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An Asymptotic Distribution Function of the Three-Dimensional Shifted van der Corput Sequence

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Abstract

In this paper, we apply the Weyl's limit relation to calculate the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F\left(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\right)=\int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z),$$

where $\gamma_q(n)$ is the van der Corput sequence in base q, $g(x,y,z)$ is the asymptotic distribution function of $\left(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\right)$, and $F(x,y,z)=\max(x,y,z)$, $\min(x,y,z)$ and xyz, respectively.

Keywords

Sequences, Arithmetic Means, Riemann-Stieltjes Integraion

1. Introduction

In this paper we apply the Weyl's limit relation [1] (p. 1-61)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(\boldsymbol{x}_n)=\int_{[0,1]^s}f(\boldsymbol{x})\,\mathrm{d}g(\boldsymbol{x}),\tag{1.1}$$

to the sequence $\boldsymbol{x}_n=\left(\gamma_q(n),\gamma_q(n+1),\cdots,\gamma_q(n+s-1)\right)$, where $\gamma_q(n)$ is the van der Corput sequence in base

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q and $g(\boldsymbol{x})$ is the asymptotic distribution function (abbreviated a.d.f.) of $\boldsymbol{x}_n$ and $s=3$. The van der Corput sequence in base q is defined as follows: Let $n=n_k q^k+\cdots+n_0$ be the q-adic expression of a positive integer n. Then

$$\gamma_q(n)=\frac{n_0}{q}+\frac{n_1}{q^2}+\cdots+\frac{n_k}{q^{k+1}},\ n=0,1,\cdots. \tag{1.2}$$

It is well-known that this sequence is uniformly distributed (abbreviated u.d.), see [1] (2.11, p. 2-102), [2] (Theorem 3.5, p. 127), [3] (p. 41).

For $s=2$ a motivation for the study of the distribution function (abbreviated d.f.) $g(x,y)$ of $(\gamma_q(n),\gamma_q(n+1))$, $n=0,1,\cdots$ is a result of Pillichshammer and Steinerberger in [4] which states that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\left|\gamma_q(n)-\gamma_q(n+1)\right|=\frac{2(q-1)}{q^2}, \tag{1.3}$$

while in J. Fialová and O. Strauch [5] the relation (1.3) was proved applying (1.1) as

$$\int_0^1\int_0^1|x-y|\,\mathrm{d}g(x,y)=\frac{2(q-1)}{q^2}.$$

Moreover, in the Unsolved Problems [6] (1.12), the following problem is stated: Find the d.f. $g(x_1,x_2,\cdots,x_s)$ of the sequence $(\gamma_q(n),\cdots,\gamma_q(n+s-1))$, $n=0,1,2,\cdots$, in $[0,1]^s$. Ch. Aistleitner and M. Hofer [7] gave the following theoretical solution:

Theorem 1 *Let* T *denote the von Neuman-Kakutani transformation described in* **Figure 1**. *Define the* s*-dimensional curve* $\{\gamma(t);t\in[0,1)\}$, *where* $\gamma(t)=(t,T(t),T^2(t),\cdots,T^{s-1}(t))$. *Then the searched a.d.f. is*

$$g(x_1,x_2,\cdots,x_s)=\left|\{t\in[0,1];\gamma(t)\in[0,x_1]\times[0,x_2]\times\cdots\times[0,x_s]\}\right|,$$

where $|X|$ is the Lebesgue measure of a set X.

The paper consists of the following parts: After definitions (Part 2) we derive the a.d.f. of $(\gamma_q(n),\gamma_q(n+1))$

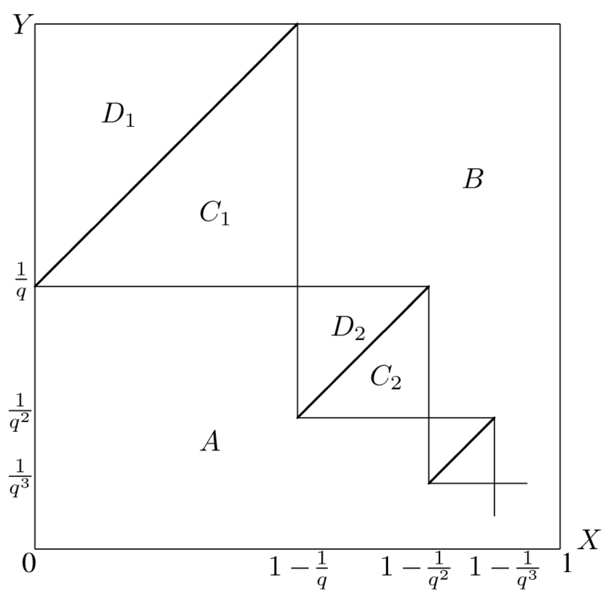

Figure 1. Line segments containing $(\gamma_q(n),\gamma_q(n+1))$ $n=1,2,\cdots$ The graph of the von Neumann-Kakutani transformation T.

Haoshangban (Part 3), the a.d.f. of $\left(\gamma_q(n),\gamma_q(n+2)\right)$ (Part 4), intervals containing $\left(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\right)$ in diagonals (Part 5) and an explicit form of a.d.f. $g(x,y,z)$ (Part 6). As an application (Part 7) we compute the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F\left(\left(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\right)\right)=\int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z) \tag{1.4}$$

for $F(x,y,z)=\max(x,y,z)$, $\min(x,y,z)$ and xyz, respectively, see (0.39), (0.41) and (0.46).

2. Definitions and Notations

Let x_n, $n=1,2,\cdots$ be a sequence in the unit interval $[0,1)$. Denote

- $$F_N(x)=\frac{\#\{n\le N; x_n\in[0,x)\}}{N}$$

the step distribution function (step d.f.) of the finite sequence $x_1,\cdots,x_N$ in $[0,1)$, while $F_N(1)=1$.

- A function $g:[0,1]\to[0,1]$ is a distribution function (d.f.) if

(i) $g(x)$ is nondecreasing;

(ii) $g(0)=0$ and $g(1)=1$.

- A d.f. $g(x)$ is a d.f. of the sequence x_n, $n=0,1,2,\cdots$ if an increasing sequence of positive integers $N_1,N_2,\cdots$ exists such that $\lim_{k\to\infty}F_{N_k}(x)=g(x)$ a.e. on $[0,1]$.
- A d.f. $g(x)$ is an asymptotic d.f. (a.d.f.) of the sequence x_n, $n=1,2,\cdots$ if $\lim_{N\to\infty}F_{N_k}(x)=g(x)$ a.e. on $[0,1]$.
- The sequence x_n is uniformly distributed (abbreviating u.d.) if its a.d.f. is $g(x)=x$.
- Similar definitions take place for $s=2,3$ and s-dimensional sequence $\boldsymbol{x}_n$, $n=0,1,2,\cdots$, in $[0,1)^s$, cf. [1] (1.11, pp. 1-60).
- In the sequel the 3-dimensional interval I we denote by $I=I_X\times I_Y\times I_Z$, where I_X,I_Y,I_Z are projections on X,Y,Z axes, respectively.

3. a.d.f. of $\left(\gamma_q(n),\gamma_q(n+1)\right), n=1,2,\cdots$

Let $q\ge 2$ be an integer.

Lemma 1 *Every point* $\left(\gamma_q(n),\gamma_q(n+1)\right)$, $n=0,1,2,\cdots$, *lies on the diagonals of intervals*

$$\left[0,1-\frac{1}{q}\right]\times\left[\frac{1}{q},1\right] \tag{1.5}$$

$$\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q^{i+1}},\frac{1}{q^i}\right],\quad i=1,2,\cdots \tag{1.6}$$

Proof. Express an integer n in the base q

$$n=n_kq^k+n_{k-1}q^{k-1}+\cdots+n_1q+n_0,$$

where $n_i<q$ and $n_k>0$. We consider the following two cases:

$$1^0.\ n_0<q-1,$$

$$2^0.\ n_0=q-1.$$

Let

1^0. $n_0<q-1$ Then

$$n=n_kq^k+\cdots+n_0,$$

$n+1=n_k q^k+\cdots+n_0+1$ and by (0.2)

$\gamma_q(n+1)-\gamma_q(n)=\frac{1}{q}$. In this case

$$\gamma_q(n)=\frac{n_0}{q}+\cdots+\frac{n_k}{q^{k+1}}\le\frac{q-2}{q}+\frac{q-1}{q^2}+\cdots=\frac{q-1}{q}.$$

Thus such $(\gamma_q(n),\gamma_q(n+1))$ lies on the line-segment

$$Y=X+\frac{1}{q},\quad X\in\left[0,1-\frac{1}{q}\right). \tag{1.7}$$

Let

2^0. $n_0=q-1$ Then

$n=n_k q^k+\cdots+n_{i+1}q^{i+1}+(q-1)q^i+(q-1)q^{i-1}+\cdots+(q-1)$ and $n_{i+1}<q-1$, where $i=0,1,2,\cdots$. Then

$n+1=n_k q^k+\cdots+(n_{i+1}+1)q^{i+1}+0\cdot q^i+0\cdot q^{i-1}+\cdots+0$. Thus

$$\gamma_q(n)=\frac{q-1}{q}+\cdots+\frac{q-1}{q^{i+1}}+\frac{n_{i+1}}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}},$$

$\gamma_q(n+1)=\frac{n_{i+1}+1}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}}$, and we have

$\gamma_q(n+1)-\gamma_q(n)=\frac{1}{q^{i+2}}-\frac{q-1}{q}\left(1+\frac{1}{q}+\cdots+\frac{1}{q^i}\right)=\frac{1}{q^{i+2}}-1+\frac{1}{q^{i+1}}$ and

$1-\frac{1}{q^{i+1}}=\frac{q-1}{q}+\cdots+\frac{q-1}{q^{i+1}}\le\gamma_q(n)$ and

$\gamma_q(n)\le\frac{q-1}{q}+\cdots+\frac{q-1}{q^{i+1}}+\frac{q-2}{q^{i+2}}+\frac{q-1}{q^{i+3}}+\cdots=1-\frac{1}{q^{i+2}}$. Thus such $(\gamma_q(n),\gamma_q(n+1))$ lies on the segment

$$Y=X-1+\frac{1}{q^{i+1}}+\frac{1}{q^{i+2}},\quad X\in\left[1-\frac{1}{q^{i+1}},1-\frac{1}{q^{i+2}}\right],\quad i=0,1,2,\cdots \tag{1.8}$$

Thus, for 1^0, terms of the sequence $(\gamma_q(n),\gamma_q(n+1))$ lie on the diagonal of the interval

$$I_X\times I_Y:=\left[0,1-\frac{1}{q}\right]\times\left[\frac{1}{q},1\right] \tag{1.9}$$

and for 2^0, after reduction $(i+1)\to i$, terms of the sequence $(\gamma_q(n),\gamma_q(n+1))$ lie on the diagonals of the intervals

$$I_X^{(i)}\times I_Y^{(i)}:=\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q^{i+1}},\frac{1}{q^i}\right],\quad i=1,2,\cdots \tag{1.10}$$

These intervals are maximal with respect to inclusion.

