = \overline{1, k}$, $l = \overline{k+1, n}$

$$\begin{aligned} 2\bar{g}_p(\bar{\nabla}_{X_i} X_j, X_l) &= X_{i_p} \bar{g}(X_j, X_l) + X_{j_p} \bar{g}(X_i, X_l) - X_{l_p} \bar{g}(X_i, X_j) + \bar{g}_p([X_i, X_j], X_l) \\ &\quad + \bar{g}_p([X_l, X_i], X_j) + \bar{g}_p(X_i, [X_l, X_j]) = -\frac{\partial \bar{g}_{ij}}{\partial x_l} = 0. \end{aligned} \quad (1.1)$$

Here we use the fact that $[X_i, X_j] = [X_l, X_i] = [X_l, X_j] = 0$ and that $\bar{g}(X_j, X_l) = \bar{g}(X_i, X_l) = 0$ because $X_l \in T(M')^\perp$.

Thus, $\bar{\nabla}_{X_i} X_j \in T(M')$ and from the remarks above the theorem follows.

QED.

Corollary 1.1. Let \bar{R} be the Riemannian curvature tensor field of $\bar{\nabla}$. Then \bar{R} vanishes on every

$$D\left(p; \frac{\varepsilon(p)}{2}\right) \text{ for } p \in M'.$$

Proof. From the formula (1.1) it is clear that $\bar{\nabla}_{X_l} X_m = 0$ for $l, m = \overline{k+1, n}$. The rest is obvious.

QED.

2. Almost Hyper Hermitian Structures (ahHs) on Tangent Bundles

0°. We follow especially close to [4].

Let (M, g) be a n -dimensional Riemannian manifold and TM be its tangent bundle. For a Riemannian connection ∇ we consider the connection map K of ∇ [5], [1], defined by the formula

$$\nabla_X Z = KZ_* X, \quad (2.1)$$

where Z is considered as a map from M into TM and the right side means a vector field on M assigning to $p \in M$ the vector $KZ_* X_p \in M_p$.

If $U \in TM$, we denote by H_U the kernel of $K|_{TM_U}$ and this n -dimensional subspace of TM_U is called the horizontal subspace of TM_U .

Let π denote the natural projection of TM onto M , then π_* is a C^∞ -map of TTM onto TM . If $U \in TM$, we denote by V_U the kernel of $\pi_*|_{TM_U}$ and this n -dimensional subspace of TM_U is called the vertical subspace of TM_U ($\dim TM_U = 2 \dim M = 2n$). The following maps are isomorphisms of corresponding vector spaces ($p = \pi(U)$)

$$\pi_*|_{H_U} : H_U \rightarrow M_p, \quad K|_{V_U} : V_U \rightarrow M_p$$

and we have

$$TM_U = H_U \oplus V_U$$

If $X \in \chi(M)$, then there exists exactly one vector field on TM called the “horizontal lift” (resp. “vertical lift”) of X and denoted by \bar{X}^h (\bar{X}^v), such that for all $U \in TM$:

$$\pi_* \bar{X}_U^h = X_{\pi(U)}, \quad K \bar{X}_U^h = 0_{\pi(U)}, \quad (2.2)$$

$$\pi_* \bar{X}_U^v = 0_{\pi(U)}, \quad K \bar{X}_U^v = X_{\pi(U)}, \quad (2.3)$$

Let R be the curvature tensor field of ∇ , then following [5] we write

$$[\bar{X}^v, \bar{Y}^v] = 0, \quad (2.4)$$

$$[\bar{X}^h, \bar{Y}^v] = (\overline{\nabla_X Y})^v \quad (2.5)$$

$$\pi_*([\bar{X}^h, \bar{Y}^h]_U) = [X, Y], \quad (2.6)$$

$$K([\bar{X}^h, \bar{Y}^h]_U) = R(X, Y)U. \quad (2.7)$$

For vector fields $\bar{X} = \bar{X}^h \oplus \bar{X}^v$ and $\bar{Y} = \bar{Y}^h \oplus \bar{Y}^v$ on TM the natural Riemannian metric $\hat{g} = \langle, \rangle$ is defined on TM by the formula

$$\langle \bar{X}, \bar{Y} \rangle = g(\pi_* \bar{X}, \pi_* \bar{Y}) + g(K \bar{X}, K \bar{Y}). \quad (2.8)$$

It is clear that the subspaces H_U and V_U are orthogonal with respect to \langle, \rangle .

It is easy to verify that $\bar{X}_1^h, \bar{X}_2^h, \dots, \bar{X}_n^h, \bar{X}_1^v, \bar{X}_2^v, \dots, \bar{X}_n^v$ are orthonormal vector fields on TM if X_1, X_2, \dots, X_n are those on M i.e. $g(X_i, X_j) = \delta_j^i$.

