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Relationships between Some k-Fibonacci Sequences

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Abstract

In this paper, we will see that some k-Fibonacci sequences are related to the classical Fibonacci sequence of such way that we can express the terms of a k-Fibonacci sequence in function of some terms of the classical Fibonacci sequence. And the formulas will apply to any sequence of a certain set of k-Fibonacci sequences. Thus we find k'-Fibonacci sequences relating to other k-Fibonacci sequences when σ'_k is linearly dependent of $\sigma_k = \frac{k+\sqrt{k^2+4}}{2}$.

Keywords

Fibonacci and Lucas Numbers, k-Fibonacci Numbers, Pascal Triangle

1. Introduction

k-Fibonacci sequence $\{F_{k,n}\}_{n\geq 0}$ was found by studying the recursive application of two geometrical transformations used in the well-known four-triangle longest-edge (4TLE) partition. This sequence generalizes the classical Fibonacci sequence [1] [2].

1.1. Definition

For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$ is defined recurrently by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$ with initial conditions $F_{k,0} = 0,\ F_{k,1} = 1$.

From this definition, the polynomial expression of the first k-Fibonacci numbers are presented in **Table 1**:

If $k = 1$, the classical Fibonacci sequence $\{0,1,1,2,3,5,8,\cdots\}$ appears and if $k = 2$, the 2-Fibonacci se-

Table 1. Polynomial expression of the first *k*-Fibonacci numbers.

$$F_{k,1} = 1$$
$$F_{k,2} = k$$
$$F_{k,3} = k^2 + 1$$
$$F_{k,4} = k^3 + 2k$$
$$F_{k,5} = k^4 + 3k^2 + 1$$
$$\ldots$$

quence is the classical Pell sequence $\{0,1,2,5,12,29,70,\cdots\}$.

1.2. Metallic Ratios

The characteristic equation of the recurrence equation of the definition of the k-Fibonacci numbers is $r^2 - kr - 1 = 0$ and its solutions are $\sigma_k = \frac{k + \sqrt{k^2+4}}{2}$ and $\sigma'_k = \frac{k - \sqrt{k^2+4}}{2}$.

As particulars cases [3]:

1) If $k = 1$, then $\sigma_1 = \frac{1+\sqrt{5}}{2}$ is known as *Golden Ratio* and it is expressed as Φ.
2) If $k = 2$, then $\sigma_2 = 1 + \sqrt{2}$ is known as *Silver Ratio*.
3) If $k = 3$, it is $\sigma_3 = \frac{3+\sqrt{13}}{2}$ and it is known as *Bronze Ratio*.

From now on, we will represent the classical Fibonacci numbers as F_n instead of $F_{1,n}$.

Binet identity takes the form [1] $F_{k,n} = \frac{\sigma_k^n - (\sigma'_k)^n}{\sigma_k - \sigma'_k}$ with $\sigma_k - \sigma'_k = \sqrt{k^2+4}$.

1.3. Theorem 1

Power σ_n^k for $n \geq 1$ is related to σ_k by mean of the formula

$$\sigma_k^n = F_{k,n}\sigma_k + F_{k,n-1} \tag{1}$$

Proof. By induction. For $n = 1$, it is obvious. Let us suppose this formula is true until: $\sigma_k^n = F_{k,n}\sigma_k + F_{k,n-1}$. Then, and taking into account $\sigma_k^2 - k\sigma_k - 1 = 0$:

$$\begin{aligned} \sigma_k^{n+1} &= \sigma_k^n \cdot \sigma_k = \left(F_{k,n}\sigma_k + F_{k,n-1}\right)\sigma_k \\ &= kF_{k,n}\sigma_k + F_{k,n-1}\sigma_k + F_{k,n} = F_{k,n+1}\sigma_k + F_{k,n} \end{aligned}$$

Obviously, the formulas found in [1] [2] can be applied to any k-Fibonacci sequence. For example, the Identities of Binet, Catalan, Simson, and D'Ocagne; the generating function; the limit of the ratio of two terms of the sequence, the sum of first "n" terms, etc. However, we will see that some k-Fibonacci sequences are related to a first k-Fibonacci sequence so that we will can express the terms of a k-Fibonacci sequence according to some terms of an initial k-Fibonacci sequence. And the formulas will be applicable to any sequence of a given set of k-Fibonacci sequences. For instance, we will express the terms of the 4-Fibonacci sequence in function of some terms of the classical Fibonacci sequence and these formulas will be applied to other k-Fibo-naccise-quences, as for example if $k = 11, 29, 76, 199, \cdots$

2. k'-Fibonacci Sequences Related to the k-Fibonacci Sequence

In this section, we try to find the relationships that can exist between the values of k' and the coefficients "a" and "b" such that $\sigma'_k = a + b\sigma_k$.