Adding the maps (1.7) and (1.8) we found the so-called von Neumann-Kakutani transformation $T:[0,1]\to[0,1]$, see **Figure 1**. Because $\gamma_q(n)$ is u.d., the sequence $(\gamma_q(n),\gamma_q(n+1))$ has a.d.f. $g(x,y)$ of the form[1]

$$\begin{aligned} g(x,y)&=\left|\mathrm{Project}_X\left(\left([0,x]\times[0,y]\right)\cap\text{graph } T\right)\right| \\ &=\min\left(\left|[0,x]\cap I_X\right|,\left|[0,y]\cap I_Y\right|\right)+\sum_{i=1}^{\infty}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|\right), \end{aligned} \tag{1.11}$$

[1] $g(x,y)$ is a copula.

where Project_X is the projection of a two dimensional set to the X-axis.

The sum (1.11) implies

$$g(x,y)=\begin{cases}0 & \text{if } (x,y)\in A,\\ 1-(1-y)-(1-x)=x+y-1 & \text{if } (x,y)\in B,\\ y-\dfrac{1}{q^i} & \text{if } (x,y)\in C_i,\\ x-1+\dfrac{1}{q^{i-1}} & \text{if } (x,y)\in D_i,\end{cases} \tag{1.12}$$

$i=1,2,\cdots$

From (1.12) it follows

$$g(x,x)=\begin{cases}0, & \text{if } x\in\left[0,\dfrac{1}{q}\right],\\ x-\dfrac{1}{q}, & \text{if } x\in\left[\dfrac{1}{q},1-\dfrac{1}{q}\right],\\ 2x-1, & \text{if } x\in\left[1-\dfrac{1}{q},1\right],\end{cases} \tag{1.13}$$

and for $q=2$, the mean equality misses.

4. a.d.f. of $(\gamma_q(n),\gamma_q(n+2)), n=1,2,\cdots$

Let $q\ge 3$ be an integer.

Lemma 2 *All terms of the sequence* $(\gamma_q(n),\gamma_q(n+2))$, $n=0,1,2,\cdots$, *lie in the diagonals of the following intervals*

$$\left[0,1-\frac{2}{q}\right]\times\left[\frac{2}{q},1\right], \tag{1.14}$$

$$\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q}+\frac{1}{q^{i+1}},\frac{1}{q}+\frac{1}{q^i}\right], i=1,2,\cdots, \tag{1.15}$$

$$\left[1-\frac{1}{q}-\frac{1}{q^k},1-\frac{1}{q}-\frac{1}{q^{k+1}}\right]\times\left[\frac{1}{q^{k+1}},\frac{1}{q^k}\right], k=1,2,\cdots, \tag{1.16}$$

Proof. Express an integer n in the base q

$$n=n_kq^k+n_{k-1}q^{k-1}+\cdots+n_1q+n_0, \tag{1.17}$$

where $n_i<q$ and $n_k>0$. We consider three following cases:

1^0. $n_0<q-2$,

2^0. $n_0=q-2$,

3^0. $n_0=q-1$.

Let

1^0. $n_0<q-2$ Then

$n=n_kq^k+\cdots+n_0$,

$n+2=n_kq^k+\cdots+n_0+2$ and

$\gamma_q(n+1)-\gamma_q(n)=\frac{2}{q}$. In this case

$0\le\gamma_q(n)=\frac{n_0}{q}+\cdots+\frac{n_k}{q^{k+1}}<\frac{q-3}{q}+\frac{q-1}{q^2}+\cdots=\frac{q-2}{q}$, and thus such $(\gamma_q(n),\gamma_q(n+1))$ lies on the line-segment

$$Z=X+\frac{2}{q},\quad X\in\left[0,1-\frac{2}{q}\right). \tag{1.18}$$

Let

2^0. $n_0=q-2$. Then

$n=n_kq^k+\cdots+n_{i+1}q^{i+1}+(q-1)q^i+(q-1)q^{i-1}+\cdots+(q-2)$ and $n_{i+1}<q-1$, then

$n+2=n_kq^k+\cdots+(n_{i+1}+1)q^{i+1}+0.q^i+0.q^{i-1}+\cdots+0$. Thus

$$\gamma_q(n)=\frac{q-2}{q}+\cdots+\frac{q-1}{q^{i+1}}+\frac{n_{i+1}}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}},$$

$\gamma_q(n+2)=\frac{n_{i+1}+1}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}}$, and we have

$$\gamma_q(n+2)-\gamma_q(n)=\frac{1}{q^{i+2}}+\frac{1}{q}-\frac{q-1}{q}\left(1+\frac{1}{q}+\cdots+\frac{1}{q^i}\right)=\frac{1}{q^{i+2}}+\frac{1}{q}-1+\frac{1}{q^{i+1}}.$$

Furthermore

$1-\frac{1}{q^{i+1}}-\frac{1}{q}=\frac{q-2}{q}+\cdots+\frac{q-1}{q^{i+1}}\le\gamma_q(n)$ and

$$\gamma_q(n)\le\frac{q-2}{q}+\cdots+\frac{q-1}{q^{i+1}}+\frac{q-2}{q^{i+2}}+\frac{q-1}{q^{i+3}}+\cdots=1-\frac{1}{q^{i+2}}-\frac{1}{q}.$$

Thus in this case $(\gamma_q(n),\gamma_q(n+2))$ lies on the line-segment

$$\begin{aligned}&Z=X+\frac{1}{q}+\frac{1}{q^{i+1}}+\frac{1}{q^{i+2}}-1,\\&X\in\left[1-\frac{1}{q}-\frac{1}{q^{i+1}},1-\frac{1}{q}-\frac{1}{q^{i+2}}\right),\quad i=0,1,\cdots.\end{aligned} \tag{1.19}$$

Let 3^0. $n_0=q-1$.

Then

$n=n_kq^k+\cdots+n_{i+1}q^{i+1}+(q-1)q^i+(q-1)q^{i-1}+\cdots+(q-1)$ and $n_{i+1}<q-1$, then

$n+2=n_kq^k+\cdots+(n_{i+1}+1)q^{i+1}+0.q^i+0.q^{i-1}+\cdots+1$. Thus

$$\gamma_q(n)=\frac{q-1}{q}+\cdots+\frac{q-1}{q^{i+1}}+\frac{n_{i+1}}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}},$$

$\gamma_q(n+2)=\frac{1}{q}+\frac{n_{i+1}+1}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}}$, and we have

$$\gamma_q(n+2)-\gamma_q(n)=\frac{1}{q}+\frac{1}{q^{i+2}}-\frac{q-1}{q}\left(1+\frac{1}{q}+\cdots+\frac{1}{q^i}\right)=\frac{1}{q^{i+2}}+\frac{1}{q}-1+\frac{1}{q^{i+1}}.$$

Furthermore

$$1-\frac{1}{q^{i+1}}=\frac{q-1}{q}+\cdots+\frac{q-1}{q^{i+1}}\le\gamma_q(n)$$ and

$$\gamma_q(n)\le\frac{q-1}{q}+\cdots+\frac{q-1}{q^{i+1}}+\frac{q-2}{q^{i+2}}+\frac{q-1}{q^{i+3}}+\cdots \\ =1-\frac{1}{q^{i+2}}.$$

This gives

$$Z=X+\frac{1}{q}+\frac{1}{q^{i+1}}+\frac{1}{q^{i+2}}-1, \\ X\in\left[1-\frac{1}{q^{i+1}},1-\frac{1}{q^{i+2}}\right),\quad i=0,1,\cdots. \tag{1.20}$$

Summary, if the n satisfies 1^0, then $(\gamma_q(n),\gamma_q(n+1))$ is contained in the diagonal of

$$I_X\times I_Z:=\left[0,1-\frac{2}{q}\right]\times\left[\frac{2}{q},1\right] \tag{1.14}$$

for 2^0 in the diagonal of

$$I_X^{(i)}\times I_Z^{(i)}:=\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q}+\frac{1}{q^{i+1}},\frac{1}{q}+\frac{1}{q^i}\right],i=1,2,\cdots, \tag{1.15}$$

and for 3^0 in the diagonal of

$$J_X^{(k)}\times J_Z^{(k)}:=\left[1-\frac{1}{q}-\frac{1}{q^k},1-\frac{1}{q}-\frac{1}{q^{k+1}}\right]\times\left[\frac{1}{q^{k+1}},\frac{1}{q^k}\right],k=1,2,\cdots, \tag{1.16}$$

Proof. Express an integer n in the base q

$$n=n_kq^k+n_{k-1}q^{k-1}+\cdots+n_1q+n_0, \tag{1.17}$$

where $n_i<q$ and $n_k>0$. We consider three following cases:
1^0. $n_0<q-2$, 2^0. $n_0=q-2$, 3^0. $n_0=q-1$.
Let

1^0. $n_0<q-2$ Then $n=n_kq^k+\cdots+n_0$, $n+2=n_kq^k+\cdots+n_0+2$ and $\gamma_q(n+1)-\gamma_q(n)=\frac{2}{q}$. In this case $0\le\gamma_q(n)=\frac{n_0}{q}+\cdots+\frac{n_k}{q^{k+1}}<\frac{q-3}{q}+\frac{q-1}{q^2}+\cdots=\frac{q-2}{q}$, and thus such $(\gamma_q(n),\gamma_q(n+1))$ lies on the line-segment

$$Z=X+\frac{2}{q},\quad X\in\left[0,1-\frac{2}{q}\right). \tag{1.18}$$

Let
2^0. $n_0=q-2$. Then $n=n_kq^k+\cdots+n_{i+1}q^{i+1}+(q-1)q^i+(q-1)q^{i-1}+\cdots+(q-2)$ and $n_{i+1}<q-1$, then $n+2=n_kq^k+\cdots+(n_{i+1}+1)q^{i+1}+0.q^i+0.q^{i-1}+\cdots+0$. Thus $\gamma_q(n)=\frac{q-2}{q}+\cdots+\frac{q-1}{q^{i+1}}+\frac{n_{i+1}}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}}$, $\gamma_q(n+2)=\frac{n_{i+1}+1}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}}$, and we have

$$\gamma_q(n+2)-\gamma_q(n)=\frac{1}{q^{i+2}}+\frac{1}{q}-\frac{q-1}{q}\left(1+\frac{1}{q}+\cdots+\frac{1}{q^i}\right)=\frac{1}{q^{i+2}}+\frac{1}{q}-1+\frac{1}{q^{i+1}}.$$

Furthermore

$$1-\frac{1}{q^{i+1}}-\frac{1}{q}=\frac{q-2}{q}+\cdots+\frac{q-1}{q^{i+1}}\le\gamma_q(n) \text{ and } \gamma_q(n)\le\frac{q-2}{q}+\cdots+\frac{q-1}{q^{i+1}}+\frac{q-2}{q^{i+2}}+\frac{q-1}{q^{i+3}}+\cdots=1-\frac{1}{q^{i+2}}-\frac{1}{q}.$$