1°. We define a tensor field J_1 on TM by the equalities

$$J_1 \bar{X}^h = \bar{X}^v, \quad J_1 \bar{X}^v = -\bar{X}^h, \quad X \in \chi(M). \quad (2.9)$$

For $X \in \chi(M)$ we get

$$J_1^2 \bar{X} = J_1 (J_1 (\bar{X}^h \oplus \bar{X}^v)) = J_1 (-\bar{X}^h \oplus \bar{X}^v) = -(\bar{X}^h \oplus \bar{X}^v) = -I\bar{X}$$

and

$$J_1^2 = -I.$$

For $X, Y \in \mathcal{X}(M)$ we obtain

$$\begin{aligned} \langle J_1 \bar{X}, J_1 \bar{Y} \rangle &= \langle -\bar{X}^h \oplus \bar{X}^v, -\bar{Y}^h \oplus \bar{Y}^v \rangle = \langle -\bar{X}^h, -\bar{Y}^h \rangle + \langle \bar{X}^v, \bar{Y}^v \rangle, \\ \langle \bar{X}, \bar{Y} \rangle &= \langle \bar{X}^h \oplus \bar{X}^v, \bar{Y}^h \oplus \bar{Y}^v \rangle = \langle \bar{X}^h, \bar{Y}^h \rangle + \langle \bar{X}^v, \bar{Y}^v \rangle \end{aligned}$$

and it follows that $(J_1 \bar{X}, J_1 \bar{Y}) = (\bar{X}, \bar{Y})$, $(TM, J_1, \langle, \rangle)$ is an almost Hermitian manifold.

Further, we want to analyze the second fundamental tensor field h^1 of the pair (J_1, \langle, \rangle) where h^1 is defined by (2.11), [6].

The Riemannian connection $\hat{\nabla}$ of the metric $\hat{g} = \langle, \rangle$ on TM is defined by the formula (see [1])

$$\begin{aligned} \langle \hat{\nabla}_{\bar{X}} \bar{Y}, \bar{Z} \rangle &= \frac{1}{2} (\bar{X} \langle \bar{Y}, \bar{Z} \rangle + \bar{Y} \langle \bar{Z}, \bar{X} \rangle - \bar{Z} \langle \bar{X}, \bar{Y} \rangle + \langle \bar{Z}, [\bar{X}, \bar{Y}] \rangle \\ &\quad + \langle \bar{Y}, [\bar{Z}, \bar{X}] \rangle + \langle \bar{X}, [\bar{Z}, \bar{Y}] \rangle), \quad X, Y, Z \in \mathcal{X}(M). \end{aligned} \quad (2.10)$$

For orthonormal vector fields $\bar{X}, \bar{Y}, \bar{Z}$ on TM we obtain

$$\begin{aligned} h_{\bar{X}\bar{Y}\bar{Z}}^1 &= \langle h_{\bar{X}}^1 \bar{Y}, \bar{Z} \rangle = \frac{1}{2} \langle \hat{\nabla}_{\bar{X}} \bar{Y} + J_1 \hat{\nabla}_{\bar{X}} J_1 \bar{Y}, \bar{Z} \rangle \\ &= \frac{1}{2} (\langle \hat{\nabla}_{\bar{X}} \bar{Y}, \bar{Z} \rangle - \langle \hat{\nabla}_{\bar{X}} J_1 \bar{Y}, J_1 \bar{Z} \rangle) \\ &= \frac{1}{4} (\langle [\bar{X}, \bar{Y}], \bar{Z} \rangle + \langle [\bar{Z}, \bar{X}], \bar{Y} \rangle + \langle [\bar{Z}, \bar{Y}], \bar{X} \rangle - \langle [\bar{X}, J_1 \bar{Y}], J_1 \bar{Z} \rangle \\ &\quad - \langle [J_1 \bar{Z}, \bar{X}], J_1 \bar{Y} \rangle - \langle [J_1 \bar{Z}, J_1 \bar{Y}], \bar{X} \rangle). \end{aligned} \quad (2.11)$$

Using (2.4)-(2.7) and (2.11) we consider the following cases for the tensor field h^1 assuming all the vector fields to be orthonormal.

$$\begin{aligned} h_{\bar{X}^h \bar{Y}^h \bar{Z}^h}^1 &= \frac{1}{4} (\langle [\bar{X}^h, \bar{Y}^h], \bar{Z}^h \rangle + \langle [\bar{Z}^h, \bar{X}^h], \bar{Y}^h \rangle + \langle [\bar{Z}^h, \bar{Y}^h], \bar{X}^h \rangle \\ &\quad - \langle [\bar{X}^h, J_1 \bar{Y}^h], J_1 \bar{Z}^h \rangle - \langle [J_1 \bar{Z}^h, \bar{X}^h], J_1 \bar{Y}^h \rangle - \langle [J_1 \bar{Z}^h, J_1 \bar{Y}^h], \bar{X}^h \rangle) \\ &= \frac{1}{4} (g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X) - \langle [\bar{X}^h, \bar{Y}^v], \bar{Z}^v \rangle \\ &\quad - \langle [\bar{Z}^v, \bar{X}^h], \bar{Y}^v \rangle - \langle [\bar{Z}^v, \bar{Y}^h], \bar{X}^h \rangle) \\ &= \frac{1}{2} g(\nabla_X Y, Z) - \frac{1}{4} (g(\nabla_X Y, Z) - g(\nabla_X Z, Y)) \\ &= \frac{1}{2} (g(\nabla_X Y, Z) - g(\nabla_X Y, Z)) = 0. \end{aligned} \quad (1.1^\circ)$$