We can write this last equation as

$$\sigma'_k = \frac{k' + \sqrt{k'^2 + 4}}{2} = a + b\frac{k + \sqrt{k^2+4}}{2} \to k' + \sqrt{k'^2+4} = 2a + b\left(k + \sqrt{k^2+4}\right)$$
$$\to \begin{cases} k' = 2a + bk \\ k'^2 + 4 = b^2\left(k^2+4\right) \to \left(k^2+4\right)b^2 - k'^2 = 4 \end{cases}$$

because $k' \in \mathbb{N}$.

Main problem is to solve the quadratic Diophantine equation $\left(k^2+4\right)b^2 - k'^2 = 4$ for "k'" and "b" for each value of "k".

2.1. Theorem 2

The positive characteristic root $b\sigma_k^{2n+1}$ generates new k-Fibonacci sequences, for $n = 1, 2$, *Proof*. From Formula (1) it is obtained $\sigma_k^{2n+1} = F_{k,2n} + F_{k,2n+1}\sigma_k$.

For $n = 1$ it is

$$\sigma_k^3 = F_{k,2} + F_{k,3}\sigma_k = k + \left(k^2+1\right)\frac{k+\sqrt{k^2+4}}{2} = \frac{1}{2}\left(k\left(k^2+3\right) + \left(k^2+1\right)\sqrt{k^2+4}\right)$$
$$= \frac{1}{2}\left(k\left(k^2+3\right) + \sqrt{\left(k\left(k^2+3\right)\right)^2 + 4}\right) = \sigma_{k\left(k^2+3\right)}$$

Then, σ_k^3 generates the $k\left(k^2+3\right)$-Fibonacci sequence.

In the same way, we can prove that σ_k^5 generates the $k\left(k^4+5k^2+5\right)$-Fibonacci sequence, σ_k^7 generates the $k\left(k^6+7k^4+14k^2+7\right)$-Fibonacci sequence, etc. Particularly, Φ $(k=1)$ generates the sequences $F_1, F_4, F_{11}, F_{29}, \cdots$.

2.2. Theorem 3

For $n \ge 2$ it is verified

$$\sigma_k^{2n+1} = \left(k^2+2\right)\sigma_k^{2n-1} - \sigma_k^{2n-3} \quad (2)$$

Proof. Taking into account both **Table 1** and Formula (1), Right Hand Side (RHS) of Equation (2) is

$$(RHS) = \left(\left(k^2+2\right)\sigma_k^2 - 1\right)\sigma_k^{2n-3} = \left(\left(k^2+2\right)\left(k\sigma_k+1\right)-1\right)\sigma_k^{2n-3} = \left(\left(k^3+2k\right)\sigma_k + \left(k^2+1\right)\right)\sigma_k^{2n-3}$$
$$= \left(F_{k,4}\sigma_k + F_{k,3}\right)\sigma_k^{2n-3} = \sigma_k^4\sigma_k^{2n-3} = \sigma_k^{2n+1}$$

It is worthy of note that Equation (2) is similar to the relationship between the elements of the k-Fibonacci sequence $F_{k,n+2} = \left(k^2+2\right)F_{k,n} - F_{k,n-2}$. Other versions of this equation will appear in this paper. Moreover, if we are looking for the characteristic roots of this equation, then we find

$$r^2 - \left(k^2+2\right)r + 1 = 0 \to r = \frac{k^2+2 \pm \sqrt{k^4+4k^2}}{2} = k\frac{k \pm \sqrt{k^2+4}}{2} + 1 = \begin{cases} k\sigma_k + 1 = \sigma_k^2 \\ k\sigma'_k + 1 = \sigma_k'^2 \end{cases}$$

And $F_{k,n+2}$ will be function of σ_k^2 with the coefficients depending of initial conditions for $n = 0$ and $n = 1$.