Thus in this case $(\gamma_q(n),\gamma_q(n+1))$ lies on the line-segment

$$Z=X+\frac{1}{q}+\frac{1}{q^{i+1}}+\frac{1}{q^{i+2}}-1,\quad X\in\left[1-\frac{1}{q}-\frac{1}{q^{i+1}},1-\frac{1}{q}-\frac{1}{q^{i+2}}\right),\quad i=0,1,\cdots. \tag{1.19}$$

Let

3^0. $n_0=q-1$. then $n=n_kq^k+\cdots+n_{i+1}q^{i+1}+(q-1)q^i+(q-1)q^{i-1}+\cdots+(q-1)$ and $n_{i+1}<q-1$, then $n+2=n_kq^k+\cdots+(n_{i+1}+1)q^{i+1}+0\cdot q^i+0\cdot q^{i-1}+\cdots+1$. Thus

$$\gamma_q(n)=\frac{q-1}{q}+\cdots+\frac{q-1}{q^{i+1}}+\frac{n_{i+1}}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}},$$

$\gamma_q(n+2)=\frac{1}{q}+\frac{n_{i+1}+1}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}}$, and we have

$$\gamma_q(n+2)-\gamma_q(n)=\frac{1}{q}+\frac{1}{q^{i+2}}-\frac{q-1}{q}\left(1+\frac{1}{q}+\cdots+\frac{1}{q^i}\right)=\frac{1}{q^{i+2}}+\frac{1}{q}-1+\frac{1}{q^{i+1}}.$$

Furthermore

$$1-\frac{1}{q^{i+1}}=\frac{q-1}{q}+\cdots+\frac{q-1}{q^{i+1}}\le\gamma_q(n) \text{ and } \gamma_q(n)\le\frac{q-1}{q}+\cdots+\frac{q-1}{q^{i+1}}+\frac{q-2}{q^{i+2}}+\frac{q-1}{q^{i+3}}+\cdots=1-\frac{1}{q^{i+2}}.$$

This gives

$$Z=X+\frac{1}{q}+\frac{1}{q^{i+1}}+\frac{1}{q^{i+2}}-1,\quad X\in\left[1-\frac{1}{q^{i+1}},1-\frac{1}{q^{i+2}}\right),\quad i=0,1,\cdots. \tag{1.20}$$

Summary, if the n satisfies 1^0, then $(\gamma_q(n),\gamma_q(n+1))$ is contained in the diagonal of

$$I_X\times I_Z:=\left[0,1-\frac{2}{q}\right]\times\left[\frac{2}{q},1\right] \quad (1.14),$$

for 2^0 in the diagonal of

$$I_X^{(i)}\times I_Z^{(i)}:=\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q}+\frac{1}{q^{i+1}},\frac{1}{q}+\frac{1}{q^i}\right],i=1,2,\cdots, \tag{1.15}$$

and for 3^0 in the diagonal of

$$J_X^{(k)}\times J_Z^{(k)}:=\left[1-\frac{1}{q}-\frac{1}{q^k},1-\frac{1}{q}-\frac{1}{q^{k+1}}\right]\times\left[\frac{1}{q^{k+1}},\frac{1}{q^k}\right],k=1,2,\cdots, \quad (1.16).$$

Composition of the maps (0.18), (0.19) and (0.20) of $Z(X)$ forms the second iteration T^2 of the von Neumann-Kakutani transformation $T(X)$. The diagonals of (1.14), (1.16) and (1.15) yield the following graph of T^2 in **Figure 2**.

Here the interval $\left[1-\frac{2}{q},1-\frac{1}{q}\right)$ on X-axis is decomposed in $\left[1-\frac{1}{q}-\frac{1}{q^k},1-\frac{1}{q}-\frac{1}{q^{k+1}}\right]$, $k=1,2,\cdots$, and the interval $\left[1-\frac{1}{q},1\right)$ is decomposed in $\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]$, $i=1,2,\cdots$. On Z-axis the interval $\left[0,\frac{1}{q}\right)$ is decomposed in $\left[\frac{1}{q^{k+1}},\frac{1}{q^k}\right]$, $k=1,2,\cdots$ and the interval $\left[\frac{1}{q},\frac{2}{q}\right)$ is decomposed in $\left[\frac{1}{q}+\frac{1}{q^{i+1}},\frac{1}{q}+\frac{1}{q^i}\right]$,

$i = 1, 2, \cdots$. Note that for $q = 2$, the interval $\left[0, 1-\frac{2}{q}\right] \times \left[\frac{2}{q}, 1\right]$ has a zero length and is missing.

Exchange for a moment the axis Z by Y. Similarly as in (1.11), we have that the a.d.f. $g(x, y)$ of the sequence $(\gamma_q(n), \gamma_q(n+2))$ is

$$\begin{aligned} g(x, y) &= \min\left(\left|[0, x] \cap I_X\right|, \left|[0, y] \cap I_Y\right|\right) + \sum_{i=1}^{\infty} \min\left(\left|[0, x] \cap I_X^{(i)}\right|, \left|[0, y] \cap I_Y^{(i)}\right|\right) \\ &\quad + \sum_{k=1}^{\infty} \min\left(\left|[0, x] \cap J_X^{(k)}\right|, \left|[0, y] \cap J_Y^{(k)}\right|\right). \end{aligned} \tag{1.21}$$

Decompose $[0,1]^2$ as the following figure shows:

Then by **Figure 3** we have

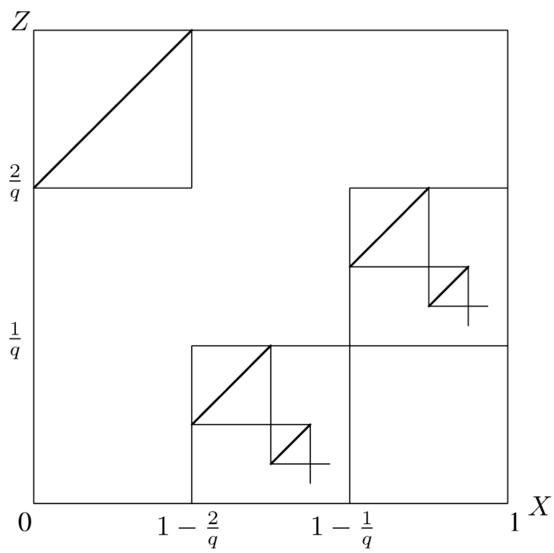

Figure 2. Straight lines containing $(\gamma_q(n), \gamma_q(n+2))$, $n = 0, 1, 2, \cdots$.

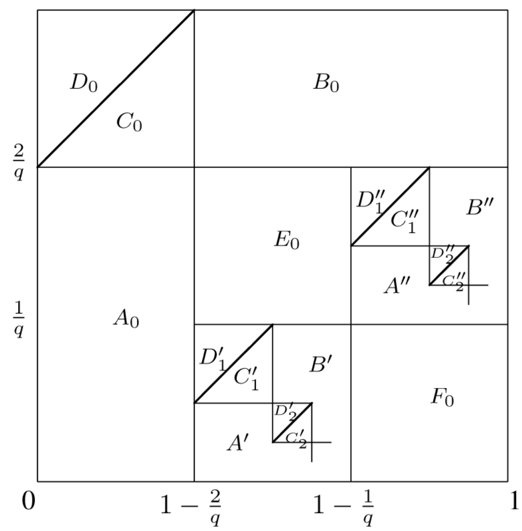

Figure 3. Decomposition of the Vunit square to parts with fixed expression of $g(x, y)$.

$$g(x,y)=\begin{cases} x & \text{if } (x,y)\in D_0, \\ y-\dfrac{2}{q} & \text{if } (x,y)\in C_0, \\ 0 & \text{if } (x,y)\in A_0, \\ y+x-1 & \text{if } (x,y)\in B_0, \\ x-1+\dfrac{2}{q} & \text{if } (x,y)\in E_0, \\ y & \text{if } (x,y)\in F_0, \\ 0 & \text{if } (x,y)\in A', \\ x+y-1+\dfrac{1}{q} & \text{if } (x,y)\in B', \\ x-1+\dfrac{1}{q}+\dfrac{1}{q^i} & \text{if } (x,y)\in D_i', \\ y-\dfrac{1}{q^{i+1}} & \text{if } (x,y)\in C_i', \\ \dfrac{1}{q} & \text{if } (x,y)\in A'', \\ x+y-1 & \text{if } (x,y)\in B'', \\ x-1+\dfrac{1}{q}+\dfrac{1}{q^i} & \text{if } (x,y)\in D_i'', \\ y-\dfrac{1}{q^{i+1}} & \text{if } (x,y)\in C_i''. \end{cases} \tag{1.22}$$

Let $q=2$.
In this case we find a.d.f $g(x,y)$ from (1.22) omitting $|D_0|=|C_0|=|A_0|=|B_0|=0$.
In Part 7. Applications we need to find $g(x,x)$ from $g(x,y)$ in (1.22):
For $q=2$

$$g(x,x)=\begin{cases} 0 & \text{if } x\in\left[0,\dfrac{1}{4}\right], \\ 2x-\dfrac{1}{2} & \text{if } x\in\left[\dfrac{1}{4},\dfrac{1}{2}\right], \\ \dfrac{1}{2} & \text{if } x\in\left[\dfrac{1}{2},\dfrac{3}{4}\right], \\ 2x-1 & \text{if } x\in\left[\dfrac{3}{4},1\right]. \end{cases} \tag{1.23}$$

For $q=3$

$$g(x,x)=\begin{cases} 0 & \text{if } x\in\left[0,\dfrac{1}{3}\right], \\ x-\dfrac{1}{3} & \text{if } x\in\left[\dfrac{1}{3},\dfrac{2}{3}\right], \\ 2x-1 & \text{if } x\in\left[\dfrac{2}{3},1\right]. \end{cases} \tag{1.24}$$

For $q\ge 4$

$$g(x,x)=\begin{cases}0 & \text{if } x\in\left[0,\frac{2}{q}\right],\\ x-\frac{2}{q} & \text{if } x\in\left[\frac{2}{q},1-\frac{2}{q}\right],\\ 2x-1 & \text{if } x\in\left[1-\frac{2}{q},1\right].\end{cases} \tag{1.25}$$

Note that for $q=4$ the term $x-2/q$ is omitted.

5. a.d.f. of $(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)), n=1,2,\cdots$

Let $q\geq 3$ be an integer.

