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Positive Definite Solutions for the System of Nonlinear Matrix Equations $X + A^*Y^{-n}A = I$, $Y + B^*X^{-m}B = I$

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Abstract

In this paper, some properties of the positive definite solutions for the nonlinear system of matrix equations $X + A^*Y^{-n}A = I, Y + B^*X^{-m}B = I$ are derived. As a matter of fact, an effective iterative method to obtain the positive definite solutions of the system is established. These solutions are based on the convergence of monotone sequences of positive definite matrices. Moreover, the necessary and sufficient conditions for the existence of the positive definite solutions are obtained. Finally, some numerical results are given.

Keywords

System of Nonlinear Matrix Equations, Iterative Methods, Monotonic Sequence, Positive Definite Matrices

1. Introduction

It is well known that algebraic discrete-type Riccati equations play a central role in modern control theory and signal processing. These equations arise in many important applications such as in optimal control theory, dynamic programming, stochastic filtering, statistics and other fields of pure and applied mathematics [1]-[3].

In the last years, the nonlinear matrix equation of the form

$$G(X) + A^*F(X)A = Q \tag{1.1}$$

where $F(X)$ and $G(X)$ maps from positive definite matrices into positive definite matrices is studied in

many papers [4]-[8]. It is well known that Equation (1.1) with $G(X) = X$ and $F(X) = X^{-1}$ is a special case of algebraic discrete-type Riccati equation of the form [2] [3]

$$X = V^*XV + Q - (V^*XB + A)(R + B^*XB)^{-1}(B^*XV + A^*)$$ (1.2)

In addition, the system (Sys.) of algebraic discrete-type Riccati equations appears in many applications [9]-[12]. Czornik and Swierniak [10] have studied the lower bounds for eigenvalues and matrix lower bound of a solution for the special case of the System:

$$X_i = V_i^*X_iV_i + Q_i - (V_i^*X_iB_i + A_i)(R_i + B_i^*X_iB_i)^{-1}(B_i^*X_iV_i + A_i^*)$$ (1.3)

where $i = 1, 2, \cdots, k$.

In the same manner, we can deduce a system of nonlinear matrix equations as matrix Equation (1.1) with $G(X) = X$ and $F(X) = X^{-1}$. For that, Al-Dubiban [13] have studied the system

$$X + A^*Y^{-1}A = I$$
$$Y + B^*X^{-1}B = I$$ (1.4)

which is a special case of Sys.(1.3). The author obtained sufficient conditions for existence of a positive definite solution of Sys.(1.4) and considered an iterative method to calculate the solution. Recently, similar kinds of Sys.(1.4) have been studied in some papers [14] [15].

In this paper we consider the system of nonlinear matrix equations that can be expressed in the form:

$$X + A^*Y^{-n}A = I$$
$$Y + B^*X^{-m}B = I$$ (1.5)

where n, m are two positive integers, X, Y are $r \times r$ unknown matrices, I is the $r \times r$ identity matrix, and A, B are nonsingular matrices. All matrices are defined over the complex field. The paper is organized as follows: in Section 2, we derive the necessary and sufficient conditions for the existence the solution to the Sys.(1.5). In Section 3, we introduce an iterative method to obtain the positive definite solutions of Sys.(1.5). We discuss the convergence of this iterative method. Section 4 discussed the error and the residual error. Some numerical examples are given to illustrate the efficiency for suggested method in Section 5.

The following notations are used throughout the rest of the paper. The notation $A \geq 0$ $(A > 0)$ means that A is positive semidefinite (positive definite), A^* denotes the complex conjugate transpose of A, and I is the identity matrix. Moreover, $A \geq B$ $(A > B)$ is used as a different notation for $A - B \geq 0$ $(A - B > 0)$. We denote by $\rho(A)$ the spectral radius of A; $\lambda_r(X), \mu_r(Y)$ means the eigenvalues of X and Y respectively. The norm used in this paper is the spectral norm of the matrix A, i.e. $\|A\| = \sqrt{\rho(AA^*)}$ unless otherwise noted.

2. Existence Conditions of the Solutions

In this section, we will discuss some properties of the solutions for Sys.(1.5) and obtain the necessary and sufficient conditions for the existence of the solutions of the Sys.(1.5).

