

Semilinear Venttsel' Problems in Fractal Domains

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Abstract

We study a semilinear parabolic problem with a semilinear dynamical boundary condition in an irregular domain with fractal boundary. Local existence, uniqueness and regularity results for the mild solution, are established via a semigroup approach. A sufficient condition on the initial datum for global existence is given.

Keywords

Energy Forms, Fractal Domains, Trace Theorems, Semigroups, Semilinear Parabolic Equations

1. Introduction

In this paper we study a semilinear problem in a fractal domain with semilinear dynamical boundary conditions. The model problem, we consider can be formally stated as follows:

$$(\bar{P}) \quad \begin{cases} u_t(t, P) - \Delta u(t, P) = J(u(t, P)) & \text{in } [0, T] \times \Omega, \\ u_t(t, P) - c_0 \Delta_F u(t, P) + b(P)u(t, P) = -\frac{\partial u(t, P)}{\partial n} + J(u(t, P)) & \text{on } [0, T] \times F, \\ u(0, P) = \phi & \text{in } \Omega, \end{cases}$$

where Ω is the (open) snowflake domain and $F = \partial\Omega$ is the union of three Koch curves (see Section 2). J is a non linear function from a subset of $L^2(\Omega, m)$ into $L^2(\Omega, m)$; m is the sum of the 2-dimensional Lebesgue measure and of the Hausdorff measure of F (see Section 2.1). Δ_F denotes the Laplace operator defined on F (see (3.4) in Section 3), c_0 is a positive constant, b is a strictly positive continuous function

in $\bar{\Omega}$, $\frac{\partial u}{\partial n}$ is the normal derivative across F , intended in a suitable sense.

More precisely, we assume that $J(u)$ is a non linear mapping from $L^{2p}(\Omega, m)$ to $L^2(\Omega, m)$ for any fixed $p > 1$, locally Lipschitz *i.e.* Lipschitz on bounded sets in $L^{2p}(\Omega, m)$ with Lipschitz constant $l(r)$ restricted to $B(0, r) \subset L^{2p}(\Omega, m)$, satisfying a suitable growth condition (see condition (g)) in Section 4). Examples of this type of non linearity include e.g. $J(u) = u|u|^{p-1}$, $p > 1$ which occurs in combustion theory (see [11]) and in the Navier Stokes system (see [2]).

Problem (\bar{P}) presents a non linear dynamical boundary condition (known also as Venttsel' boundary condition [3]). Problem (\bar{P}) models a fluid diffusion within a semipermeable membrane and heat flow subject to non linear cooling on the boundary (see [4] [5]). The literature on boundary value problems with dynamical conditions is huge, we refer to [6] for a derivation of such boundary conditions and to [7] and the references listed in. All these papers deal with smooth domains. The case of irregular domains is studied in [8]-[12].

In the present case we consider the case in which the non linearity appears both in bulk and on the boundary. We study the problem by a semigroup approach. More precisely we consider the corresponding abstract Cauchy problem:

$$(P) \quad \begin{cases} \frac{du(t)}{dt} = Au(t) + J(u(t)), & 0 \leq t \leq T \\ u(0) = \phi \end{cases} \quad (1.1)$$

where $A: \mathcal{D}(A) \subset L^2(\Omega, m) \rightarrow L^2(\Omega, m)$ is the generator associated to the energy form E introduced in (3.8), T is a fixed positive real number, ϕ is a given function in $L^2(\Omega, m)$. We assume that J is a mapping from $L^{2p}(\Omega, m) \rightarrow L^2(\Omega, m)$, $p > 1$ locally Lipschitz *i.e.* Lipschitz on bounded sets in $L^{2p}(\Omega, m)$; we let $l(r)$ denote the Lipschitz constant of J :

$$\|J(u) - J(v)\|_{L^2(\Omega, m)} \leq l(r)\|u - v\|_{L^{2p}(\Omega, m)} \quad (1.2)$$

whenever $\|u\|_{L^{2p}(\Omega, m)} \leq r, \|v\|_{L^{2p}(\Omega, m)} \leq r$.