$$\begin{aligned} h_{\bar{X}^h \bar{Y}^h \bar{Z}^v}^1 &= \frac{1}{4} (\langle [\bar{X}^h, \bar{Y}^h], \bar{Z}^v \rangle + \langle [\bar{Z}^v, \bar{X}^h], \bar{Y}^h \rangle + \langle [\bar{Z}^v, \bar{Y}^h], \bar{X}^h \rangle \\ &\quad - \langle [\bar{X}^h, J_1 \bar{Y}^h], J_1 \bar{Z}^v \rangle - \langle [J_1 \bar{Z}^v, \bar{X}^h], J_1 \bar{Y}^h \rangle - \langle [J_1 \bar{Z}^v, J_1 \bar{Y}^h], \bar{X}^h \rangle) \\ &= \frac{1}{4} (g(R(X, Y)U, Z) + \langle [\bar{Z}^h, \bar{X}^h], \bar{Y}^v \rangle) \\ &= \frac{1}{4} (g(R(X, Y)U, Z) + g(R(Z, X)U, Y)) \\ &= -\frac{1}{4} (g(R(X, Y)Z, U) + g(R(Z, X)Y, U)). \end{aligned} \quad (2.1^\circ)$$

By similar arguments we obtain

$$h_{\bar{X}^h \bar{Y}^v \bar{Z}^h}^1 = -\frac{1}{4} \left(g(R(Z, X)Y, U) + g(R(X, Y)Z, U) \right). \quad (3.1^\circ)$$

$$h_{\bar{X}^v \bar{Y}^h \bar{Z}^h}^1 = -\frac{1}{4} \left(g(R(Z, Y)X, U) \right). \quad (4.1^\circ)$$

$$h_{\bar{X}^v \bar{Y}^v \bar{Z}^v}^1 = \frac{1}{4} \left(g(R(Z, Y)X, U) \right). \quad (5.1^\circ)$$

$$h_{\bar{X}^v \bar{Y}^v \bar{Z}^h}^1 = 0. \quad (6.1^\circ)$$

$$h_{\bar{X}^v \bar{Y}^h \bar{Z}^v}^1 = 0. \quad (7.1^\circ)$$

$$h_{\bar{X}^h \bar{Y}^v \bar{Z}^v}^1 = 0. \quad (8.1^\circ)$$

It is obvious that (J_1, \hat{g}) is a Kaehlerian structure if and only if $h^1 = 0$.

2°. Now assume additionally that we have an almost Hermitian structure J on (M, g) . We define a tensor field J_2 on TM by the equalities

$$J_2 \bar{X}^h = (\overline{JX})^h, \quad J_2 \bar{X}^v = -(\overline{JX})^v, \quad X \in \chi(M). \quad (2.12)$$

For $X \in \chi(M)$ we get

$$J_2^2 \bar{X} = J_2 \left(J_2 (\bar{X}^h \oplus \bar{X}^v) \right) = J_2 \left((\overline{JX})^h \oplus -(\overline{JX})^v \right) = -(\bar{X}^h \oplus \bar{X}^v) = -\bar{X}$$

and

$$J_2^2 = -I.$$

For $X, Y \in \chi(M)$ we obtain

$$\begin{aligned} \langle J_2 \bar{X}, J_2 \bar{Y} \rangle &= \langle (\overline{JX})^h \oplus -(\overline{JX})^v, (\overline{JY})^h \oplus -(\overline{JY})^v \rangle = \langle (\overline{JX})^h, (\overline{JY})^h \rangle + \langle (\overline{JX})^v, (\overline{JY})^v \rangle \\ &= g(JX, JY) + g(JX, JY) = g(X, Y) + g(X, Y) \\ &= \langle \bar{X}^h, \bar{Y}^h \rangle + \langle \bar{X}^v, \bar{Y}^v \rangle = \langle \bar{X}^h \oplus \bar{X}^v, \bar{Y}^h \oplus \bar{Y}^v \rangle = \langle \bar{X}, \bar{Y} \rangle. \end{aligned}$$

Further, we obtain

$$J_1(J_2 \bar{X}) = J_1 \left((\overline{JX})^h \oplus -(\overline{JX})^v \right) = (\overline{JX})^h \oplus (\overline{JX})^v,$$

$$J_2(J_1 \bar{X}) = J_2 \left(-\bar{X}^h \oplus \bar{X}^v \right) = -(\overline{JX})^h \oplus -(\overline{JX})^v.$$

Thus, we get $J_1 J_2 = -J_2 J_1 = J_3$ and ahHs $(J_1, J_2, J_3, \langle, \rangle)$ on TM has been constructed.