Theorem 1 *If* λ_-, λ_+ *are the smallest and the largest eigenvalues of a solution* X *of Sys.*(1.5), *respectively, and* μ_-, μ_+ *are the smallest and the largest eigenvalues of a solution* Y *of Sys.*(1.5), *respectively,* η, ξ *are eigenvalues of A, B then*

$$\sqrt{(1-\lambda_+)\mu_-^n} \leq |\eta| \leq \sqrt{(1-\lambda_-)\mu_+^n}$$ (1.6)

$$\sqrt{(1-\mu_+)\lambda_-^m} \leq |\xi| \leq \sqrt{(1-\mu_-)\lambda_+^m}$$ (1.7)

Proof: Let v be an eigenvector corresponding to an eigenvalue η of the matrix A and $\|v\| = 1$, ω be an eigenvector corresponding to an eigenvalue ξ of the matrix B and $\|\omega\| = 1$. Since the solution (X, Y) of Sys.(1.5) is a positive definite solution then $0 < \lambda_- \leq \lambda_+ < 1$ and $0 < \mu_- \leq \mu_+ < 1$.

From the Sys.(1.5), we have

$$\langle Xv, v \rangle + \langle (A^*Y^{-n}A)v, v \rangle = \langle Iv, v \rangle$$

$$\langle Xv,v \rangle + \langle Y^{-n}Av, Av \rangle = \langle v,v \rangle$$

$$\langle Xv,v \rangle + |\eta|^2 \langle Y^{-n}v,v \rangle = 1,$$

i.e

$$\lambda_- + \frac{|\eta|^2}{\mu_+^n} \le 1 \le \lambda_+ + \frac{|\eta|^2}{\mu_-^n}.$$

Hence

$$\sqrt{(1-\lambda_+)\mu_-^n} \le |\eta| \le \sqrt{(1-\lambda_-)\mu_+^n}.$$

Also, from the Sys.(1.5), we have

$$\langle Y\omega,\omega \rangle + \langle (B^*X^{-m}B)\omega,\omega \rangle = \langle I\omega,\omega \rangle$$

$$\langle Y\omega,\omega \rangle + \langle X^{-m}B\omega, B\omega \rangle = \langle \omega,\omega \rangle$$

$$\langle Y\omega,\omega \rangle + |\xi|^2 \langle X^{-m}\omega,\omega \rangle = 1,$$

i.e

$$\mu_- + \frac{|\xi|^2}{\lambda_+^m} \le 1 \le \mu_+ + \frac{|\xi|^2}{\lambda_-^m}.$$

Hence

$$\sqrt{(1-\mu_+)\lambda_-^m} \le |\xi| \le \sqrt{(1-\mu_-)\lambda_+^m}$$

Theorem 2 *If Sys.*(1.5) *has a positive definite solution* (X,Y), *then*

$$\sqrt[n]{AA^*} < Y < I - B^*B \tag{1.8}$$

$$\sqrt[m]{BB^*} < X < I - A^*A \tag{1.9}$$

Proof: Since (X,Y) be a positive definite solution of Sys.(1.5), then

$$X < I, \quad A^*Y^{-n}A < I, \quad Y < I, \quad B^*X^{-m}B < I.$$

From the inequality $X < I$, we have $B^*X^{-m}B > B^*B$, therefore

$$Y = I - B^*X^{-m}B < I - B^*B$$

From the inequality $A^*Y^{-n}A < I$, we have $Y > \sqrt[n]{AA^*}$, then

$$\sqrt[n]{AA^*} < Y < I - B^*B.$$

And from the inequality $Y < I$, we have $A^*Y^{-n}A > A^*A$, therefore

$$X = I - A^*Y^{-n}A < I - A^*A$$

From the inequality $B^*X^{-m}B < I$, we have $X > \sqrt[m]{BB^*}$, hence

$$\sqrt[m]{BB^*} < X < I - A^*A,$$

which complete the proof.

Corollary 1 *If Sys.*(1.5) *has a positive definite solution* (X,Y), *then*

$$\sqrt[n]{AA^*} + B^*B < I \tag{1.10}$$

$$\sqrt[m]{BB^*} + A^*A < I \tag{1.11}$$

Theorem 3 *Sys.* (1.5) *has a positive definite solution* (X,Y) *if and only if the matrices A, B have the factori-*

zation

$$A = \left(P^*P\right)^{\frac{n}{2}} N, \qquad B = \left(Q^*Q\right)^{\frac{m}{2}} M \tag{1.12}$$

where P, Q are nonsingular matrices satisfying the following system

$$\begin{aligned} Q^*Q + N^*N &= I \\ P^*P + M^*M &= I \end{aligned} \tag{1.13}$$

In this case the solution is $\left(Q^*Q, P^*P\right)$.