A is the generator of the analytic contraction positivity preserving semigroup $T(t)$ from $L^2(\Omega, m)$ into $L^2(\Omega, m)$, associated to E . We study problem (P) via the corresponding integral equation

$$u(t) = T(t)\phi + \int_0^t T(t-s)J(u(s))ds. \quad (1.3)$$

In order to prove the existence of the solutions to (1.3) the usual way is to use a contraction argument in suitable Banach spaces see e.g. [13]. Usually the functional setting is that of an interpolation space between the domain of the generator A and $L^2(\Omega, m)$ or the domain of a fractional power of $-A$, we refer the reader to [13]-[17]. In our fractal case we do not know the domain of A . We stress the fact that it is not neither known a characterization of the domain of the fractal Laplacian Δ_F . To overcome this difficulty we adapt the abstract approach in [18] to prove local existence and uniqueness results for the mild solution. The key tool in [18] is an assumption on the estimate of the semigroup $T(t)$ as a bounded operator from $L^2(\Omega)$ to $L^{2p}(\Omega)$ (see (2.1) in [18]). In the present case we take into account that our problem has a probabilistic interpretation [19]; this, in turn, allows us to deduce an analogue estimate of $T(t)$ as a bounded map from $L^2(\Omega, m)$ to $L^{2p}(\Omega, m)$ see (3.15). We then deal with the strong formulation of the B.V.P. satisfied by the mild solution, which is of course of great interest in the applications, actually we prove that the solution of problem (P) solves in a suitable sense Problem (\bar{P}) see Theorems 5.1 and 5.2.

The layout of the paper is the following in Section 2 we recall the preliminaries on the geometry and the functional spaces. In Section 3 we consider the energy forms and the associated semigroups. In Section 4 we consider the abstract Cauchy problem (P) and we prove local and global existence results. Finally in Section 5 we prove that the solution of the abstract Cauchy problem (P) solves problem (\bar{P}) in a suitable sense.

2. Preliminaries

2.1. Geometry

In the paper we denote by $P = (x_1, x_2)$ points in \mathbb{R}^2 , by $|P - P_0|$ the Euclidean distance and by $B(P_0, r) = \{P \in \mathbb{R}^2 : |P - P_0| < r\}$, $P_0 \in \mathbb{R}^2, r > 0$ the Euclidean balls. By the Koch snowflake F , we will denote

the union of three coplanar Koch curves (see [20]) K_1 , K_2 and K_3 as shown in Figure 1. We assume that the junction points A_1 , A_3 and A_5 are the vertices of a regular triangle with unit side length, i.e. $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$. From now on we assume that a clockwise orientation is given on F .

The Hausdorff dimension of the Koch snowflake is given by $d_f = \frac{\ln 4}{\ln 3}$. This fractal is no longer self-similar (and hence, not nested).

One can define, in a natural way, a finite Borel measure μ supported on F by

$$\mu := \mu_1 + \mu_2 + \mu_3, \quad (2.1)$$

where μ_i denotes the normalized d_f -dimensional Hausdorff measure, restricted to K_i , $i = 1, 2, 3$.

The measure μ has the property that there exist two positive constants c_1 , c_2 such that

$$c_1 r^d \leq \mu(B(P, r) \cap F) \leq c_2 r^d, \quad \forall P \in F, \quad (2.2)$$

where $d = d_f = \frac{\log 4}{\log 3}$ and where $B(P, r)$ denotes the Euclidean ball in \mathbb{R}^2 . As μ is supported on F , it

is not ambiguous to write in (2.2) $\mu(B(P, r))$ in place of $\mu(B(P, r) \cap F)$. In the terminology of the following section we say that F is a d -set with $d = d_f$ according to [21].

Remark 2.1. The Koch snowflake can be also regarded as a fractal manifold (see [22]).

We denote by Ω the (open) snowflake domain.

2.2. Functional Spaces

By $L^2(\cdot)$ we denote the Lebesgue space with respect to the Lebesgue measure \mathcal{L}_2 on subsets of \mathbb{R}^2 , which will be left to the context whenever that does not create ambiguity. By $L^2(F)$ we denote the Hilbert space of square summable functions on F with respect to the invariant measure μ . Let \mathcal{S} be a closed set of \mathbb{R}^2 , by $C(\mathcal{S})$ we denote the space of continuous functions on \mathcal{S} , by $C_0(\mathcal{S})$ we denote the space of continuous functions vanishing on $\partial\mathcal{S}$. Let \mathcal{G} be an open set of \mathbb{R}^2 , by $H^s(\mathcal{G})$, where $s \in \mathbb{R}^+$ we denote the usual (possibly fractional) Sobolev spaces (see [23]); $H_0^s(\mathcal{G})$ is the closure of $\mathcal{D}(\mathcal{G})$, (the infinitely differentiable functions with compact support on \mathcal{G}), with respect to the $\|\cdot\|_{H^s}$ -norm.

We now recall a trace theorem.