For orthonormal vector fields $\bar{X}, \bar{Y}, \bar{Z}$ on TM we obtain

$$\begin{aligned} h_{\bar{X}\bar{Y}\bar{Z}}^2 &= \langle h_{\bar{X}}^2 \bar{Y}, \bar{Z} \rangle = \frac{1}{2} \langle \hat{\nabla}_{\bar{X}} \bar{Y} + J_2 \hat{\nabla}_{\bar{X}} J_2 \bar{Y}, \bar{Z} \rangle = \frac{1}{2} \left(\langle \hat{\nabla}_{\bar{X}} \bar{Y}, \bar{Z} \rangle - \langle \hat{\nabla}_{\bar{X}} J_2 \bar{Y}, J_2 \bar{Z} \rangle \right) \\ &= \frac{1}{4} \left(\langle [\bar{X}, \bar{Y}], \bar{Z} \rangle + \langle [\bar{Z}, \bar{X}], \bar{Y} \rangle + \langle [\bar{Z}, \bar{Y}], \bar{X} \rangle - \langle [\bar{X}, J_2 \bar{Y}], J_2 \bar{Z} \rangle \right. \\ &\quad \left. - \langle [J_2 \bar{Z}, \bar{X}], J_2 \bar{Y} \rangle - \langle [J_2 \bar{Z}, J_2 \bar{Y}], \bar{X} \rangle \right). \end{aligned} \quad (2.13)$$

Using (2.4)-(2.7) and (2.13) we consider the following cases for the tensor field h^2 assuming all the vector fields to be orthonormal.

$$\begin{aligned}
h_{\bar{X}^h \bar{Y}^h \bar{Z}^h}^2 &= \frac{1}{4} \left(\langle [\bar{X}^h, \bar{Y}^h], \bar{Z}^h \rangle + \langle [\bar{Z}^h, \bar{X}^h], \bar{Y}^h \rangle + \langle [\bar{Z}^h, \bar{Y}^h], \bar{X}^h \rangle \right. \\
&\quad \left. - \langle [\bar{X}^h, J_2 \bar{Y}^h], J_2 \bar{Z}^h \rangle - \langle [J_2 \bar{Z}^h, \bar{X}^h], J_2 \bar{Y}^h \rangle - \langle [J_2 \bar{Z}^h, J_2 \bar{Y}^h], \bar{X}^h \rangle \right) \\
&= \frac{1}{4} \left(g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X) \right. \\
&\quad \left. - g([X, JY], JZ) - g([JZ, X], JY) - g([JZ, JY], X) \right) \\
&= \frac{1}{2} (g(\nabla_X Y, Z) - g(\nabla_X JY, JZ)) = h_{XYZ}.
\end{aligned} \tag{1.2°}$$

$$\begin{aligned}
h_{\bar{X}^h \bar{Y}^h \bar{Z}^v}^2 &= \frac{1}{4} \left(\langle [\bar{X}^h, \bar{Y}^h], \bar{Z}^v \rangle + \langle [\bar{Z}^v, \bar{X}^h], \bar{Y}^h \rangle + \langle [\bar{Z}^v, \bar{Y}^h], \bar{X}^h \rangle \right. \\
&\quad \left. - \langle [\bar{X}^h, J_2 \bar{Y}^h], J_2 \bar{Z}^v \rangle - \langle [J_2 \bar{Z}^v, \bar{X}^h], J_2 \bar{Y}^h \rangle - \langle [J_2 \bar{Z}^v, J_2 \bar{Y}^h], \bar{X}^h \rangle \right) \\
&= \frac{1}{4} (g(R(X, Y)U, Z) + g(R(X, JY)U, JZ)) \\
&= -\frac{1}{4} (g(R(X, Y)Z, U) + g(R(X, JY)JZ, U)).
\end{aligned} \tag{2.2°}$$

By similar arguments we obtain

$$h_{\bar{X}^h \bar{Y}^v \bar{Z}^h}^2 = -\frac{1}{4} (g(R(X, Z)Y, U) + g(R(X, JZ)JY, U)). \tag{3.2°}$$

$$h_{\bar{X}^v \bar{Y}^h \bar{Z}^h}^2 = -\frac{1}{4} (g(R(Z, Y)X, U) + g(R(JZ, JY)X, U)). \tag{4.2°}$$

$$h_{\bar{X}^v \bar{Y}^v \bar{Z}^v}^2 = 0. \tag{5.2°}$$

$$h_{\bar{X}^v \bar{Y}^h \bar{Z}^h}^2 = 0. \tag{6.2°}$$

$$h_{\bar{X}^h \bar{Y}^h \bar{Z}^v}^2 = 0. \tag{7.2°}$$

$$h_{\bar{X}^h \bar{Y}^v \bar{Z}^v}^2 = \frac{1}{2} (g(\nabla_X Y, Z) - g(\nabla_X JY, JZ)) = h_{XYZ}. \tag{8.2°}$$

Here h is the second fundamental tensor field of the pair (J, g) on M .