Proof: Let Sys.(1.5) has a positive definite solution $\left(X, Y\right)$, then $X = Q^*Q, Y = P^*P$, where Q, P are non-singular matrices. Furthermore Sys.(1.5) can be rewritten as

$$Q^*Q + A^*\left(P^*P\right)^{-n} A = I$$

$$P^*P + B^*\left(Q^*Q\right)^{-m} B = I$$

$$Q^*Q + A^*\left(P^*P\right)^{\frac{-n}{2}} \left(P^*P\right)^{\frac{-n}{2}} A = I$$

$$P^*P + B^*\left(Q^*Q\right)^{\frac{-m}{2}} \left(Q^*Q\right)^{\frac{-m}{2}} B = I$$

Let $N = \left(P^*P\right)^{\frac{-n}{2}} A$, $M = \left(Q^*Q\right)^{\frac{-m}{2}} B$, then $A = \left(P^*P\right)^{\frac{n}{2}} N$, $B = \left(Q^*Q\right)^{\frac{m}{2}} M$, and Sys. (1.5) turns into Sys. (1.13).

Conversely, if A, B have the factorization (1.12) and satisfying Sys.(1.13), let $X = Q^*Q, Y = P^*P$, then X, Y are positive definite matrices, and we have

$$X + A^*Y^{-n}A = Q^*Q + A^*\left(P^*P\right)^{-n} A = Q^*Q + A^*\left(P^*P\right)^{\frac{-n}{2}} \left(P^*P\right)^{\frac{-n}{2}} A = Q^*Q + N^*N = I$$

$$Y + B^*X^{-m}B = P^*P + B^*\left(Q^*Q\right)^{-m} B = P^*P + B^*\left(Q^*Q\right)^{\frac{-m}{2}} \left(Q^*Q\right)^{\frac{-m}{2}} B = P^*P + B^*\left(Q^*Q\right)^{\frac{-m}{2}} \left(Q^*Q\right)^{\frac{-m}{2}} B = I$$

Hence Sys.(1.5) has a positive definite solution.

3. Iterative Method for the System

In this section, we will investigate the iterative solution of the Sys.(1.5). From this section to the end of the paper we will consider the matrices A, B are normal satisfing $A^{-1}B = BA^{-1}$ and $A^{-1}B^* = B^*A^{-1}$.

Let us consider the iterative processes

$$\begin{aligned} X_0 &= I, \qquad Y_0 = I \\ X_{s+1} &= I - A^*Y_s^{-n}A \\ Y_{s+1} &= I - B^*X_s^{-m}B \qquad s = 0, 1, 2, \cdots. \end{aligned} \tag{1.14}$$

Lemma 1 *For the Sys.(1.5), we have*

$$\begin{aligned} AX_s &= X_sA, \qquad BY_s = Y_sB \\ A^{-1}Y_s &= Y_sA^{-1}, \qquad B^{-1}X_s = X_sB^{-1} \end{aligned} \tag{1.15}$$

where $X_s, Y_s, s = 0, 1, 2, \cdots$ are matrices generated from the sequences (1.14).

Proof: Since $X_0 = Y_0 = I$, then

$$\begin{aligned} AX_0 &= X_0A, \qquad BY_0 = Y_0B \\ A^{-1}Y_0 &= Y_0A^{-1}, \qquad B^{-1}X_0 = X_0B^{-1} \end{aligned}$$

Using the conditions $AA^* = A^*A, BB^* = B^*B$, we obtain

$$AX_1 = A\left(I - A^*A\right) = \left(I - A^*A\right)A = X_1A.$$

Also, we have

$$BY_1 = Y_1 B.$$

Using the conditions $A^{-1}B = BA^{-1}, A^{-1}B^* = B^*A^{-1}$, we obtain

$$A^{-1}Y_1 = A^{-1}\left(I - B^*B\right) = A^{-1} - B^*BA^{-1} = \left(I - B^*B\right)A^{-1} = Y_1 A^{-1}.$$

By the same manner, we get

$$B^{-1}X_1 = X_1 B^{-1}.$$

Further, assume that for each k, we have

$$AX_k = X_k A, BY_k = Y_k B$$
$$A^{-1}Y_k = Y_k A^{-1}, B^{-1}X_k = X_k B^{-1} \tag{1.16}$$