2.3. *k*-Fibonacci Sequences Related to an Initial *f*-Fibonacci Sequence

From two previous theorems, the k-Fibonacci sequences related to an initial k-Fibonacci sequence have as the positive characteristic root σ_k^{2n+1} or that is the same, the sequence of characteristic roots

$\left\{\sigma_k^{2n+1}\right\} = \left\{\sigma_k, \sigma_k^3, \sigma_k^5, \cdots\right\}$ generates the k-Fibonacci sequences related to the first k-Fibonacci sequence.

The values of the parameter of these sequences are

$\{k_n\} = \left\{k, k\left(k^2+3\right), k\left(k^4+5k^2+5\right), k\left(k^5+7k^4+14k^2+7\right), \cdots\right\}$ and Equation (2) for this sequence takes the similar form $k_{n+1} = \left(k^2+2\right)k_n - k_{n-1}$.

Next we present the first few values of the parameter k_n:

a) $k_1 = k$

b) $k_2 = k^3 + 3k$

c) $k_3 = k^5 + 5k^3 + 5k$

d) $k_4 = k^7 + 7k^5 + 14k^3 + 7k$

e) $k_5 = k^9 + 9k^7 + 27k^5 + 30k^3 + 9k$

But these polynomials verify the relationship

$$k_n = F_{k,2n} + F_{k,2n-2} \tag{3}$$

where $F_{k,n}$ are expressed in **Table 1**.

The coefficients of these polynomials generate the triangle in **Table 2**:

Last column is the sum by row of the coefficients, and it is a bisection of the classical Lucas sequence $\{2,1,3,4,7,11,18,29,47,\cdots\}$ and we will see again in this paper.

If $a_{r,c}$ is a term of this table, then $a_{r,c} = a_{r-1,c-1} + \sum_{j=0}^{c} a_{r-1-j,c-j}$. For instance, $1+5+14+30$ of the second diagonal plus 27 of the row 5 is the 77 of the row 6.

All the first diagonal sequences are listed in [4], from now on OEIS, but the unique antidiagonal sequences listed in OEIS are:

a) $\{1,1,1,1,1,\cdots\}: A000012$

b) $\{3,5,7,9,11,\cdots\}: A005408-\{1\}$

c) $\{5,14,27,44,65,\cdots\}: A014106$

d) $\{7,30,77,156,275,\cdots\}: A030440$

From this study, it is easy to find the values of "b" mentioned at the beginning of this section, because

$$b_n = \sqrt{\frac{k_n^2+4}{k^2+4}} = F_{k,2n-1}.$$

Sequence $\{b_n\}$ also verifies the recurrence law given in Equation (2): $b_{n+1} = \left(k^2+2\right)b_n - b_{n-1}$.

In this case, the triangle of coefficients is in **Table 3** and the formto generate these numbers is the same as in table of k_n. This triangle is formed by the odd rows of 2-Pascal triangle of [2]. The sequence of the last column is a bisection of the classical Fibonacci sequence $\{1,1,2,3,5,8,13,21,\cdots\}$.

First diagonal sequences and the antidiagonal sequences are listed in OEIS.

Finally, for the values of a_n is enough to do $a_n = \frac{k_n - b_n k}{2}$ and therefore, applying Formula (3) and the definition of the k-Fibonacci numbers, $a_n\$ = F_{k,2n-2}$.

Table 2. Triangle of the coefficients of k_n.

1						1							1
2					1		3						4
3				1		5		5					11
4			1		7		14		7				29
5		1		9		27		30		9			76
6	1		11		44		77		55		11		199

In this case, the triangle of the coefficients of the expressions of a_n is in **Table 4**.
Last column is the other bisection of the classical Fibonacci sequence.

The diagonal sequence $\{1, n, \cdots\}$ indicates the number of terms in the expansion of $(x_1 + x_2 + \cdots + x_n)^j$ and it is $a_{j,n} = \binom{n+j-1}{j}$.

In this table, it is verified:

a) $a_{r,c} = \sum_{j=0}^{r} a_{r-j,c-j-1} + a_{r-1,c-1}$

b) $\sum a_{2n+1} - \sum a_{2n} = 0, 1, -1$, if $n \equiv 0, 1, 2 \pmod 3$, respectively.

c) The diagonal sequences are listed in OEIS.

d) The elements of rth diagonal sequence, for $r = 0, 1, 2, \cdots$ verify the relation $a_{n,r} = \binom{n+2r}{2r+1}$

Then we will apply the results to the k-Fibonacci sequences, for $k = 1, 2, 3, 4$.