3. Embeddings of Almost Hermitian Manifolds in Almost Hyper Hermitian Those

For an almost Hermitian manifold (M, J, g) we have constructed in Section 2 ahHs (J_1, J_2, J_3, \hat{g}) on TM . The manifold M can be considered as the null section O_M in TM ($p \leftrightarrow o_p \in O_M \subset TM$) and it is clear from (2.8) that $\hat{g}|_M = g$. All the results of 1 can be applied to a submanifold M in (TM, \hat{g}) , see [7]. So, we can consider the normal tubular neighborhoods $Tb\left(M, \frac{\varepsilon(p)}{2}\right) \subset Tb(M, \varepsilon(p)) \subset TM$ and the deformations $\bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{g}$ of the tensor fields J_1, J_2, J_3, \hat{g} respectively.

Theorem 2. Let (M, J, g) be an almost Hermitian manifold and $Tb(M, \varepsilon(p))$ be the corresponding normal tubular neighborhood with respect to $\hat{g} = \langle, \rangle$ on TM . Then $M(O_M)$ is a totally geodesic submanifold of the almost hyper Hermitian manifold $\left(Tb\left(M, \frac{\varepsilon(p)}{2}\right), \bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{g}\right)$, where the ahHs $(\bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{g})$ is the deformation of the structure $(\bar{J}_1, \bar{J}_2, \bar{J}_3, \hat{g})$ obtained in 2°, Section 1. The structure (\bar{J}_1, \bar{g}) is Kaehlerian one.

Proof. It follows from Theorem 1 that M is a totally geodesic submanifold of the Riemannian manifold

$$\left(Tb\left(M, \frac{\varepsilon(p)}{2}\right), \bar{g}\right).$$

Let \tilde{W}_0 be a coordinate neighborhood in TM considered in 1° , Section 1. A point $x \in \tilde{W}_0$ has the coordinates $x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}$ where x_1, \dots, x_n are coordinates of the point p in $\tilde{V}_0 \subset M$ and x_{n+1}, \dots, x_{2n} are normal coordinates of x in $D\left(p, \frac{\varepsilon(p)}{2}\right)$.

We denote $X_i = \frac{\partial}{\partial x_i}, i = \overline{1, 2n}$, $\hat{\nabla}_{X_i} X_j = \sum_k \hat{\Gamma}_{ij}^k X_k$, $\bar{\nabla}_{X_i} X_j = \sum_k \bar{\Gamma}_{ij}^k X_k$, $JX_j = \sum_k J_j^k X_k$, $\bar{J}X_j = \sum_k \bar{J}_j^k X_k$, $\hat{g}_{ij} = \hat{g}(X_i, X_j)$, $\bar{g}_{ij} = \bar{g}(X_i, X_j)$ where $\hat{\nabla}$ and $\bar{\nabla}$ are Riemannian connections of metrics \hat{g} and \bar{g} , J is any tensor field from J_1, J_2, J_3 .

Using the construction in 2° , Section 1 we have $\bar{g}_{ij}(x) = \hat{g}_{ij}(p)$, $\bar{J}_j^i(x) = J_j^i(p)$ on $Tb\left(M, \frac{\varepsilon(p)}{2}\right) \cap \tilde{W}_0$.

According to [2] we can write

$$\sum_l \bar{g}_{lk} \bar{\Gamma}_{ij}^l = \frac{1}{2} \left(\frac{\partial \bar{g}_{kj}}{\partial x_i} + \frac{\partial \bar{g}_{ik}}{\partial x_j} - \frac{\partial \bar{g}_{ij}}{\partial x_k} \right) \quad (3.1)$$

It follows from (3.1) that $\bar{\Gamma}_{ij}^l(x) = \bar{\Gamma}_{ij}^l(p)$ and $\bar{\Gamma}_{ij}^l(x) = 0$ i.e. $\bar{\nabla}_{X_i} X_j = 0$ for $i = \overline{n+1, 2n}$. Further, we get

$$\begin{aligned} (\bar{\nabla}_{X_i} \bar{J}) X_j &= \bar{\nabla}_{X_i} \bar{J}X_j - \bar{J} \bar{\nabla}_{X_i} X_j = \sum_k \bar{\nabla}_{X_i} \bar{J}_j^k X_k - \bar{J} \left(\sum_k \bar{\Gamma}_{ij}^k X_k \right) \\ &= \sum_k \left(\bar{J}_j^k \bar{\nabla}_{X_i} X_k + (X_i \bar{J}_j^k) X_k \right) - \sum_{k,l} \bar{\Gamma}_{ij}^l \bar{J}_l^k X_k \\ &= \sum_{k,l} \left(\bar{J}_j^l \bar{\Gamma}_{il}^k - \bar{\Gamma}_{ij}^l \bar{J}_l^k + X_i \bar{J}_j^k \right) X_k, \\ ((\bar{\nabla}_{X_i} \bar{J}) X_j)(x) &= \sum_{k,l} \left(\bar{J}_j^l \bar{\Gamma}_{il}^k - \bar{\Gamma}_{ij}^l \bar{J}_l^k + X_i \bar{J}_j^k \right)(x) X_{k|x} \\ &= \sum_{k,l} \left((\bar{J}_j^l \bar{\Gamma}_{il}^k - \bar{\Gamma}_{ij}^l \bar{J}_l^k)(p) + (X_i \bar{J}_j^k)(x) \right) X_{k|x}. \end{aligned}$$

It follows that $\bar{\nabla}_{X_i} \bar{J} = 0$ for $i = \overline{n+1, 2n}$.