Now, by induction, we will prove

$$AX_{k+1} = X_{k+1}A, \quad BY_{k+1} = Y_{k+1}B$$
$$A^{-1}Y_{k+1} = Y_{k+1}A^{-1}, \quad B^{-1}X_{k+1} = X_{k+1}B^{-1}$$

Since the two matrices A, B are normal, then by using the equalities (0.16), we have

$$AX_{k+1} = A\left(I - A^*Y_k^{-n}A\right) = A - AA^*Y_k^{-n}A = A - A^*Y_k^{-n}AA = \left(I - A^*Y_k^{-n}A\right)A = X_{k+1}A.$$

Similarly

$$BY_{k+1} = Y_{k+1}B.$$

By using the conditions $A^{-1}B = BA^{-1}, A^{-1}B^* = B^*A^{-1}$ and the equalities (1.16), we have

$$A^{-1}Y_{k+1} = A^{-1}\left(I - B^*X_k^{-m}B\right) = A^{-1} - A^{-1}B^*X_k^{-m}B = A^{-1} - B^*A^{-1}X_k^{-m}B = A^{-1} - B^*X_k^{-m}A^{-1}B$$
$$= A^{-1} - B^*X_k^{-m}BA^{-1} = \left(I - B^*X_k^{-m}B\right)A^{-1} = Y_{k+1}A^{-1}.$$

Also, we can prove

$$B^{-1}X_{k+1} = X_{k+1}B^{-1}.$$

Therefore, the equalities (1.15) are true for all $s = 0,1,2,\cdots$.

Lemma 2 *For the Sys.*(1.5), *we have*

$$X_s X_{s+1} = X_{s+1}X_s, \quad Y_s Y_{s+1} = Y_{s+1}Y_s \tag{1.17}$$

where $X_s, Y_s, s = 0,1,2,\cdots$ are matrices generated from the sequences (1.14).

Proof: Since $X_0 = Y_0 = I$ then $X_0 X_1 = X_1 X_0, Y_0 Y_1 = Y_1 Y_0$
By using the equalities (1.15), we have

$$X_1 X_2 = \left(I - A^*A\right)\left(I - A^*Y_1^{-n}A\right) = I - A^*A - A^*Y_1^{-n}A + A^*AA^*Y_1^{-n}A = I - A^*A - A^*Y_1^{-n}A + A^*AY_1^{-n}A^*A$$
$$= I - A^*A - A^*Y_1^{-n}A + A^*Y_1^{-n}AA^*A = \left(I - A^*Y_1^{-n}A\right)\left(I - A^*A\right) = X_2 X_1.$$

Similarly we get

$$Y_1 Y_2 = Y_2 Y_1.$$

Further, assume that for each k it is satisfied

$$X_{k-1}X_k = X_k X_{k-1}, \quad Y_{k-1}Y_k = Y_k Y_{k-1} \tag{1.18}$$

Now, by induction, we will prove

$$X_k X_{k+1} = X_{k+1}X_k, \quad Y_k Y_{k+1} = Y_{k+1}Y_k$$

From the equalities (1.18), we have

$$X_{k-1}^{-m}X_k^{-m} = X_k^{-m}X_{k-1}^{-m}, Y_{k-1}^{-n}Y_k^{-n} = Y_k^{-n}Y_{k-1}^{-n} \tag{1.19}$$

By using the equalities (1.15) and (1.19), we have

$$X_k X_{k+1} = \left(I - A^* Y_{k-1}^{-n} A\right)\left(I - A^* Y_k^{-n} A\right) = I - A^* Y_{k-1}^{-n} A - A^* Y_k^{-n} A + A^* Y_{k-1}^{-n} A A^* Y_k^{-n} A$$

$$= I - A^* Y_{k-1}^{-n} A - A^* Y_k^{-n} A + A^* A Y_{k-1}^{-n} Y_k^{-n} A^* A = I - A^* Y_{k-1}^{-n} A - A^* Y_k^{-n} A + A^* A Y_k^{-n} Y_{k-1}^{-n} A^* A$$

$$= I - A^* Y_{k-1}^{-n} A - A^* Y_k^{-n} A + A^* Y_k^{-n} A A^* Y_{k-1}^{-n} A = \left(I - A^* Y_k^{-n} A\right)\left(I - A^* Y_{k-1}^{-n} A\right) = X_{k+1} X_k.$$

By the same manner, we can prove

$$Y_k Y_{k+1} = Y_{k+1} Y_k.$$

Therefore, the equalities (1.17) are true for all $s = 0, 1, 2, \cdots$.