Lemma 3 *Every point* $(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2))$ *is contained in diagonals of the intervals*

$$I=\left[0,1-\frac{2}{q}\right]\times\left[\frac{1}{q},1-\frac{1}{q}\right]\times\left[\frac{2}{q},1\right], \tag{1.26}$$

$$I^{(i)}=\left[1-\frac{1}{q^i},1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q^{i+1}},\frac{1}{q^i}\right]\times\left[\frac{1}{q}+\frac{1}{q^{i+1}},\frac{1}{q}+\frac{1}{q^i}\right], i=1,2,\cdots, \tag{1.27}$$

$$J^{(k)}=\left[1-\frac{1}{q}-\frac{1}{q^k},1-\frac{1}{q}-\frac{1}{q^{k+1}}\right]\times\left[1-\frac{1}{q^k},1-\frac{1}{q^{k+1}}\right]\times\left[\frac{1}{q^{k+1}},\frac{1}{q^k}\right],$$
$$k=1,2,\cdots, \tag{1.28}$$

where $|I|=0$ if $q=2$. These intervals are maximal with respect to inclusion.

Proof. Every maximal 3-dimensional interval I containing points $(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2))$ will be written as $I=I_X\times I_Y\times I_Z$, where I_X,I_Y,I_Z are projections of I to the X,Y,Z, axes, respectively. Moreover if $\gamma_q(n)\in I_X$ then $\gamma_q(n+1)\in I_Y$ and $\gamma_q(n+2)\in I_Z$. From u.d. of $\gamma_q(n)$ follows that the lengths $|I_X|=|I_Y|=|I_Z|$. Combining intervals (1.5), (1.14), (1.15), (1.16), (1.6) of equal lengths by following **Figure 3**.

We find (1.26), (1.27), and (1.28).

Now, let T be the union of diagonals of (1.27), (1.28) and (1.26). Again, as in (1.11), the a.d.f. $g(x,y,z)$ has[2] the form

$$g(x,y,z)=\left|\text{Project}_X\left([0,x]\times[0,y]\times[0,z]\cap T\right)\right| \tag{1.29}$$

and it can be rewritten as

$$\begin{aligned}g(x,y,z)&=\min\left(\left|[0,x]\cap I_X\right|,\left|[0,y]\cap I_Y\right|,\left|[0,z]\cap I_Z\right|\right)\\&+\sum_{i=1}^{\infty}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|,\left|[0,z]\cap I_Z^{(i)}\right|\right)\\&+\sum_{k=1}^{\infty}\min\left(\left|[0,x]\cap J_X^{(k)}\right|,\left|[0,y]\cap J_Y^{(k)}\right|,\left|[0,z]\cap J_Z^{(k)}\right|\right).\end{aligned} \tag{1.30}$$

To calculate minimums in (1.30) we can use the following **Figure 4** (here $q=3$):

As an example of application of (1.30) and **Figure 4**, we compute $g(x,x,x)$ for $q\geq 3$ without using the knowledge of $g(x,y,z)$,[3]

$$g(x,x,x)=\begin{cases}0 & \text{if } x\in\left[0,\frac{2}{q}\right],\\ x-\frac{2}{q} & \text{if } x\in\left[\frac{2}{q},1-\frac{1}{q}\right],\\ 3x-2 & \text{if } x\in\left[1-\frac{1}{q},1\right].\end{cases} \tag{1.31}$$

Proof.

1. Let $x \in \left[0, \frac{1}{q}\right]$.

Then $\left|[0,x] \cap I_Z\right| = 0$, $\left|[0,x] \cap I_Z^{(i)}\right| = 0$, $\left|[0,x] \cap I_Y^{(k)}\right| = 0$, consequently $g(x,x,x) = 0$.

2. Let $x \in \left[\frac{1}{q}, \frac{2}{q}\right]$. Then $\left|[0,x] \cap I_Z\right| = 0$, $\left|[0,x] \cap I_Y^{(k)}\right| = 0$, $\left|[0,x] \cap I_Z^{(i)}\right| = 0$, consequently $g(x,x,x) = 0$.

3. Let $x \in \left[\frac{2}{q}, 1 - \frac{1}{q}\right]$. Then $\left|[0,x] \cap I_X^{(i)}\right| = 0$, $\left|[0,x] \cap I_Y^{(k)}\right| = 0$, consequently

$$g(x,x,x) = \min\left(1 - \frac{2}{q}, x - \frac{1}{q}, x - \frac{2}{q}\right) = x - \frac{2}{q}.$$

4. Let $x \in \left[1 - \frac{1}{q}, 1\right]$.

Specify $x \in I_X^{(k_1)}$, $x \in J_Y^{(k_1)}$. Then $\left|[0,x] \cap I_X^{(k)}\right| = 0$, $\left|[0,x] \cap J_Y^{(k)}\right| = 0$ for $k > k_1$. Thus (1.30) implies

$$\begin{aligned} g(x,x,x) &= \min\left(1 - \frac{2}{q}, 1 - \frac{1}{q} - \frac{1}{q}, x - \frac{2}{q}\right) + \sum_{i=1}^{k_1} \min\left(\left|[0,x] \cap I_X^{(i)}\right|, \left|[0,y] \cap I_Y^{(i)}\right|, \left|[0,z] \cap I_Z^{(i)}\right|\right) \\ &\quad + \sum_{k=1}^{k_1} \min\left(\left|[0,x] \cap J_X^{(k)}\right|, \left|[0,y] \cap J_Y^{(k)}\right|, \left|[0,z] \cap J_Z^{(k)}\right|\right) \\ &= x - \frac{2}{q} + \sum_{i=1}^{k_1-1}\left(\frac{1}{q^i} - \frac{1}{q^{i+1}}\right) + x - 1 + \frac{1}{q^{k_1}} + \sum_{k=1}^{k_1-1}\left(\frac{1}{q^k} - \frac{1}{q^{k+1}}\right) + x - 1 + \frac{1}{q^{k_1}} \\ &= 3x - 2. \end{aligned}$$

For $q = 2$ we have

$$g(x,x,x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right], \\ x - \frac{1}{2} & \text{if } x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ 3x - 2 & \text{if } x \in \left[\frac{3}{4}, 1\right]. \end{cases} \tag{1.32}$$

6. Explicit Form of $g(x,y,z)$

Let $q \geq 3$ be an integer.

Motivated by the **Figure 4** we decompose the unit interval $[0,1]$ on X, Y and Z axes in the **Figure 5**

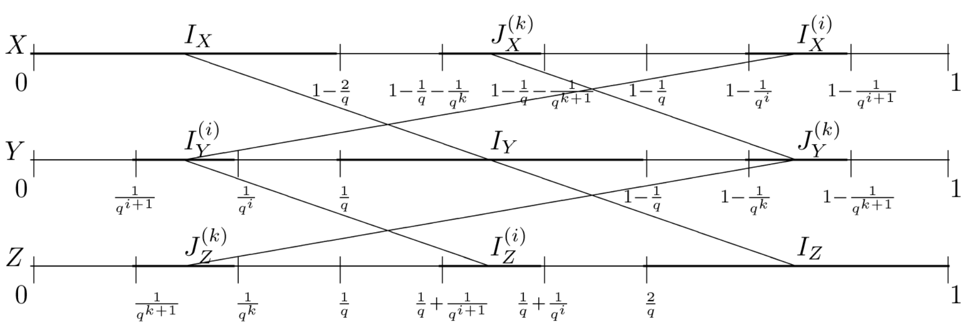

Figure 4. Projections of intervals $I, I^{(i)}, J^{(k)}$ on axes X, Y, Z.

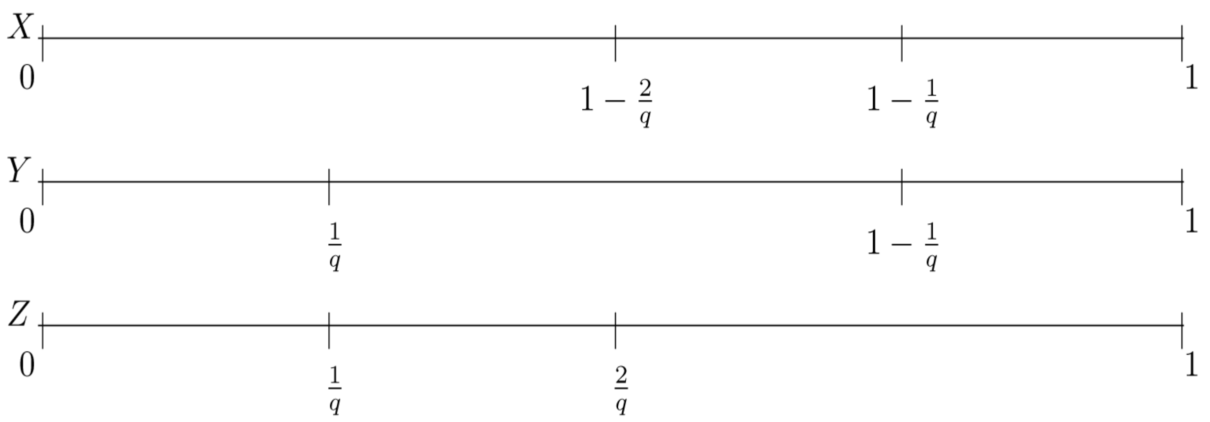

Figure 5. Divisions of the unit intervals.

intervals (here $q = 4$):

In this decomposition, for $(x, y, z) \in [0,1]^3$, we have 27 possibilities. We shall order choices of (x, y, z) from the left to the right. Detailed proofs are included only in non-trivial cases.

1. Let

$x \in \left[0, 1-\frac{2}{q}\right], y \in \left[0, \frac{1}{q}\right], z \in \left[0, \frac{1}{q}\right]$. Then

$$g(x, y, z) = 0 .$$

Proof. We have $\left|[0, x] \cap I_X^{(i)}\right| = 0$, $\left|[0, x] \cap J_X^{(k)}\right| = 0$, $i, k = 1, 2, \cdots$. Then, by (1.30),

$g(x, y, z) = \min\left(\left|[0, x] \cap I_X\right|, \left|[0, y] \cap I_Y\right|, \left|[0, z] \cap I_Z\right|\right)$.

Similarly, in the following cases 2-9.

2. Let $x \in \left[0, 1-\frac{2}{q}\right], y \in \left[0, \frac{1}{q}\right], z \in \left[0, \frac{2}{q}\right]$. Then

$$g(x, y, z) = 0 .$$

3. Let $x \in \left[0, 1-\frac{2}{q}\right], y \in \left[0, \frac{1}{q}\right], z \in \left[\frac{2}{q}, 1\right]$. Then

$$g(x, y, z) = 0 .$$

4. Let $x \in \left[0, 1-\frac{2}{q}\right], y \in \left[\frac{1}{q}, 1-\frac{1}{q}\right], z \in \left[0, \frac{1}{q}\right]$. Then

$$g(x, y, z) = 0 .$$

5. Let $x \in \left[0, 1-\frac{2}{q}\right], y \in \left[\frac{1}{q}, 1-\frac{1}{q}\right], z \in \left[\frac{1}{q}, \frac{2}{q}\right]$. Then

$$g(x, y, z) = 0 .$$

6. Let $x \in \left[0, 1-\frac{2}{q}\right], y \in \left[\frac{1}{q}, 1-\frac{1}{q}\right], z \in \left[\frac{2}{q}, 1\right]$. Then

$$g(x, y, z) = \min(x, y - \frac{1}{q}, z - \frac{2}{q}) .$$

7. Let $x\in\left[0,1-\frac{2}{q}\right], y\in\left[\frac{1}{q},1-\frac{1}{q}\right], z\in\left[0,\frac{1}{q}\right]$. Then

$$g(x,y,z)=0.$$

8. Let $x\in\left[0,1-\frac{2}{q}\right], y\in\left[\frac{1}{q},1-\frac{1}{q}\right], z\in\left[\frac{1}{q},\frac{2}{q}\right]$. Then

$$g(x,y,z)=0.$$

9. Let $x\in\left[0,1-\frac{2}{q}\right], y\in\left[1-\frac{1}{q},1\right], z\in\left[\frac{2}{q},1\right]$. Then

$$g(x,y,z)=\min\left(x,z-\frac{2}{q}\right).$$

Proof. We use $z-\frac{2}{q}\le 1-\frac{2}{q}$.