For $i = \overline{1, n}$ $(X_i \bar{J}_j^k)(x) = (X_i J_j^k)(p)$ and we obtain

$$((\bar{\nabla}_{X_i} \bar{J}) X_j)(x) = \sum_{k,l} (J_j^l \hat{\Gamma}_{il}^k - \hat{\Gamma}_{ij}^l J_l^k + X_i J_j^k)(p) X_{k|x}.$$

From the other side we can write

$$((\hat{\nabla}_{X_i} \bar{J}) X_j)(p) = \sum_{k,l} (J_j^l \hat{\Gamma}_{il}^k - \hat{\Gamma}_{ij}^l J_l^k + X_i J_j^k)(p) X_{k|p}.$$

According to [6] we have $(\bar{\nabla}_{X_i} J) X_j = (2h_{X_i} JX_j)(p)$ where the second fundamental tensor field h is defined by (2.11). From (1.1°)-(8.1°) it follows that $h_p^i = 0$ for any $p \in M$ ($U = o_p \in O_M$). Thus, we have obtained $\bar{\nabla} J_1 = 0$ and the structure (\bar{J}_1, \bar{g}) is Kaehlerian one on $Tb\left(M, \frac{\varepsilon(p)}{2}\right)$.

QED.

As a corollary we have got the following:

Theorem 3 [8]. Let (M, g) be a smooth Riemannian manifold and $Tb(M, \varepsilon(p))$ be the corresponding normal tubular neighborhood with respect to $g = \langle, \rangle$ on TM . Then $M(O_M)$ is a totally geodesic submanifold of the Kaehlerian manifold $\left(Tb\left(M, \frac{\varepsilon(p)}{2}\right), \bar{J}_1, \bar{g}\right)$.

The classification given in [9] can be rewritten in terms of the second fundamental tensor field h (Table 1),

Table 1. Classification of almost Hermitian structures.

Class	Defining condition
K	$h = 0$
$U_1 = NK$	$h_X X = 0$
$U_2 = AK$	$\sigma h_{XYZ} = 0$
$U_3 = SK \cap H$	$h_{XYZ} - h_{JXJYZ} = \beta(Z) = 0$
U_4	$h_{XYZ} = \frac{1}{2(n-1)} [\langle X, Y \rangle \beta(Z) - \langle X, Z \rangle \beta(Y) - \langle X, JY \rangle \beta(JZ) + \langle X, JZ \rangle \beta(JY)]$
$U_1 \oplus U_2 = QK$	$h_{XYZ} = h_{JXJYZ}$
$U_3 \oplus U_4 = H$	$N(J) = 0$ or $h_{XYZ} = -h_{JXJYZ}$
$U_1 \oplus U_3$	$h_{XXY} - h_{JXJXY} = \beta(Z) = 0$
$U_2 \oplus U_4$	$\sigma \left[h_{XYZ} - \frac{1}{(n-1)} \langle JX, Y \rangle \beta(Z) \right] = 0$
$U_1 \oplus U_4$	$h_{XXY} = -\frac{1}{2(n-1)} [\langle X, Y \rangle \beta(X) - \ X\ ^2 \beta(Y) - \langle X, JY \rangle \beta(JX)]$
$U_2 \oplus U_3$	$\sigma [h_{XYZ} + h_{JXJYZ}] = \beta(Z) = 0$
$U_1 \oplus U_2 \oplus U_3 = SK$	$\beta = 0$
$U_1 \oplus U_2 \oplus U_4$	$h_{XYZ} - h_{JXJYZ} = \frac{1}{(n-1)} [\langle X, Y \rangle \beta(JZ) - \langle X, Z \rangle \beta(JY) + \langle X, JY \rangle \beta(Z) - \langle X, JZ \rangle \beta(Y)]$
$U_1 \oplus U_3 \oplus U_4$	$h_{XXY} + h_{JXJXY} = 0$
$U_2 \oplus U_3 \oplus U_4$	$\sigma [h_{XYZ} + h_{JXJYZ}] = 0$
U	No condition

see chapter 5 of monograph [6].

Let $\dim M \geq 6$ and $2\beta(X) = \delta\Phi(JX)$, where $\Phi(X, Y) = g(JX, Y)$, then we have Table 1.