Theorem 4 *If A, B are satisfying the following conditions*:

(i) $\quad A^* A \le \dfrac{(\alpha - 1)}{n\alpha^{(2n+1)}} I$

(ii) $\quad B^* B \le \dfrac{(\alpha - 1)}{m\alpha^{(2m+1)}} I$

where $\alpha > 1$, then the Sys.(1.5) has a positive definite solution.

Proof: We consider the sequences (1.14). For X_0 we have $X_0 = I > \left(\dfrac{1}{\alpha}\right) I$.

For X_1 we obtain

$$X_1 = I - A^* Y_0^{-n} A = I - A^* A < I = X_0.$$

Applying the condition (i) we obtain

$$X_1 = I - A^* A \ge I - \frac{(\alpha - 1)}{n\alpha^{(2n+1)}} I > I - \frac{(\alpha - 1)}{\alpha} I = \frac{1}{\alpha} I.$$

i.e.

$$X_0 > X_1 > \frac{1}{\alpha} I$$

Also, we can prove that

$$Y_0 > Y_1 > \frac{1}{\alpha} I$$

So, assume that

$$X_{s-1} > X_s > \frac{1}{\alpha} I, \quad Y_{s-1} > Y_s > \frac{1}{\alpha} I. \tag{1.20}$$

Now, we will prove $X_s > X_{s+1} > \dfrac{1}{\alpha} I$ and $Y_s > Y_{s+1} > \dfrac{1}{\alpha} I.$

By using the in equalities (1.20) we have

$$X_{s+1} = I - A^* Y_s^{-n} A < I - A^* Y_{s-1}^{-n} A = X_s.$$

Similarly

$$Y_{s+1} < Y_s.$$

Also, by using the conditions $A^{-1} B = B A^{-1}$, $A^{-1} B^* = B^* A^{-1}$ and the equalities (1.20), we have

$$X_{s+1} = I - A^* Y_s^{-n} A > I - \alpha^n A^* A \ge I - \alpha^n \frac{(\alpha - 1)}{n\alpha^{(2n+1)}} I = I - \frac{(\alpha - 1)}{n\alpha^{(n+1)}} I > I - \frac{(\alpha - 1)}{\alpha} I = \frac{1}{\alpha} I.$$

Similarly, we have

$$Y_{s+1} > \frac{1}{\alpha} I.$$

Therefore, the inequalities (1.20) are true for all $s = 1, 2, \cdots$. Hence $\{X_s\}$ is monotonically decreasing and bounded from below by the matrix $\frac{1}{\alpha} I$. Consequently the sequence $\{X_s\}$ converges to a positive definite solution X. Also, the sequence $\{Y_s\}$ is monotonically decreasing and bounded from below by the matrix $\frac{1}{\alpha} I$ and converges to a positive definite solution Y. So (X, Y) is a positive definite solution of Sys.(1.5).

4. Estimation of the Errors

Theorem 5 *If A, B are satisfying the following conditions*

(i) $A^* A \leq \dfrac{(\alpha - 1)}{n\alpha^{(2n+1)}} I$

(ii) $B^* B \leq \dfrac{(\alpha - 1)}{m\alpha^{(2m+1)}} I$

then

$$\|X_s - X\| < \left(\frac{\alpha - 1}{\alpha}\right)^2 \|X_{s-2} - X\| \tag{1.21}$$

$$\|Y_s - Y\| < \left(\frac{\alpha - 1}{\alpha}\right)^2 \|Y_{s-2} - Y\| \tag{1.22}$$

where $\alpha > 1$, $X_s, Y_s, s = 2, 3, 4, \cdots$ are matrices generated from the sequences (1.14).