10. Let $x\in\left[1-\frac{2}{q},1-\frac{1}{q}\right], y\in\left[0,\frac{1}{q}\right], z\in\left[0,\frac{1}{q}\right]$. Then

$$g(x,y,z)=0.$$

Proof. We use $\left|[0,x]\cap I_X^{(i)}\right|=0$, $\left|[0,y]\cap J_Y^{(k)}\right|=0$, $\left|[0,z]\cap I_Z\right|=0$. Similarly,

11. Let $x\in\left[1-\frac{2}{q},1-\frac{1}{q}\right], y\in\left[0,\frac{1}{q}\right], z\in\left[\frac{1}{q},\frac{2}{q}\right]$. Then

$$g(x,y,z)=0.$$

12. Let $x\in\left[1-\frac{2}{q},1-\frac{1}{q}\right], y\in\left[0,\frac{1}{q}\right], z\in\left[\frac{2}{q},1\right]$. Then

$$g(x,y,z)=0.$$

Proof. We use $\left|[0,x]\cap I_X^{(i)}\right|=0$, $\left|[0,y]\cap J_Y^{(k)}\right|=0$, $\left|[0,y]\cap J_Y\right|=0$.

13. Let $x\in\left[1-\frac{2}{q},1-\frac{1}{q}\right], y\in\left[\frac{1}{q},1-\frac{1}{q}\right], z\in\left[0,\frac{1}{q}\right]$. Then

$$g(x,y,z)=0.$$

14. Let $x\in\left[1-\frac{2}{q},1-\frac{1}{q}\right], y\in\left[\frac{1}{q},1-\frac{1}{q}\right], z\in\left[\frac{1}{q},\frac{2}{q}\right]$. Then

$$g(x,y,z)=0.$$

15. Let $x\in\left[1-\frac{2}{q},1-\frac{1}{q}\right], y\in\left[\frac{1}{q},1-\frac{1}{q}\right], z\in\left[\frac{2}{q},1\right]$. Then

$$g(x,y,z)=\min\left(y-\frac{1}{q},z-\frac{2}{q}\right).$$

16. Let $x\in\left[1-\frac{2}{q},1-\frac{1}{q}\right], y\in\left[1-\frac{1}{q},1\right], z\in\left[0,\frac{1}{q}\right]$.

Specify $x\in J_X^{(k_1)}$, $y\in J_Y^{(k_2)}$, $z\in J_Z^{(k_3)}$. Then

$$g(x,y,z)=\begin{cases}0 & \text{if } \min(k_1,k_2)<k_3,\\ \min\left(x-1+\frac{1}{q}+\frac{1}{q^{k_3}},y-1+\frac{1}{q^{k_3}},z-\frac{1}{q^{k_3+1}}\right) & \text{if } k_3=k_1=k_2,\\ \min\left(x-1+\frac{1}{q}+\frac{1}{q^{k_3}},z-\frac{1}{q^{k_3+1}}\right) & \text{if } k_3=k_1<k_2,\\ \min\left(y-1+\frac{1}{q^{k_3}},z-\frac{1}{q^{k_3+1}}\right) & \text{if } k_3=k_2<k_1,\\ z+x-1+\frac{1}{q} & \text{if } k_3<k_1<k_2,\\ z+\min\left(x-1+\frac{1}{q},y-1\right) & \text{if } k_3<k_1=k_2,\\ z+y-1 & \text{if } k_3<k_2<k_1.\end{cases}$$

Proof. First observe that $\left|[0,z]\cap I_Z\right|=0$, and $\left|[0,x]\cap I_X^{(i)}\right|=0$. Thus $\min\left(\left|[0,x]\cap I_X\right|,\left|[0,y]\cap I_Y\right|,\left|[0,z]\cap I_Z\right|\right)=0$ and $\sum_{i=1}^{\infty}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|,\left|[0,z]\cap I_Z^{(i)}\right|\right)=0.$ Further, for $k=1,2,\cdots$, we have

$$\left|[0,x]\cap J_X^{(k)}\right|=\begin{cases}0 & \text{if } k>k_1,\\ x-\left(1-\frac{1}{q}-\frac{1}{q^{k_1}}\right) & \text{if } k=k_1,\\ \frac{1}{q^k}-\frac{1}{q^{k+1}} & \text{if } k<k_1;\end{cases}$$

$$\left|[0,y]\cap J_Y^{(k)}\right|=\begin{cases}0 & \text{if } k>k_2,\\ x-\left(1-\frac{1}{q^{k_2}}\right) & \text{if } k=k_2,\\ \frac{1}{q^k}-\frac{1}{q^{k+1}} & \text{if } k<k_2;\end{cases}$$

$$\left|[0,z]\cap J_Z^{(k)}\right|=\begin{cases}0 & \text{if } k>k_3,\\ z-\frac{1}{q^{k_3+1}} & \text{if } k=k_3,\\ \frac{1}{q^k}-\frac{1}{q^{k+1}} & \text{if } k<k_3;\end{cases}$$

Thus, using (1.30), we find

$$g(x,y,z)=\sum_{k=k_3}^{\min(k_1,k_2)}\min\left(\left|[0,x]\cap J_X^{(k)}\right|,\left|[0,y]\cap J_Y^{(k)}\right|,\left|[0,z]\cap J_Z^{(k)}\right|\right).$$

17. Let $x\in\left[1-\frac{2}{q},1-\frac{1}{q}\right], y\in\left[1-\frac{1}{q},1\right], z\in\left[\frac{1}{q},\frac{2}{q}\right]$.

Specify $x\in J_X^{(k_1)}$, $y\in J_Y^{(k_2)}$. Then

$$g(x,y,z)=\begin{cases}\min\left(x-1+\frac{2}{q},y-1+\frac{1}{q}\right) & \text{if } k_1=k_2,\\ x-1+\frac{2}{q} & \text{if } k_1<k_2,\\ y-1+\frac{1}{q} & \text{if } k_1>k_2.\end{cases}$$

Proof. We have $\left|[0,x]\cap I_X^{(i)}\right|$, $\left|[0,z]\cap I_Z\right|$, $\left|[0,z]\cap J_Z^{(k)}\right|=\frac{1}{q^k}-\frac{1}{q^{k+1}}$, then

$$g(x,y,z)=\sum\nolimits_{k\le\min(k_1,k_2)}\min\left(\left|[0,x]\cap J_X^{(k)}\right|,\left|[0,y]\cap J_Y^{(k)}\right|\right).$$

18. Let $x\in\left[1-\frac{2}{q},1-\frac{1}{q}\right], y\in\left[1-\frac{1}{q},1\right], z\in\left[\frac{2}{q},1\right]$.

Specify $x\in J_X^{(k_1)}$, $y\in J_Y^{(k_2)}$. Then

$$g(x,y,z)=\begin{cases}\min\left(x-1+\frac{2}{q},y-1+\frac{1}{q}\right)+z-\frac{2}{q} & \text{if } k_1=k_2,\\ x+z-1 & \text{if } k_1<k_2,\\ y+z-1-\frac{1}{q} & \text{if } k_1>k_2.\end{cases}$$

Proof. We have

$$g(x,y,z)=\min\left(1-\frac{2}{q},1-\frac{2}{q},z-\frac{2}{q}\right)+\sum\nolimits_{k\le\min(k_1,k_2)}\min\left(\left|[0,x]\cap J_X^{(k)}\right|,\left|[0,y]\cap J_Y^{(k)}\right|,\left|[0,z]\cap J_Z^{(k)}\right|\right).$$

19. Let $x\in\left[1-\frac{1}{q},1\right], y\in\left[0,\frac{1}{q}\right], z\in\left[0,\frac{1}{q}\right]$. Then

$$g(x,y,z)=0.$$

20. Let $x\in\left[1-\frac{1}{q},1\right], y\in\left[0,\frac{1}{q}\right], z\in\left[\frac{1}{q},\frac{2}{q}\right]$.

Specify $x\in I_X^{(i_1)}$, $y\in I_Y^{(i_2)}$, $z\in I_Z^{(i_3)}$. Then we have

$$g(x,y,z)=\begin{cases}0 & \text{if } i_1<\max(i_2,i_3),\\ \min\left(x-1+\frac{1}{q^{i_1}},y-\frac{1}{q^{i_1+1}},z-\frac{1}{q}-\frac{1}{q^{i_1+1}}\right) & \text{if } i_3=i_2=i_1,\\ \min\left(y,z-\frac{1}{q}\right)+x-1 & \text{if } i_3=i_2<i_1,\\ \min\left(x-1+\frac{1}{q^{i_1}},y-\frac{1}{q^{i_1+1}}\right) & \text{if } i_3<i_2=i_1,\\ \min\left(x-1+\frac{1}{q^{i_1}},z-\frac{1}{q}-\frac{1}{q^{i_1+1}}\right) & \text{if } i_2<i_3=i_1,\\ x+z-1-\frac{1}{q} & \text{if } i_2<i_3<i_1,\\ x+y-1 & \text{if } i_3<i_2<i_1.\end{cases}$$

Proof. We have $\left|[0,y]\cap J_Y^{(k)}\right|$, $\left|[0,z]\cap I_Z\right|=0$ and

$$g(x,y,z)=\sum_{i=\max(i_2,i_3)}^{i_1}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|,\left|[0,z]\cap I_Z^{(i)}\right|\right).$$

21. Let $x\in\left[1-\frac{1}{q},1\right], y\in\left[0,\frac{1}{q}\right], z\in\left[\frac{2}{q},1\right]$.

Specify $x\in I_X^{(i_1)}$, $y\in I_Y^{(i_2)}$. Then

$$g(x,y,z)=\begin{cases}0 & \text{if } i_1<i_2,\\ \min\left(x-1+\frac{1}{q^{i_1}}, y-\frac{1}{q^{i_1+1}}\right) & \text{if } i_1=i_2,\\ x+y-1 & \text{if } i_2<i_1.\end{cases}$$

Proof. $\left|[0,y]\cap I_Y\right|=0$, $\left|[0,y]\cap J_Y^{(k)}\right|=0$, $\left|[0,z]\cap I_Z^{(i)}\right|=\frac{1}{q^i}-\frac{1}{q^{i+1}}$,

$$g(x,y,z)=\sum_{i_2\le i\le i_1}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|\right).$$

22. Let $x\in\left[1-\frac{1}{q},1\right], y\in\left[\frac{1}{q},1-\frac{1}{q}\right], z\in\left[0,\frac{1}{q}\right]$. Then

$$g(x,y,z)=0.$$

23. Let $x\in\left[1-\frac{1}{q},1\right], y\in\left[\frac{1}{q},1-\frac{1}{q}\right], z\in\left[\frac{1}{q},\frac{2}{q}\right]$.