3. k-Fibonacci Sequences Related to the Classical Fibonacci Sequence

In this section we try to find the relations that could exist between the values of "k" and "a" and "b" in order that the positive characteristic root σ_k is $\sigma_k = a + b\Phi$.

In this case, Equation (2) takes the form $\begin{cases} k = 2a + b \\ k^2 + 4 = 5b^2 \to 5b^2 - k^2 = 4 \end{cases}$.

3.1. Integer Solutions of Equation $5b^2 - k^2 = 4$

The integer solutions of Equation $5b^2 - k^2 = 4$ are $b_n = F_{2n+1}$, $k_n = L_{2n+1}$, being L_n the classical Lucas sequence $\{2, 1, 3, 4, 7, 11, 18, 29, 47, \cdots\}$.

Proof. Applying Binnet Identity, and taking into account $L_n = F_{n+1} + F_{n-1} \to L_n = \Phi^n + (-\Phi)^{-n}$, it is

$$5b^2 - 4 = \left(\Phi^{2n+1} - (-\Phi)^{-2n-1}\right)^2 - 4 = \Phi^{4n+2} - 2(-1)^{2n+1} + \Phi^{-4n-2} - 4 = \Phi^{4n+2} - 2 + \Phi^{-4n-2}$$

$$= \left(\Phi^{2n+1} + (-\Phi)^{-2n-1}\right)^2 = L_{2n+1}^2$$

Table 3. Triangle of the coefficients of b_n.

1						1						1
2					1		1					2
3				1		3		1				5
4			1		5		6		1			13
5		1		7		15		10		1		34
6	1		9		28		35		15		1	89

Table 4. Triangle of the coefficients of a_n.

1						1						1
2					1		2					3
3				1		4		3				8
4			1		6		10		4			21
5		1		8		21		20		5		55
6	1		10		36		56		35		6	144

Consequently, the values of the parameter "k" can also be expressed as $k_n = F_{2n} + F_{2n+2} = L_{2n+1}$.

Integer solutions of this equation are expressed in **Table 5**, where $\sigma_1 = \frac{1+\sqrt{5}}{2} = \Phi$ is the Golden Ratio.

3.2. On the Sequences $\{a_n\}$, $\{b_n\}$, and $\{k_n\}$

We will show some properties of the sequences of **Table 5**.

- The sequence of values of "a", $\{0,1,3,8,21,\cdots\}, A001906$ is the sequence $\{F_{2n}\}$ of even Fibonacci numbers, and is known as Bisection of Fibonacci sequence. Its elements, a_n, have the property that $5a_n^2 + 4$ are perfect squares and these numbers form the sequence $\{2,3,7,18,47,\cdots\}, A005248$ that is the Bisection of the classical Lucas sequence. The sequence of sums of two consecutive terms of this sequence is 5 times the following sequence.
- The sequence of values of "b", $\{1,2,5,13,34,\cdots\}, A001519$ is the sequence of odd Fibonacci numbers, $\{F_{2n+1}\}$, and is also known as Bisection of Fibonacci sequence. The sequence of sums of two consecutive terms of this sequence is the preceding sequence $\text{A005248} - \{2\}$.
- The sequence of values of "k", $\{1,4,11,29,76,\cdots\}, A002878$ is the sequence of odd Lucas numbers, or, that is the same, is the sum of two even consecutive Fibonacci numbers, $\{F_{2n} + F_{2n+2}\}$ and is known as Bisection Lucas Sequence. The sequence of sums of two consecutive terms of this sequence is 5 times the preceding sequence $\text{A001906} - \{0\}$.
- All these sequences verify the recurrence law given in Equation (2), $p_{n+1} = 3p_n - p_{n-1}$.

As a consequence of this situation, if we represent as $\{\sigma_{1,n}\}_{n\in\mathbb{N}}$ the sequence of values of σ, then, Equation (2) is the relation $\sigma_{1.n} = F_{2n+1}\sigma + F_{2n}$.