Proposition 4. Let (J, g) be from some class from the Table 1. Then the structure (\bar{J}_2, \bar{g}) has the analogous class on $Tb\left(M, \frac{\varepsilon(p)}{2}\right)$.

Proof. From $(1.2^\circ)-(8.2^\circ)$ it follows that $h_{XYZ}^2 = 2h_{XYZ}$. The rest is obvious from the table.

QED.

4. Complex and Hypercomplex Numbers in Differential Geometry

For the manifold M we consider the products $M^2 = M \times M = \{(x; y) | x, y \in M\}$,

$M^4 = M^2 \times M^2 = \{(x; y; u; v) | x, y, u, v \in M\}$ and the diagonals $\Delta(M^2) = \{(x; x) \in M^2\}$,

$\Delta(M^4) = \{(x; x; x; x) \in M^4\}$. It is obvious that the manifold $\Delta(M^2)$ and $\Delta(M^4)$ are diffeomorphic to M ($\Delta(M^2) \cong \Delta(M^4) \cong M$).

Theorem 5 [1]. Let (M, ∇) be a manifold with a connection ∇ and $\pi: TM \rightarrow M$ be the canonical projection. Then there exists such a neighborhood N_0 of the null section O_M in TM that the mapping

$$\varphi: \pi \times \exp: X \rightarrow (\pi(X), \exp_{\pi(X)} X)$$

is the diffeomorphic of N_0 on a neighborhood N_Δ of the diagonal $\Delta(M^2)$.

Further, ∇ is a Riemannian connection of the Riemannian metric g . Combining the Theorems 3 and 5 we have obtained the following.

Theorem 6. The diffeomorphism φ induces the Kaehlerian structure (\bar{J}_1, \bar{g}) on the neighborhood N_Δ of the diagonal $\Delta(M^2)$ and $\Delta(M^2) \cong M$ is a totally geodesic submanifold of the Kaehlerian manifold $(N_\Delta, \bar{J}_1, \bar{g})$.

Remark. Generally speaking, the complex structure of the Kaehlerian manifold $(N_\Delta, \bar{J}_1, \bar{g})$ is not compatible with the product structure of M^2 . It means that if $z_l, l=1, n$ are the complex coordinates of a point $(x, y) \in N_\Delta$, then, generally speaking, we can not find such real coordinates $x_l, y_l, l=1, n$ of the points $x, y \in M$ respectively that $z_l = x_l + iy_l$ where $i^2 = -1$.

Combining the Theorems 2, 3, 4, 5 and 6 we have obtained the following.

Theorem 7. There exists the hyper Kaehlerian structure $(\bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{g})$ on a neighborhood \bar{N}_Δ of the diagonal $\Delta(M^4)$ and $\Delta(M^4) \cong M$ is a totally geodesic submanifold of the hyper Kaehlerian manifold $(N_\Delta, \bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{g})$.

Remark. Generally speaking, the hypercomplex structure of the hyper Kaehlerian manifold $(\bar{N}_\Delta, \bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{g})$ is not compatible with the product structure of M^4 . It means that if $q_l, l=1, n$ are the hypercomplex coordinates of a point $(x; y; u; v) \in \bar{N}_\Delta$, then, generally speaking we can not find such real coordinates

$x_l, y_l, u_l, v_l, l=1, n$ of the points $x; y; u; v \in M$ respectively that $q_l = x_l + iy_l + ju_l + kv_l$ where $i^2 = j^2 = k^2 = -1, ij = -ji = k$.

5. A Local Construction of Kaehlerian and Riemannian Metrics

1°. We consider a Riemannian manifold (M, g) as a totally geodesic submanifold of the Kaehlerian manifold

$Tb\left(M, \frac{\varepsilon(p)}{2}, \bar{J} = J_1, \bar{g}\right)$ (see Theorem 3) then $\bar{g}|_M = g$.

Let x_1, \dots, x_n be coordinates in some coordinate neighborhood $U \subset M$ and $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ be the corresponding vector fields. We can choose a neighborhood $\bar{U} = U \times D = \bigcup_{p \in U} D(p; \varepsilon) \subset Tb\left(M, \frac{\varepsilon(p)}{2}\right)$ where $\varepsilon \leq \frac{\varepsilon(p)}{2}$ for every point $p \in U$. It is clear from 3°, 1 that $U \times D$ is a Riemannian product with respect the metric \bar{g} . For every point $x \in \bar{U}$ where $\pi(x) = p$ we denote $Y_{jx} = \bar{J} \frac{\partial}{\partial x_{jx}}, j = \overline{1, n}$ and the vector fields

Y_j define the coordinates y_1, \dots, y_n on $D_{(p; \varepsilon)}$ hence $Y_j = \frac{\partial}{\partial y_j}$ is tangent to $D_{(p; \varepsilon)}$ for $j = \overline{1, n}$.