Specify $x\in I_X^{(i_1)}$, $z\in I_Z^{(i_3)}$. Then

$$g(x,y,z)=\begin{cases}0 & \text{if } i_1<i_2,\\ \min\left(x-1+\frac{1}{q^{i_1}}, z-\frac{1}{q}-\frac{1}{q^{i_1+1}}\right) & \text{if } i_1=i_2,\\ y+z-1-\frac{1}{q} & \text{if } i_2<i_1.\end{cases}$$

Proof. $\left|[0,z]\cap I_Z\right|=0$, $\left|[0,y]\cap J_Y^{(k)}\right|=0$, $\left|[0,y]\cap I_Y^{(i)}\right|=\frac{1}{q^i}-\frac{1}{q^{i+1}}$,

$$g(x,y,z)=\sum_{i_2\le i\le i_1}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|,\left|[0,z]\cap I_Z^{(i)}\right|\right).$$

24. Let $x\in\left[1-\frac{1}{q},1\right], y\in\left[\frac{1}{q},1-\frac{1}{q}\right], z\in\left[\frac{2}{q},1\right]$. Then

$$g(x,y,z)=\min\left(y-\frac{1}{q}, z-\frac{2}{q}\right)+x-1+\frac{1}{q}.$$

Proof. We have $\left|[0,y]\cap J_Y^{(k)}\right|=0$. Specify $x\in I_X^{(i_1)}$. Then

$$\begin{aligned}g(x,y,z)=&\min\left(\left|[0,x]\cap I_X\right|,\left|[0,y]\cap I_Y\right|,\left|[0,z]\cap I_Z\right|\right)\\ &+\sum_{i\le i_1}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|,\left|[0,z]\cap I_Z^{(i)}\right|\right).\end{aligned}$$

The first term is $\min\left(y-\frac{1}{q}, z-\frac{2}{q}\right)$ and the second is $x-1+\frac{1}{q}$.

25. Let $x \in \left[1-\frac{1}{q},1\right], y \in \left[1-\frac{1}{q},1\right], z \in \left[0,\frac{1}{q}\right]$.

Specify $y \in J_Y^{(k_1)}$, $z \in J_Z^{(k_2)}$. Then

$$g(x,y,z)=\begin{cases} 0 & \text{if } k_1 < k_2, \\ \min\left(y-1+\frac{1}{q^{k_1}}, z-\frac{1}{q^{k_1+1}}\right) & \text{if } k_1 = k_2, \\ y+z-1 & \text{if } k_2 < k_1. \end{cases}$$

Proof.

$$g(x,y,z)=\sum_{k_2 \le k \le k_1} \min\left(\left|[0,x]\cap J_X^{(k)}\right|, \left|[0,y]\cap J_Y^{(k)}\right|, \left|[0,z]\cap J_Z^{(k)}\right|\right).$$

26. Let $x \in \left[1-\frac{1}{q},1\right], y \in \left[1-\frac{1}{q},1\right], z \in \left[\frac{1}{q},\frac{2}{q}\right]$.

Specify $x \in I_X^{(i_1)}$, $y \in J_Y^{(k_1)}$ and $z \in J_Z^{(k_2)}$. Then

$$g(x,y,z)=\begin{cases} y-1+\frac{1}{q} & \text{if } i_1 < i_2, \\ y-1+\frac{1}{q}+\min\left(x-1+\frac{1}{q^{i_1}}, z-\frac{1}{q}-\frac{1}{q^{i_1+1}}\right) & \text{if } i_1 = i_2, \\ x+y+z-2 & \text{if } i_2 < i_1. \end{cases}$$

Proof. We have $\left|[0,z]\cap I_Z\right|=0$, $\left|[0,x]\cap J_X^{(k)}\right|=\left|J_X^{(k)}\right|$, $\left|[0,z]\cap J_Z^{(k)}\right|=\left|J_Z^{(k)}\right|$, $\left|[0,y]\cap I_Y^{(i)}\right|=\left|I_Y^{(i)}\right|$.

Moreover

$$\left|[0,x]\cap I_X^{(i)}\right|=\begin{cases} 0 & \text{if } i > i_1, \\ x-1+\frac{1}{q^i} & \text{if } i = i_1, \\ \frac{1}{q^i}-\frac{1}{q^{i+1}} & \text{if } i < i_1, \end{cases}$$

$$\left|[0,z]\cap I_Z^{(i)}\right|=\begin{cases} 0 & \text{if } i < i_2, \\ z-\frac{1}{q}-\frac{1}{q^{i+1}} & \text{if } i = i_2, \\ \frac{1}{q^i}-\frac{1}{q^{i+1}} & \text{if } i > i_2, \end{cases}$$

$$\left|[0,y]\cap J_Y^{(k)}\right|=\begin{cases} 0 & \text{if } k > k_1, \\ y-1+\frac{1}{q^k} & \text{if } k = k_1, \\ \frac{1}{q^k}-\frac{1}{q^{k+1}} & \text{if } k < k_1 \end{cases}$$

which gives

$$\begin{aligned} g(x,y,z) &= 0+\sum_{i_2\le i\le i_1}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|,\left|[0,z]\cap I_Z^{(i)}\right|\right) \\ &+\sum_{k\le k_1}\min\left(\left|[0,x]\cap J_X^{(k)}\right|,\left|[0,y]\cap J_Y^{(k)}\right|,\left|[0,z]\cap J_Z^{(k)}\right|\right) \\ &= \sum_{i_2\le i\le i_1}\left(\frac{1}{q^i}-\frac{1}{q^{i+1}}\right)+x-1+\frac{1}{q^{i_1}}+z-\frac{1}{q}-\frac{1}{q^{i_2+1}} \\ &+\sum_{k<k_1}\left(\frac{1}{q^k}-\frac{1}{q^{k+1}}\right)+y-1+\frac{1}{q^{k_1}}=x+y+z-2 \end{aligned}.$$

The final equation holds if $i_1-i_2>1$ and $k_1>1$. It can be seen that it holds also for $i_1=i_2+1$ and $k_1=1$. For $i_1=i_2$ and $i_1<i_2$ we need to compute this sum separately.

27. Let $x\in\left[1-\frac{1}{q},1\right], y\in\left[1-\frac{1}{q},1\right], z\in\left[\frac{2}{q},1\right]$. Then

$$g(x,y,z)=x+y+z-2.$$

Proof. First observe

$\left|[0,x]\cap I_X\right|=1-\frac{2}{q}$, $\left|[0,y]\cap I_Y\right|=1-\frac{2}{q}$, $\left|[0,z]\cap I_Z\right|=1-\frac{2}{q}$. Thus

$$\min\left(\left|[0,x]\cap I_X\right|,\left|[0,y]\cap I_Y\right|,\left|[0,z]\cap I_Z\right|\right)=z-\frac{2}{q}.$$

New specify $x\in I_X^{(i_1)}$ and $y\in J_Y^{(k_1)}$. Then we have

$$\left|[0,x]\cap I_X^{(i)}\right|=\begin{cases}0 & \text{if } i>i_1,\\ x-1+\frac{1}{q^i} & \text{if } i=i_1,\\ \frac{1}{q^i}-\frac{1}{q^{i+1}} & \text{if } i<i_1,\end{cases}$$

$$\left|[0,y]\cap J_Y^{(k)}\right|=\begin{cases}0 & \text{if } k>k_1,\\ y-1+\frac{1}{q^k} & \text{if } k=k_1,\\ \frac{1}{q^k}-\frac{1}{q^{k+1}} & \text{if } k<k_1,\end{cases}$$

and $\left|[0,z]\cap I_Z^{(i)}\right|=\frac{1}{q^i}-\frac{1}{q^{i+1}}$, $\left|[0,z]\cap J_Z^{(k)}\right|=\frac{1}{q^k}-\frac{1}{q^{k+1}}$, $\left|[0,y]\cap I_Y^{(i)}\right|=\frac{1}{q^i}-\frac{1}{q^{i+1}}$. Then for the sums in (1.30) we have

$$\begin{aligned} &\sum_{i=1}^{i_1}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|,\left|[0,z]\cap I_Z^{(i)}\right|\right) \\ &+\sum_{k=1}^{k_1}\min\left(\left|[0,x]\cap J_X^{(k)}\right|,\left|[0,y]\cap J_Y^{(k)}\right|,\left|[0,z]\cap J_Z^{(k)}\right|\right) \\ &=\frac{1}{q}-\frac{1}{q^{i_1}}+x-1+\frac{1}{q^{i_1}}+\frac{1}{q}-\frac{1}{q^{k_1}}+y-1+\frac{1}{q^{k_1}} \end{aligned}$$

which gives $g(x,y,z)=x+y+z-2$.

The above computation of $g(x,y,z)$ holds for $q\ge 3$.

Let $q = 2$.

We have $|I| = 0$ and

$$g(x,y,z) = \sum_{i=1}^{\infty}\min\left(\left|[0,x]\cap I_X^{(i)}\right|,\left|[0,y]\cap I_Y^{(i)}\right|,\left|[0,z]\cap I_Z^{(i)}\right|\right) + \sum_{k=1}^{\infty}\min\left(\left|[0,x]\cap J_X^{(k)}\right|,\left|[0,y]\cap J_Y^{(k)}\right|,\left|[0,z]\cap J_Z^{(k)}\right|\right). \quad (1.33)$$

Thus $g(x,y,z)$ for $q = 2$ directly follows from $g(x,y,z)$ for $q \geq 3$ if we use only such items in 1 - 27 for which $x \notin I_X$, $y \notin I_Y$, $z \notin I_Z$. These are 10, 11, 16, 17, 19, 20, 25, 26, *i.e.*, $2\times 2\times 2 = 8$.