3.3. Relationships between the k-Fibonacci Sequences If $k = L_{2n+1}$ and the Classical Fibonacci Sequence

Applying Subsection 2.3 when $k=1$ in Equation (3), the sequence $\{\Phi,\Phi^3,\Phi^5,\cdots\} = \{\Phi^{2n+1}\}_{n\in\mathbb{N}}$ is the sequence $\{\sigma_1,\sigma_4,\sigma_{11},\sigma_{29},\cdots\}$.

Consequently:

$$F_{4,n} = \frac{\sigma_4^n - (-\sigma_4)^{-n}}{\sqrt{20}} = \frac{\Phi^{3n} - (-\Phi)^{-3n}}{2\sqrt{5}} = \frac{F_{3n}}{F_3} \to F_4 = \frac{1}{2}\{0,2,8,34,144,\cdots\}$$

$$F_{11,n} = \frac{\sigma_{11}^n - (-\sigma_{11})^{-n}}{\sqrt{125}} = \frac{\Phi^{5n} - (-\Phi)^{-5n}}{5\sqrt{5}} = \frac{F_{5n}}{F_5} \to F_{11} = \frac{1}{5}\{0,5,55,610,\cdots\}$$

$$F_{29,n} = \frac{\sigma_{29}^n - (-\sigma_{29})^{-n}}{\sqrt{845}} = \frac{\Phi^{7n} - (-\Phi)^{-7n}}{13\sqrt{5}} = \frac{F_{7n}}{F_7} \to F_{29} = \frac{1}{13}\{0,13,377,10946,\cdots\}$$

4. k-Fibonacci Sequences Related with the Pell Sequence

Repeating the previous process, we can solve the Diophantine equation $8b^2 - k^2 = 4$ and being $k = 2a + 2b$.

Table 5. Integer solutions of the Diophantine equation $5b^2 - k^2 = 4$.

$k_n = L_{2n+1}$	$b_n = F_{2n+1}$	$a_n = F_{2n}$	$\sigma_{1,n}$
1	1	0	$\sigma_1 = 0 + 1\sigma_1$
4	2	1	$\sigma_4 = 1 + 2\sigma_1$
11	5	3	$\sigma_{11} = 3 + 5\sigma_1$
29	13	8	$\sigma_{29} = 8 + 13\sigma_1$
76	34	21	$\sigma_{76} = 21 + 34\sigma_1$

The values obtained are showed in Table 6:

4.1. On These Quences $\{a_n\}$, $\{b_n\}$, and $\{k_n\}$.

We will show some properties of the sequences of Table 4.

- $\{a_n\}=\{0,2,12,70,408,\cdots\}, A001542$ is the sequence of even Pell numbers. Its elements have the property that $8a_n^2+4$ are perfect squares, being $\left\{\sqrt{8a_n^2+4}\right\}=\{2,6,34,198,1154,\cdots\}, A003499$. The sequence of sums of two consecutive terms of this sequence is the sequence $\{8b_n\}$.
- $\{b_n\}=\{1,5,29,169,985,\cdots\}, A001653$ is the sequence of odd Pell numbers. Its elements have the property that $2p_n^2-1$ are perfect squares.
- $\{k_n\}=\{2,14,82,478,2786,\cdots\}, A077444$. Its elements are the Pell-Lucas numbers, $k_n=P_{2n}+P_{2n+2}=LP_{2n+1}$. This sequence can be obtained by summing up two consecutive terms of the sequence A001542.
- Much more interesting is the sequence obtained by dividing by 2: $\{1,7,41,239,1393,\cdots\}, A002315$. This sequence has been studied in [5] and has been determined as the values whose square coincide with the sum of the $4n+1$ first Pell numbers, $\sum_{j=0}^{n} S_{4j+1}=a_n^2$ and it is known as the Newman-Shanks-Williams Primes. It verifies the recurrence law $a_{2,n+1}=6a_{2,n}-a_{2.n-1}$ with initial conditions $a_{2,1}=1$ and $a_{2,2}=7$. The sequence of sums of two consecutive terms of this sequence is 8 times $\{6,35,204,1189,\cdots\}, A001109$. Its elements verify the property $8s_n^2+1$ are perfect squares, $\{17,99,577,\cdots\}, A001541-\{1,3\}$.
- All these sequences verify the recurrence law (2), $p_{n+1}=6p_n-p_{n-1}$.