So, \bar{U} is an coordinate neighborhood of the Kaehlerian manifold $\left(Tb\left(M, \frac{\varepsilon(p)}{2}\right), \bar{J}, \bar{g}\right)$, with complex coordinates $z_j = x_j + iy_j, j = \overline{1, n}, i^2 = -1$, and the vector fields $\frac{\partial}{\partial z_\alpha} = \frac{1}{2} \left(\frac{\partial}{\partial x_\alpha} - i \frac{\partial}{\partial y_\alpha} \right)$,

$\frac{\partial}{\partial \bar{z}_\beta} = \frac{1}{2} \left(\frac{\partial}{\partial x_\alpha} + i \frac{\partial}{\partial y_\alpha} \right), \alpha, \beta = \overline{1, n}$. It is known [3] that the Kaehlerian metric \bar{g}^c has on \bar{U} the following decomposition

$$ds^2 = 2 \sum_{\alpha, \beta} \bar{g}_{\alpha\beta}^c dz^\alpha d\bar{z}^\beta, \quad \bar{g}_{\alpha\beta}^c = \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\beta},$$

where u is a real-valued function on \bar{U} .

We have

$$\begin{aligned} \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\beta} &= \frac{1}{4} \left\{ \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} - i \left(\frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} \right) \right\} = 0, \\ \frac{\partial^2 u}{\partial \bar{z}_\alpha \partial z_\beta} &= \frac{1}{4} \left\{ \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} + i \left(\frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} \right) \right\} = 0. \end{aligned}$$

It follows that

$$\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} = \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}, \quad \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} = -\frac{\partial^2 u}{\partial y_\alpha \partial x_\beta}.$$

Further, we obtain

$$\begin{aligned} \bar{g}_{\alpha\bar{\beta}}^c &= \frac{\partial^2 u}{\partial \bar{z}_\alpha \partial \bar{z}_\beta} = \frac{1}{4} \left\{ \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} + i \left(\frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} \right) \right\} = \frac{1}{2} \left(\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + i \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} \right), \\ \bar{g}_{\bar{\alpha}\beta}^c &= \frac{\partial^2 u}{\partial \bar{z}_\alpha \partial z_\beta} = \frac{1}{4} \left\{ \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} - i \left(\frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} \right) \right\} = \frac{1}{2} \left(\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} - i \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} \right). \end{aligned}$$

Finally, we get

$$\begin{aligned} \bar{g} \left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right) &= \frac{1}{2} \operatorname{Re} \bar{g}^c \left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right) = \frac{1}{2} \operatorname{Re} \bar{g}^c \left(\frac{\partial}{\partial \bar{z}_\alpha} + \frac{\partial}{\partial \bar{z}_\beta}, \frac{\partial}{\partial \bar{z}_\beta} + \frac{\partial}{\partial \bar{z}_\alpha} \right) \\ &= \operatorname{Re} \left(\bar{g}_{\alpha\bar{\beta}}^c + \bar{g}_{\bar{\alpha}\bar{\beta}}^c + \bar{g}_{\alpha\bar{\beta}}^c + \bar{g}_{\bar{\alpha}\bar{\beta}}^c \right) = \operatorname{Re} \left(\bar{g}_{\alpha\bar{\beta}}^c + \bar{g}_{\bar{\alpha}\bar{\beta}}^c \right) = \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta}. \end{aligned}$$

We can consider the restriction of \bar{g} and the function u on the neighborhood U . So, we have obtained.

Theorem 8. Let (M, g) be a Riemannian manifold and x_1, \dots, x_n be coordinates is some coordinate neighborhood $U \subset M$. There exists a smooth function $u: U \rightarrow \mathbf{R}$ that $g_{ij} = g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 u}{\partial x_i \partial x_j}$ on U .

2°. Let (M, J, g) be a Kaehlerian manifold $x_1, \dots, x_n, y_1, \dots, y_n$, be coordinates is some coordinate neighborhood $U \subset M$, where $\frac{\partial}{\partial y_\alpha} = J \frac{\partial}{\partial x_\alpha}, \alpha = \overline{1, n}$. We consider a function $u: U \rightarrow \mathbf{R}$ from Theorem 5. Then, we have the following conditions on this function.

$$\begin{aligned} \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} &= g \left(\frac{\partial}{\partial x_\alpha}, J \frac{\partial}{\partial x_\beta} \right) = -g \left(J \frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right) = -\frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}; \\ \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} &= g \left(J \frac{\partial}{\partial x_\alpha}, J \frac{\partial}{\partial x_\beta} \right) = g \left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right) = \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}, \quad \alpha, \beta = \overline{1, n}. \end{aligned}$$

6. Conclusion

We consider such mappings in the category of Riemannian manifolds that metrics are invariant with respect to them. It follows that only totally geodesic submanifolds are “naturally good”. Theorems 6 and 7 allow considering any Riemannian manifold as a totally geodesic submanifold of a Kaehlerian (hyper Kaehlerian) one *i.e.* to apply the results of Kaehlerian (hyper Kaehlerian) geometry to Riemannian metrics. We remark that Whitney embeddings are not suitable in this context.

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