The non-zero values of $g(x,y,z) \neq 0$ can also be seen in the following table.

x	**y**	**z**	$\boldsymbol{g(x,y,z)}$
$\left[0,1-\frac{2}{q}\right]$	$\left[\frac{1}{q},1-\frac{1}{q}\right]$	$\left[\frac{2}{q},1\right]$	$\min\left(x,y-\frac{1}{q},z-\frac{2}{q}\right)$
	$\left[1-\frac{1}{q},1\right]$	$\left[\frac{2}{q},1\right]$	$\min\left(x,z-\frac{2}{q}\right)$
$\left[1-\frac{1}{q}-\frac{1}{q^k},1-\frac{1}{q}-\frac{1}{q^{k+1}}\right]_{k=1,2,\cdots}$	$\left[\frac{1}{q},1-\frac{1}{q}\right]$	$\left[\frac{2}{q},1\right]$	$\min\left(y-\frac{1}{q},z-\frac{2}{q}\right)$
	$\left[1-\frac{1}{q^l},1-\frac{1}{q^{l+1}}\right]_{1\le l<k}$	$\left[\frac{1}{q^{l+1}},\frac{1}{q^l}\right]$	$\min\left(y-1+\frac{1}{q^l},z-\frac{1}{q^{l+1}}\right)$
		$\left[\frac{1}{q^l},\frac{1}{q}\right]$	$y+z-1$
		$\left[\frac{1}{q},\frac{2}{q}\right]$	$y-1+\frac{1}{q}$
		$\left[\frac{2}{q},1\right]$	$y+z-1-\frac{1}{q}$
	$\left[1-\frac{1}{q^k},1-\frac{1}{q^{k+1}}\right]$	$\left[\frac{1}{q^{k+1}},\frac{1}{q^k}\right]$	$\min\left(x-1+\frac{1}{q}+\frac{1}{q^k},y-1+\frac{1}{q^k},z-\frac{1}{q^{k+1}}\right)$
		$\left[\frac{1}{q^k},\frac{1}{q}\right]$	$\min\left(x+\frac{1}{q},y\right)-1+z$
		$\left[\frac{1}{q},\frac{2}{q}\right]$	$\min\left(x+\frac{1}{q},y\right)-1+\frac{1}{q}$
		$\left[\frac{2}{q},1\right]$	$\min\left(x+\frac{1}{q},y\right)-1+z-\frac{1}{q}$
	$\left[1-\frac{1}{q^{k+1}},1\right]$	$\left[\frac{1}{q^{k+1}},\frac{1}{q^k}\right]$	$\min\left(x-1+\frac{1}{q}+\frac{1}{q^k},z-\frac{1}{q^{k+1}}\right)$
		$\left[\frac{1}{q^k},\frac{1}{q}\right]$	$x+z-1+\frac{1}{q}$
		$\left[\frac{1}{q},\frac{2}{q}\right]$	$x-1+\frac{2}{q}$

Continued

x	y	z	$g(x,y,z)$
		$\left[\frac{2}{q},1\right]$	$x+z-1$
$\left[1-\frac{1}{q^k},1-\frac{1}{q^{k+1}}\right]_{k=1,2,\cdots}$	$\left[\frac{1}{q^{k+1}},\frac{1}{q^k}\right]$	$\left[\frac{1}{q}+\frac{1}{q^{k+1}},\frac{1}{q}+\frac{1}{q^k}\right]$	$\min\left(x-1+\frac{1}{q^k},y-\frac{1}{q^{k+1}},z-\frac{1}{q}-\frac{1}{q^{k+1}}\right)$
		$\left[\frac{1}{q}+\frac{1}{q^k},1\right]$	$\min\left(x-1+\frac{1}{q^k},y-\frac{1}{q^{k+1}}\right)$
	$\left[\frac{1}{q^k},\frac{1}{q}\right]$	$\left[\frac{1}{q}+\frac{1}{q^{k+1}},\frac{1}{q}+\frac{1}{q^k}\right]$	$\min\left(x-1+\frac{1}{q^k},z-\frac{1}{q}-\frac{1}{q^{k+1}}\right)$
		$\left[\frac{1}{q}+\frac{1}{q^k},\frac{2}{q}\right]$	$\min\left(y,z-\frac{1}{q}\right)+x-1$
		$\left[\frac{2}{q},1\right]$	$x+y-1$
	$\left[\frac{1}{q},1-\frac{1}{q}\right]$	$\left[\frac{1}{q}+\frac{1}{q^{k+1}},\frac{1}{q}+\frac{1}{q^k}\right]$	$\min\left(x-1+\frac{1}{q^k},z-\frac{1}{q}-\frac{1}{q^{k+1}}\right)$
		$\left[\frac{1}{q}+\frac{1}{q^k},\frac{2}{q}\right]$	$x+z-1-\frac{1}{q}$
		$\left[\frac{2}{q},1\right]$	$\min\left(y,z-\frac{1}{q}\right)+x-1$
	$\left[1-\frac{1}{q^l},1-\frac{1}{q^{l+1}}\right]_{l\geq 1}$	$\left[\frac{1}{q^{l+1}},\frac{1}{q^l}\right]$	$\min\left(y-1+\frac{1}{q^l},z-\frac{1}{q^{l+1}}\right)$
		$\left[\frac{1}{q^l},\frac{1}{q}\right]$	$y+z-1$
		$\left[\frac{1}{q},\frac{1}{q}+\frac{1}{q^{k+1}}\right]$	$y-1+\frac{1}{q}$
		$\left[\frac{1}{q}+\frac{1}{q^{k+1}},\frac{1}{q}+\frac{1}{q^k}\right]$	$\min\left(x-1+\frac{1}{q^k},z-\frac{1}{q}-\frac{1}{q^{k+1}}\right)+y-1+\frac{1}{q}$
		$\left[\frac{1}{q}+\frac{1}{q^k},1\right]$	$x+y+z-2$

In all other cases $g(x,y,z)=0$.

7. Applications

The knowledge of the a.d.f. $g(x,y,z)$ of the sequence $\left(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\right)$ $n=1,2,\cdots$ allows us to compute the following limit by the Weyl limit relation (1.1) in dimension $s=3$.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F\left(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\right)$$
$$=\int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_x g(x,y,z), \tag{1.34}$$

where $F(x,y,z)$ is an arbitrary continuous function defined in $[0,1]^3$. For computing (1.34) we use the following two methods.

7.1. Method I

In the first method in the Riemann-Stieltjes integral (1.34) we apply integration by parts.

Lemma 4 *Assume that* $F(x,y,z)$ *is a continuous in* $[0,1]^3$ *and* $g(x,y,z)$ *is a.d.f. Then*

$$\begin{aligned}\int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z) &= F(1,1,1)-\int_0^1 g(1,1,z)\,\mathrm{d}_z F(1,1,z)\\ &\quad -\int_0^1 g(1,y,1)\,\mathrm{d}_y F(1,y,1)\\ &\quad -\int_0^1 g(x,1,1)\,\mathrm{d}_x F(x,1,1)+\int_0^1\int_0^1 g(1,y,z)\,\mathrm{d}_y\mathrm{d}_z F(1,y,z)\\ &\quad +\int_0^1\int_0^1 g(x,1,z)\,\mathrm{d}_x\mathrm{d}_z F(x,1,z)+\int_0^1\int_0^1 g(x,y,1)\,\mathrm{d}_x\mathrm{d}_y F(x,y,1)\\ &\quad -\int_0^1\int_0^1\int_0^1 g(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z).\end{aligned} \tag{1.35}$$

Here

$$\begin{aligned}\mathrm{d}_x\mathrm{d}_y F(x,y) &= F(x+\mathrm{d}x,y+\mathrm{d}y)+F(x,y)\\ &\quad -F(x+\mathrm{d}x,y)-F(x,y+\mathrm{d}y),\\ \mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z) &= F(x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z)\\ &\quad -F(x,y,z)+F(x+\mathrm{d}x,y,z)\\ &\quad +F(x,y+\mathrm{d}y,z)+F(x,y,z+\mathrm{d}z)\\ &\quad -F(x+\mathrm{d}x,y+\mathrm{d}y,z)-F(x,y+\mathrm{d}y,z+\mathrm{d}z)\\ &\quad -F(x+\mathrm{d}x,y,z+\mathrm{d}z).\end{aligned} \tag{1.36}$$

Note that

$$\mathrm{d}_x\mathrm{d}_y F(x,y)=\frac{\partial^2 F(x,y)}{\partial x\partial y}\mathrm{d}x\mathrm{d}y,$$

$$\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z)=\frac{\partial^3 F(x,y,z)}{\partial x\partial y\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

if the partial derivatives exist.

Exercise 1 *Put* $F(x,y,z)=\max(x,y,z)$. *We have*

$$\mathrm{d}_x F(x,1,1)=\mathrm{d}_y F(1,y,1)=\mathrm{d}_z F(1,1,z)=0,$$

$$\mathrm{d}_x\mathrm{d}_y F(x,y,1)=\mathrm{d}_x\mathrm{d}_z F(x,1,z)=\mathrm{d}_y\mathrm{d}_z F(1,y,z)=0,$$

The differential $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z)$ is non-zero if and only if $x=y=z$ and in this case $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z)=\mathrm{d}x$.

Proof: For every interval $J=\left[x_1^{(1)},x_2^{(1)}\right]\times\left[x_1^{(2)},x_2^{(2)}\right]\times\cdots\times\left[x_1^{(s)},x_2^{(s)}\right]\subset[0,1]^s$ and every continuous $F(x_1,x_2,\cdots,x_s)$ the differential $\Delta(F,J)$ is defined as

$$\Delta(F,J)=\sum_{\varepsilon_1=1}^{2}\cdots\sum_{\varepsilon_s=1}^{2}(-1)^{\varepsilon_1+\cdots+\varepsilon_s}F\left(x_{\varepsilon_1}^{(1)},\cdots,x_{\varepsilon_s}^{(s)}\right). \tag{1.37}$$

Putting $F(x_1,x_2,\cdots,x_s)=\max(x_1,x_2,\cdots,x_s)$, $x_1^{(i)}=x$, $x_2^{(i)}=x+\mathrm{d}x$ we have

$$\Delta(F,J) = (-1)^{1+1+\cdots+1} x + \sum_{\varepsilon_1=1}^{2}\cdots\sum_{\varepsilon_s=1}^{2}(-1)^{\varepsilon_1+\cdots+\varepsilon_s}(x+\mathrm{d}x)$$
$$= \sum_{\varepsilon_1=1}^{2}\cdots\sum_{\varepsilon_s=1}^{2}(-1)^{\varepsilon_1+\cdots+\varepsilon_s}(x+\mathrm{d}x) - (-1)^{1+1+\cdots+1}\mathrm{d}x$$
$$= (-1)^{s+1}\mathrm{d}x.$$

Then by (1.35)

$$\int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z)$$
$$= 1 - \int_0^1\int_0^1\int_0^1 g(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z) = 1 - \int_0^1 g(x,x,x)\,\mathrm{d}x. \tag{1.38}$$

For $q \ge 3$ and by (1.31) we have

$$\int_0^1 g(x,x,x)\,\mathrm{d}x = \int_{\frac{2}{q}}^{1-\frac{1}{q}}\left(x-\frac{2}{q}\right)\mathrm{d}x + \int_{1-\frac{1}{q}}^{1}(3x-2)\,\mathrm{d}x = \frac{1}{2}-\frac{2}{q}+\frac{3}{q^2}.$$

For $q = 2$ and by (1.32) we have

$$\int_0^1 g(x,x,x)\,\mathrm{d}x = \int_{\frac{1}{2}}^{\frac{3}{4}}\left(x-\frac{1}{2}\right)\mathrm{d}x + \int_{\frac{3}{4}}^{1}(3x-2)\,\mathrm{d}x = \frac{3}{16}.$$

Therefore for $q \ge 3$, by (1.34) and by (1.38) we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\max\left(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\right) = \frac{1}{2}+\frac{2}{q}-\frac{3}{q^2}. \tag{1.39}$$

Note that the same result follows from (1.44).