As in the preceding section, if we represent the sequence of values of "σ" as $\{\sigma_{2,n}\}$, then these terms verify the recurrence relation $\sigma_{2,n}=P_{2n+1}\sigma_2+P_{2n}$, being $\sigma_2=1+\sqrt{2}$ the Silver Ratio.

4.2. Relationships between the k-Fibonacci Sequences for $k=2,14,82,478,\cdots$ and the Pell Sequence

Taking into account $\sigma_2^2-2\sigma_2-1=0$, it is easy to prove $\{\sigma_2,\sigma_{14},\sigma_{82},\sigma_{478},\cdots\}$ is the geometric sequence $\{\sigma_2,\sigma_2^3,\sigma_2^5,\cdots\}=\{\sigma_2^{2n+1}\}_{n\in\mathbb{N}}$.

Consequently:

$$F_{14,n}=\frac{\sigma_{14}^n-(-\sigma_{14})^{-n}}{\sqrt{200}}=\frac{\sigma_2^{3n}-(-\sigma_2)^{-3n}}{5\sqrt{8}}=\frac{F_{2,3n}}{F_{2,3}}=\frac{P_{3n}}{P_3}\to F_{14}=\frac{1}{5}\{0,5,70,985,\cdots\}$$

$$F_{82,n}=\frac{\sigma_{82}^n-(-\sigma_{82})^{-n}}{\sqrt{1682}}=\frac{\sigma_2^{5n}-(-\sigma_2)^{-5n}}{29\sqrt{8}}=\frac{F_{2,5n}}{F_{2,5}}=\frac{P_{5n}}{P_5}\to F_{82}=\frac{1}{29}\{0,29,2378,195025,\cdots\}$$

$$F_{478,n}=\frac{\sigma_{478}^n-(-\sigma_{478})^{-n}}{\sqrt{228488}}=\frac{F_{2,7n}}{F_{2,7}}=\frac{P_{7n}}{P_7}\to F_{478}=\frac{1}{169}\{0,169,80782,\cdots\}$$

5. k-Fibonacci Sequences Related to the 3-Fibonacci Sequence

Repeating the previous process, we can solve the Diophantine equation $13b^2-k^2=4$ being $k=2a+3b$.

The values obtained are showed in Table 7.

Table 6. Integer solutions of the Diophantine equation $8b^2-k^2=4$.

$k_n=P_{2n}+P_{2n+2}$	$b_n=P_{2n+1}$	$a_n=P_{2n}$	$\sigma_{2,n}$
2	1	0	$\sigma_2=0+1\sigma_2$
14	5	2	$\sigma_{14}=2+5\sigma_2$
82	29	12	$\sigma_{82}=12+29\sigma_2$
478	169	70	$\sigma_{478}=70+169\sigma_2$

5.1. On These Quences $\{a_n\}$, $\{b_n\}$, and $\{k_n\}$

We will show some properties of the sequences of Table 7.

- $\{a_n\} = \{0, 3, 33, 360, 3927, \cdots\}$, $A075835$, is the sequence of even 3-Fibonacci numbers. Its elements have the property that $13a_n^2 + 4$ are perfect squares, $\{2, 11, 119, 1298, \cdots\}$, $A057076$. The sequence of sums of two consecutive terms is 13 times the following sequence.
- $\{b_n\} = \{1, 10, 109, 1189, \cdots\}$, $A078922$, is the sequence of the odd 3-Fibonacci numbers.
- $\{k_n\} = \{3, 36, 393, 4287, 46764, \cdots\}$ is the sequence of the odd 3-Lucas numbers $k_n = F_{3,2n} + F_{3,2n+2} = L_{3,2n+1}$. This sequence can also be expressed as 3 times the sequence $\{1, 12, 131, 1429, \ldots\}$, $A097783$.
- All these sequences verify the recurrence law (Equation (2)), $p_{n+1} = 11p_n - p_{n-1}$.
- The sequence $\{\sigma_{3,n}\}$ verify the relationship $\sigma_{3,n} = F_{3,2n+1}\sigma_3 + F_{3,2n}$ being $\sigma_3 = \frac{3+\sqrt{13}}{2}$ the Bronze Ratio [3].