Proof: From Theorem 4 it follows that the sequences (1.14) are convergent to a positive definite solution of Sys.(1.5). We consider the spectral norm of the matrices $X_s - X, Y_s - Y$.

$$\|X_s - X\| = \left\|A^* Y^{-n} A - A^* Y_{s-1}^{-n} A\right\| = \left\|A^* \left(Y^{-n} - Y_{s-1}^{-n}\right) A\right\|$$

$$\leq \|A\|^2 \left\|Y^{-n} \left(Y_{s-1}^n - Y^n\right) Y_{s-1}^{-n}\right\| \leq \|A\|^2 \left\|Y^{-n}\right\| \left\|Y_{s-1}^{-n}\right\| \left\|Y_{s-1}^n - Y^n\right\| = \|A\|^2 \left\|Y^{-n}\right\| \left\|Y_{s-1}^{-n}\right\| \left\|(Y_{s-1} - Y) \sum_{i=1}^{n} Y_{s-1}^{n-i} Y^{i-1}\right\|$$

$$\leq \|A\|^2 \left\|Y^{-n}\right\| \left\|Y_{s-1}^{-n}\right\| \|Y_{s-1} - Y\| \left\|\sum_{i=1}^{n} Y_{s-1}^{n-i} Y^{i-1}\right\| \leq \|A\|^2 \left\|Y^{-n}\right\| \left\|Y_{s-1}^{-n}\right\| \|Y_{s-1} - Y\| \sum_{i=1}^{n} \left\|Y_{s-1}^{n-i}\right\| \left\|Y^{i-1}\right\|$$

According to Theorem 4 we have

$$Y_{s-1}^n > \frac{1}{\alpha^n} I, \ Y^n \geq \frac{1}{\alpha^n} I, \ Y \leq Y_s < Y_0 = I.$$

Consequently

$$\left\|Y_{s-1}^{-n}\right\| < \alpha^n, \left\|Y^{-n}\right\| \leq \alpha^n, \left\|Y_{s-1}^{n-i}\right\| \leq 1, \left\|Y^{i-1}\right\| \leq 1$$

Then we get

$$\|X_s - X\| < n\alpha^{2n} \|A\|^2 (Y_{s-1} - Y) \leq n\alpha^{2n} \frac{(\alpha - 1)}{n\alpha^{(2n+1)}} \|Y_{s-1} - Y\| = \frac{(\alpha - 1)}{\alpha} \|Y_{s-1} - Y\| \tag{1.23}$$

Also, we have

$$\|Y_s - Y\| = \left\|B^* X^{-m} B - B^* X_{s-1}^{-m} B\right\| = \left\|B^* \left(X^{-m} - X_{s-1}^{-m}\right) B\right\| \leq \|B\|^2 \left\|X^{-m} \left(X_{s-1}^m - X^m\right) X_{s-1}^{-m}\right\|$$

$$\leq \|B\|^2 \left\|X^{-m}\right\| \left\|X_{s-1}^{-m}\right\| \left\|X_{s-1}^m - X^m\right\| \leq \|B\|^2 \left\|X^{-m}\right\| \left\|X_{s-1}^{-m}\right\| \|X_{s-1} - X\| \left\|\sum_{i=1}^{m} X_{s-1}^{m-i} X^{i-1}\right\|$$

$$\leq \|B\|^2 \left\|X^{-m}\right\| \left\|X_{s-1}^{-m}\right\| \|X_{s-1} - X\| \sum_{i=1}^{m} \left\|X_{s-1}^{m-i}\right\| \left\|X^{i-1}\right\|$$

According to Theorem 4 we have

$$X_{s-1}^m > \frac{1}{\alpha^m}I, \ X^m \geq \frac{1}{\alpha^m}I, \ X \leq X_s < X_0 = I$$

Consequently

$$\left\|X_{s-1}^{-m}\right\| < \alpha^m, \left\|X^{-m}\right\| \leq \alpha^m, \left\|X_{s-1}^{m-i}\right\| \leq 1, \left\|X^{i-1}\right\| \leq 1$$

then we get

$$\left\|Y_s - Y\right\| < m\alpha^{2m}\left\|B\right\|^2\left\|X_{s-1} - X\right\| \leq m\alpha^{2m}\frac{(\alpha-1)}{m\alpha^{(2m+1)}}\left\|X_{s-1} - X\right\| = \frac{(\alpha-1)}{\alpha}\left\|X_{s-1} - X\right\| \tag{1.24}$$

By using (1.24) in (1.23), we have

$$\left\|X_s - X\right\| < \frac{(\alpha-1)}{\alpha}\left\|Y_{s-1} - Y\right\| < \left(\frac{\alpha-1}{\alpha}\right)^2\left\|X_{s-2} - X\right\|.$$

Similarly, by using (1.23) in (1.24), we have

$$\left\|Y_s - Y\right\| < \frac{(\alpha-1)}{\alpha}\left\|X_{s-1} - X\right\| < \left(\frac{\alpha-1}{\alpha}\right)^2\left\|Y_{s-2} - Y\right\|.$$