Exercise 2 *Put* $F(x,y,z) = \min(x,y,z)$. *Since* $\mathrm{d}_x F(x,1,1) = \mathrm{d}x, \mathrm{d}_y F(1,y,1) = \mathrm{d}y, \mathrm{d}_z F(1,1,z) = \mathrm{d}z$, $\mathrm{d}_x\mathrm{d}_y F(x,y,1) = \mathrm{d}x$ if $x = y$, $\mathrm{d}_x\mathrm{d}_z F(x,1,z) = \mathrm{d}x$ if $x = z$, $\mathrm{d}_y\mathrm{d}_z F(1,y,z) = \mathrm{d}y$ if $y = z$, and $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z) = \mathrm{d}x$ if $x = y = z$ and 0 otherwise, applying (1.34) and (1.35) we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F\left(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\right) = 1 - 3\times\frac{1}{2} + 2\cdot\int_0^1 g(x,x,1)\,\mathrm{d}x + \int_0^1 g(x,1,x)\,\mathrm{d}x - \int_0^1 g(x,x,x)\,\mathrm{d}x. \tag{1.40}$$

Here we have $g(x,x,1) = g(1,x,x) = g(x,x)$ in (1.13) for $q \ge 2$. For $g(x,1,x)$ we use $g(x,x)$ in (1.23) if $q = 2$, (1.24) if $q = 3$ and (1.25) if $q \ge 4$ and for $g(x,x,x)$ we use (1.31) if $q \ge 3$ and (1.32) if $q = 2$. Thus we have:

a) $2\int_0^1 g(x,x,1)\,\mathrm{d}x = 1 - \dfrac{2(q-1)}{q^2}$ for $q \ge 2$;

b) $\int_0^1 g(x,1,x)\,\mathrm{d}x = \int_{\frac{2}{q}}^{1-\frac{2}{q}}\left(x-\dfrac{2}{q}\right)\mathrm{d}x + \int_{1-\frac{2}{q}}^{1}(2x-1)\,\mathrm{d}x = \dfrac{1}{2}-\dfrac{2}{q}+\dfrac{4}{q^2}$ for $q \ge 4$;

c) $\int_0^1 g(x,1,x)\,\mathrm{d}x = \int_{\frac{1}{3}}^{\frac{2}{3}}\left(x-\dfrac{1}{3}\right)\mathrm{d}x + \int_{\frac{2}{3}}^{1}(2x-1)\,\mathrm{d}x = \dfrac{5}{18}$ for $q = 3$;

d) $\int_0^1 g(x,1,x)\,\mathrm{d}x = \int_{\frac{1}{4}}^{\frac{1}{2}}\left(2x-\dfrac{1}{2}\right)\mathrm{d}x + \int_{\frac{1}{2}}^{\frac{3}{4}}\dfrac{1}{2}\mathrm{d}x + \int_{\frac{3}{4}}^{1}(2x-1)\,\mathrm{d}x = \dfrac{3}{8}$ for $q = 2$.

e) $\int_0^1 g(x,x,x)\,\mathrm{d}x = \dfrac{1}{2}-\dfrac{2}{q}+\dfrac{3}{q^2}$ for $q \ge 3$;

f) $\int_0^1 g(x,x,x)\,\mathrm{d}x = \dfrac{3}{16}$ for $q = 2$.

Putting a)-f) into (1.40) yields

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\min\left(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\right)=\begin{cases}\frac{1}{2}-\frac{2}{q}+\frac{3}{q^2} & \text{if } q\ge 4,\\ \frac{1}{6} & \text{if } q=3,\\ \frac{3}{16} & \text{if } q=2.\end{cases} \tag{1.41}$$

7.2. Method II

In the second method we compute the differential $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z)$ directly. It is nonzero only for $(x,y,z)\in I\cup\left(\cup_{i=1}^{\infty}I^{(i)}\right)\cup\left(\cup_{k=1}^{\infty}J^{(k)}\right)$. For such (x,y,z) the a.d.f. $g(x,y,z)$ has the form

$$g(x,y,z)=\begin{cases}\min\left(x,y-\frac{1}{q},z-\frac{2}{q}\right) & \text{if } (x,y,z)\in I,\\ \min\left(x-1+\frac{1}{q^i},y-\frac{1}{q^{i+1}},z-\frac{1}{q}-\frac{1}{q^{i+1}}\right) & \text{if } (x,y,z)\in I^{(i)},\\ \min\left(x-1+\frac{1}{q}+\frac{1}{q^k},y-1+\frac{1}{q^k},z-\frac{1}{q^{k+1}}\right) & \text{if } (x,y,z)\in J^{(k)}.\end{cases} \tag{1.42}$$

Thus $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z)\ne 0$ and moreover $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z)=\mathrm{d}x=\mathrm{d}y=\mathrm{d}z$ only on the following straight lines

$$x=y-\frac{1}{q}=z-\frac{2}{q},$$

$$x-1+\frac{1}{q^i}=y-\frac{1}{q^{i+1}}=z-\frac{1}{q}-\frac{1}{q^{i+1}},$$

$$x-1+\frac{1}{q}+\frac{1}{q^k}=y-1+\frac{1}{q^k}=z-\frac{1}{q^{k+1}}.$$

Considering three possible cases, calculate

$$\begin{aligned}
&(\mathrm{i})\ x,\ y=x+\frac{1}{q},\ z=x+\frac{2}{q},\quad x\in I_X,\ \text{ and}\\
&\int_I F(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z)=\int_{I_X}F\left(x,x+\frac{1}{q},x+\frac{2}{q}\right)\mathrm{d}x;\\
&(\mathrm{ii})\ x,\ y=x-1+\frac{1}{q^i}+\frac{1}{q^{i+1}},\ z=x-1+\frac{1}{q}+\frac{1}{q^{i+1}},\quad x\in I_X^{(i)},\ \text{ and}\\
&\int_{I^{(i)}}F(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z)\\
&=\int_{I_X^{(i)}}F\left(x,x-1+\frac{1}{q^i}+\frac{1}{q^{i+1}},x-1+\frac{1}{q}+\frac{1}{q^i}+\frac{1}{q^{i+1}}\right)\mathrm{d}x;\\
&(\mathrm{iii})\ x,\ y=x+\frac{1}{q},\ z=x-1+\frac{1}{q}+\frac{1}{q^k}+\frac{1}{q^{k+1}},\quad x\in J_X^{(k)}\ \text{ and}\\
&\int_{J^{(k)}}F(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z)\\
&=\int_{J_X^{(k)}}F\left(x,x+\frac{1}{q},x-1+\frac{1}{q}+\frac{1}{q^k}+\frac{1}{q^{k+1}}\right)\mathrm{d}x.
\end{aligned} \tag{1.43}$$

Summary

$$\int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z) = \int_0^{1-\frac{2}{q}} F\left(x, x+\frac{1}{q}, x+\frac{2}{q}\right)\mathrm{d}x$$

$$+\sum_{i=1}^{\infty}\int_{1-\frac{1}{q^i}}^{1-\frac{1}{q^{i+1}}} F\left(x, x-1+\frac{1}{q^i}+\frac{1}{q^{i+1}}, x-1+\frac{1}{q}+\frac{1}{q^i}+\frac{1}{q^{i+1}}\right)\mathrm{d}x \quad (1.44)$$

$$+\sum_{k=1}^{\infty}\int_{1-\frac{1}{q}-\frac{1}{q^k}}^{1-\frac{1}{q}-\frac{1}{q^{k+1}}} F\left(x, x+\frac{1}{q}, x-1+\frac{1}{q}+\frac{1}{q^k}+\frac{1}{q^{k+1}}\right)\mathrm{d}x.$$

Exercise 3 *Put* $F(x,y,z) = xyz$. *By Method I, we have* $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z) = \mathrm{d}x\mathrm{d}y\mathrm{d}z$, *similarly* $\mathrm{d}_x\mathrm{d}_y F(x,y,1) = \mathrm{d}x\mathrm{d}y$, *etc., and by* (1.35) *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} F\left(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)\right)$$

$$= 1 - 3\times\frac{1}{2} + 2\cdot\int_0^1\int_0^1 g(x,y,1)\,\mathrm{d}x\mathrm{d}y \quad (1.45)$$

$$+\int_0^1\int_0^1 g(x,1,z)\,\mathrm{d}x\mathrm{d}z - \int_0^1\int_0^1\int_0^1 g(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Since a computation of (1.45) is complicated we use Method II, by (1.44) we have

$$\int_0^{1-\frac{2}{q}} x\left(x+\frac{1}{q}\right)\left(x+\frac{2}{q}\right)\mathrm{d}x = \frac{1}{4}\left(1-\frac{2}{q}\right)^2,$$

$$\sum_{i=1}^{\infty}\int_{1-\frac{1}{q^i}}^{1-\frac{1}{q^{i+1}}} x\left(x-1+\frac{1}{q^i}+\frac{1}{q^{i+1}}\right)\left(x-1+\frac{1}{q}+\frac{1}{q^i}+\frac{1}{q^{i+1}}\right)\mathrm{d}x$$

$$= \frac{10q^5+7q^4+7q^3-3q-3}{12q^8+12q^7+24q^6+12q^5+12q^4},$$

$$\sum_{k=1}^{\infty}\int_{1-\frac{1}{q}-\frac{1}{q^1}}^{1-\frac{1}{q}-\frac{1}{q^{k+1}}} x\left(x+\frac{1}{q}\right)\left(x-1+\frac{1}{q}+\frac{1}{q^k}+\frac{1}{q^{k+1}}\right)\mathrm{d}x$$

$$= \frac{6q^6-4q^5-7q^4-q^3-6q^2+3q+3}{12q^8+12q^7+24q^6+12q^5+12q^4}.$$

Inserting these formulas into (1.44) we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2) = \frac{q^4-3q^3+3q^2+2q+2}{4q^4+4q^3+4q^2} \quad (1.46)$$

for $q \geq 3$.

8. Conclusion

The problems solved in this paper is significantly more complicated in higher dimensions $s > 3$. For example, in dimension $s = 4$, to compute the d.f. $g(x,y,z,u)$ of the sequence $\left(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2), \gamma_q(n+3)\right)$, it is necessary to investigate 4^4 cases analogous to 3^3 cases for the explicit form $g(x,y,z)$ in the part 0.6. Also **Figure 3** would have to be converted to the dimension 4. Finally, we would need the third iteration of von Neumann-Kakutani transformation.

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