5.2. Relationships between the *k*–Fibonacci Sequences for $k = 3, 36, 393, 4287, \cdots$ and the 3-Fibonacci Sequence

Taking into account $\sigma_3^2 - 3\sigma_3 - 1 = 0$, it is easy to prove $\{\sigma_3, \sigma_{36}, \sigma_{393}, \sigma_{4287}, \cdots\}$ is the geometric sequence $\{\sigma_3, \sigma_3^3, \sigma_3^5, \cdots\} = \{\sigma_3^{2n+1}\}_{n\in\mathbb{N}}$.

Consequently:

$$F_{36,n} = \frac{\sigma_{36}^n - (-\sigma_{36})^{-n}}{\sqrt{1300}} = \frac{\sigma_3^{3n} - (\sigma_3)^{-3n}}{10\sqrt{13}} = \frac{F_{3,3n}}{F_{3,3}} \to F_{36} = \frac{1}{10}\{0, 10, 360, 12970, \cdots\}$$

$$F_{393,n} = \frac{\sigma_{393}^n - (-\sigma_{393})^{-n}}{\sqrt{154453}} = \frac{\sigma_3^{5n} - (\sigma_3)^{-5n}}{109\sqrt{13}} = \frac{F_{3,5n}}{F_{3,5}} \to F_{393} = \frac{1}{109}\{0, 109, 11881, \cdots\}$$

$$F_{4287,n} = \frac{F_{3,7n}}{F_{3,3}} \to F_{4287} = \frac{1}{1189}\{0, 1189, 5097243, \cdots\}$$

6. Conclusions

There are infinite k-Fibonacci sequences related to an initial k-Fibonacci sequence for a fixed value of "k". Between these sequences, the following relations are verified:

1) The relationship $\sigma_{k,n} = a + b\sigma_k$ is verified if and only if both following relations happen:
   Relationship between "a", "b", and "k": $k_n = 2a + kb$
   Diophantine equation: $(k^2+4)b^2 - k^2 = 4$
2) Relationship between the positive characteristic root $\sigma_{k,n}$ and the k–Fibonacci numbers: $\sigma_k^{n+1} = F_{k,n+1}\sigma_k + F_{k,n}$
3) Second sequence related to the k–Fibonacci sequence: $k' = k(k^2+3)$
4) Two first values of "b" are $b_1 = 1 = F_{k,1}$ and $b_2 = k^2 + 1 = F_{k,3}$
5) Two first values of "a" are $a_0 = 0 = F_{k,0}$ and $a_1 = k = F_{k,2}$
6) Recurrence law for the sequences $\{a_n\}, \{b_n\}$, and $\{k_n\}$: $p_{n+1} = (k^2+2)p_n - p_{n-1}$

Table 7. Integer solutions of the Diophantine equation 13 $b^2 - k^2 = 4$.

k_n	$b_n = F_{3,2n+1}$	$a_n = F_{3,2n}$	$\sigma_{3,n}$
3	1	0	$\sigma_3 = 0 + 1\sigma_3$
36	10	3	$\sigma_{36} = 3 + 10\sigma_3$
393	109	33	$\sigma_{393} = 33 + 109\sigma_3$
4287	1189	360	$\sigma_{4287} = 360 + 1189\sigma_3$

It is worthy of remarking the fact the last sequence $\left\{\sigma_k^{2n+1}\right\}_{n\in\mathbb{N}}$ indicates the k_n-Fibonacci sequence related to the initial k-Fibonacci sequence $\{F_1, F_2, F_3, F_4, \cdots\}$ generated by the respective positive characteristic root, σ_k^{2n+1}. From this sequence, we can obtain the sequence of k-Fibonacci sequences related to F_n: taking into account the positive characteristic root of this sequence is σ_k^{2n+1}, the sequence of r-Fibonacci sequences related to this has as positive characteristic root, $\sigma_k^{r(2n+1)}$ for $r \geq 1$. For instance: from the sequence of k-Fibonacci sequences related with the classical Fibonacci sequence (see Section 2), $F_1, F_4, F_{11}, F_{29,} F_{76}, \cdots$ we can obtain the sequences of k-Fibonacci sequences related to

- 4-Fibonacci sequence: $\{F_4, F_{76}.F_{1364}, F_{24476}, \cdots\}$
- 11-Fibonacci sequence: $\{F_{11}, F_{1364}, \cdots\}$
- 29-Fibonacci sequence: $\{F_{29}, F_{24476}, \cdots\}$.

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