Theorem 6 *If A, B are satisfying the following conditions*:

(i) $\quad A^*A \leq \dfrac{(\alpha-1)}{n\alpha^{(2n+1)}}I$

(ii) $\quad B^*B \leq \dfrac{(\alpha-1)}{m\alpha^{(2m+1)}}I$

where $\alpha > 1$, and after *s* iterative steps of the iterative process (1.14), we have
$\left\|I - Y_s^n Y_{s-1}^{-n}\right\| < \varepsilon, \left\|I - X_s^m X_{s-1}^{-m}\right\| < \varepsilon$, then

$$\left\|X_s + A^*Y_s^{-n}A - I\right\| < \frac{(\alpha-1)\varepsilon}{\alpha} \tag{1.25}$$

$$\left\|Y_s + B^*X_s^{-m}B - I\right\| < \frac{(\alpha-1)\varepsilon}{\alpha} \tag{1.26}$$

Proof: Since

$$X_s + A^*Y_s^{-n}A - I = X_s + A^*Y_s^{-n}A - X_s - A^*Y_{s-1}^{-n}A = A^*\left(Y_s^{-n} - Y_{s-1}^{-n}\right)A.$$

Taking the norm of both sides, we have

$$\left\|X_s + A^*Y_s^{-n}A - I\right\| = \left\|A^*\left(Y_s^{-n} - Y_{s-1}^{-n}\right)A\right\| \leq \left\|A\right\|^2\left\|Y_s^{-n}\left(I - Y_s^n Y_{s-1}^{-n}\right)\right\| \leq \frac{(\alpha-1)}{n\alpha^{(2n+1)}}\left\|Y_s^{-n}\right\|\left\|I - Y_s^n Y_{s-1}^{-n}\right\|$$

$$< \frac{(\alpha-1)}{n\alpha^{(2n+1)}}\alpha^n\left\|I - Y_s^n Y_{s-1}^{-n}\right\| = \frac{(\alpha-1)}{n\alpha^{(n+1)}}\left\|I - Y_s^n Y_{s-1}^{-n}\right\| < \frac{(\alpha-1)}{\alpha}\left\|I - Y_s^n Y_{s-1}^{-n}\right\| < \frac{(\alpha-1)\varepsilon}{\alpha}.$$

Also,

$$Y_s + B^*X_s^{-m}B - I = Y_s + B^*X_s^{-m}B - Y_s - B^*X_{s-1}^{-m}B = B^*\left(X_s^{-m} - X_{s-1}^{-m}\right)B.$$

Taking the norm of both sides, we get

$$\left\| Y_s + B^* X_s^{-m} B - I \right\| = \left\| B^* \left(X_s^{-m} - X_{s-1}^{-m} \right) B \right\| \le \left\| B \right\|^2 \left\| X_s^{-m} \left(I - X_s^m X_{s-1}^{-m} \right) \right\| \le \frac{(\alpha-1)}{m\alpha^{(2m+1)}} \left\| X_s^{-m} \right\| \left\| I - X_s^m X_{s-1}^{-m} \right\|$$

$$< \frac{(\alpha-1)}{m\alpha^{(2m+1)}} \alpha^m \left\| I - X_s^m X_{s-1}^{-m} \right\| = \frac{(\alpha-1)}{m\alpha^{(m+1)}} \left\| I - X_s^m X_{s-1}^{-m} \right\| < \frac{(\alpha-1)}{\alpha} \left\| I - X_s^m X_{s-1}^{-m} \right\| < \frac{(\alpha-1)\varepsilon}{\alpha}.$$

5. Numerical Examples

In this section the numerical examples are given to display the flexibility of the method. The solutions are computed for some different matrices A, B with different orders. In the following examples we denote X, Y the solutions which are obtained by iterative method (1.14) and

$$\varepsilon_1(X) = \left\| X - X_s \right\|_\infty, \varepsilon_2(X) = \left\| X_s + A^* Y_s^{-n} A - I \right\|_\infty, \varepsilon_1(Y) = \left\| Y - Y_s \right\|_\infty, \varepsilon_2(Y) = \left\| Y_s + B^* X_s^{-m} B - I \right\|_\infty .$$

Example 1 *Consider Sys.*(1.5) *with* $n = 3, m = 2$ *and normal matrices*

$$A = 0.1 \begin{pmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.1 \\ 0.1 & 0.1 & 0.4 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0.5 & 0.1 & -0.1 \\ 0.1 & 0.1 & 0.4 \\ -0.1 & 0.4 & 0.2 \end{pmatrix}$$

By computation, we get

$$X = \begin{pmatrix} 0.998324 & -0.00348299 & -0.00231391 \\ -0.00348299 & 0.992508 & -0.00420267 \\ -0.00231391 & -0.00420267 & 0.995275 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0.728832 & -0.0215192 & 0.0280578 \\ -0.0215192 & 0.817073 & -0.113007 \\ 0.0280578 & -0.113007 & 0.786525 \end{pmatrix}$$

$$\lambda_r(X) = \{0.999998, 0.998302, 0.987807\}$$

$$\mu_r(Y) = \{0.92206, 0.723913, 0.686458\}$$

The results are given in the **Table 1**.

Example 2 *Consider Sys.*(1.5) *with* $n = 15, m = 10$ *and matrices*

$$A = 0.2 \begin{pmatrix} 1 & 2 & 1 & 5 \\ 5 & 2 & 5 & 1 \\ 2 & 8 & 1 & 4 \\ 1 & 4 & 1 & 3 \end{pmatrix}$$

and

$$B = 0.2 \begin{pmatrix} 2 & 4 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 4 \\ -6 & 1 & 4 & 2 \end{pmatrix}$$

By computation, we get

$$X = 10^{-2} \begin{pmatrix} 98.2384 & -2.19737 & -1.61786 & -1.42586 \\ -2.19737 & 94.6191 & -1.73678 & -3.28209 \\ -1.61786 & -1.73678 & 98.4733 & -1.2119 \\ -1.42586 & -3.28209 & -1.2119 & 97.0555 \end{pmatrix}$$

Table 1. Error analysis for Example 1.

s	$\varepsilon_1(X)$	$\varepsilon_1(Y)$	$\varepsilon_2(X)$	$\varepsilon_2(Y)$
0	7.49206E−03	2.71168E−01	3.00000E−03	2.70000E−01
1	4.49206E−03	3.47475E−03	4.30417E−03	1.21171E−03
2	1.87889E−04	2.26304E−03	6.19423E−05	2.16047E−03
3	1.25947E−04	1.02573E−04	1.20048E−04	3.39126E−05
4	5.89833E−06	6.86609E−05	1.95510E−06	6.54498E−05
5	3.94324E−06	3.21107E−06	3.75901E−06	1.06385E−06
6	1.84224E−07	2.14722E−06	6.10146E−08	2.04688E−06
7	1.23210E−07	1.00339E−07	1.17451E−07	3.32336E−08
8	5.75853E−09	6.71056E−08	1.90736E−09	6.39693E−08

Table 2. Error analysis for Example 2.

s	$\varepsilon_1(X)$	$\varepsilon_1(Y)$	$\varepsilon_2(X)$	$\varepsilon_2(Y)$
0	5.38087E−02	2.53842E−02	3.52000E−02	2.28000E−02
3	5.89949E−03	2.21295E−03	2.26209E−03	1.03086E−03
6	1.28148E−03	4.23324E−04	5.63284E−04	1.49409E−04
9	2.63304E−04	1.00522E−04	9.15826E−05	4.36395E−05
12	6.35004E−05	2.10415E−05	2.75006E−05	7.29690E−06
15	1.33387E−05	5.09274E−06	4.62298E−06	2.20407E−06
18	3.23064E−06	1.07076E−06	1.39800E−06	3.71047E−07
21	6.79370E−07	2.59387E−07	2.35413E−07	1.12241E−07
24	1.64580E−07	5.45489E−08	7.12162E−08	1.89019E−08
27	3.46115E−08	1.32148E−08	1.19933E−08	5.71823E−09

$$Y = 10^{-3} \begin{pmatrix} 974.616 & -2.71766 & 8.04079 & -1.70724 \\ -2.71766 & 988.835 & -8.8286 & -7.26638 \\ 8.04079 & -8.8286 & 984.322 & -12.1189 \\ -1.70724 & -7.26638 & -12.1189 & 985.411 \end{pmatrix}$$

$$\lambda_r(X) = \{0.999969, 0.993351, 0.985387, 0.905156\}$$

$$\mu_r(Y) = \{0.999774, 0.994647, 0.973194, 0.965569\}$$

The results are given in **Table 2**.

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