For f in $H^s(\mathcal{G})$, we put

$$\gamma_0 f(P) = \lim_{r \rightarrow 0} \frac{1}{|B(P, r) \cap \mathcal{G}|} \int_{B(P, r) \cap \mathcal{G}} f(Q) d\mathcal{L}_2 \quad (2.3)$$

at every point $P \in \bar{\mathcal{G}}$ where the limit exists. It is known that the limit (2.3) exists at *quasi every* $P \in \bar{\mathcal{G}}$ with respect to the $(s, 2)$ -capacity [24].

Definition 2.2. Let $\mathcal{S} \subset \mathbb{R}^D$ be a closed non-empty subset. It is a d -set ($0 < d \leq D$) if there exists a Borel

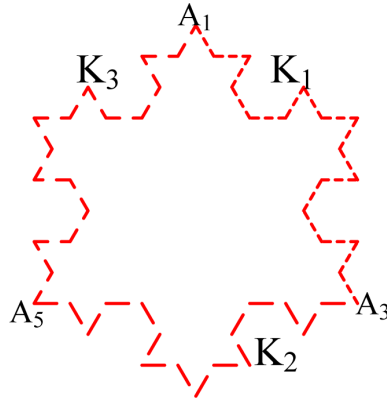


Figure 1. The snowflake domain Ω .

measure $\tilde{\mu}$ with $\text{supp } \tilde{\mu} = \mathcal{S}$ such that for some constants $c_1 = c_1(\mathcal{S}) > 0$ and $c_2 = c_2(\mathcal{S}) > 0$

$$c_1 r^d \leq \tilde{\mu}(B(P, r)) \leq c_2 r^d \quad (P \in \mathcal{S}, 0 < r \leq 1). \quad (2.4)$$

Such a $\tilde{\mu}$ is called a d -measure on \mathcal{S} .

Proposition 2.3. The set F is a d -set with $d = d_f$. The measure μ is a d -measure.

See [22] and [25].

Throughout the paper c will denote possibly different constants.

We now come to the definition of the Besov spaces.

Actually there are many equivalent definitions of these spaces see for instance [21] and [26]. We recall here the one which best fits our aims and we will restrict ourselves to the case $0 < \alpha < 1$, $p = q = 2$; the general setting being much more involved see [18]. By $B_\alpha^{2,2}(F)$ we denote the space of functions

$$B_\alpha^{2,2}(F) = \left\{ u \in L^2(F) : \|u\|_{B_\alpha^{2,2}(F)} < +\infty \right\}$$

where

$$\|u\|_{B_\alpha^{2,2}(F)} = \|u\|_{L^2(F)} + \left(\iint_{|x-y| \leq 1} \frac{|u(x) - u(y)|^2}{|x-y|^{2\alpha+d_f}} d\mu(x) d\mu(y) \right)^{\frac{1}{2}}$$

Theorem 2.4. Let $\alpha = \frac{d_f}{2}$, then $B_\alpha^{2,2}(F)$ is the trace space to F of $H^1(\Omega)$ in the following sense:

- 1) γ_0 is a continuous linear operator from $H^1(\Omega)$ to $B_\alpha^{2,2}(F)$,
- 2) there is a continuous linear operator Ext from $B_\alpha^{2,2}(F)$ to $H^1(\Omega)$ such that $\gamma_0 \circ \text{Ext}$ is the identity operator in $B_\alpha^{2,2}(F)$.

For the proof we refer to Theorem 1 of Chapter VII in [21], see also [26].

From now on we denote $\gamma_0 u$ by $u|_F$.

3. Energy Forms and Semigroups Associated

3.1. The Energy Form E

In Definition 4.5 of [22] a Lagrangian measure \mathcal{L}_F on F and the corresponding energy form E_F as

$$E_F(u, v) = \int_F d\mathcal{L}_F(u, v) \quad (3.1)$$

with domain $\mathcal{D}(F)$ have been introduced. The domain $\mathcal{D}(F)$, which is a Hilbert space with norm

$$\|u\|_{\mathcal{D}(F)} = \left(\|u\|_{L^2(F)}^2 + E_F(u, u) \right)^{\frac{1}{2}} \quad (3.2)$$

has been characterized in terms of the domains of the energy forms on K_i (see [22] Theorem 4.6).

In the following we will omit the subscript F , the Lagrangian measure will be simply denoted by $\mathcal{L}(u, v)$ and we will set $\mathcal{L}[u] = \mathcal{L}(u, u)$, an analogous notation will be adopted for the energies.

In the following we shall also use the form $E_F(u, v)$ which is obtained from $E_F[u]$ by the polarization identity:

$$E_F(u, v) = \frac{1}{2} \{ E_F[u+v] - E_F[u] - E_F[v] \}, \quad u, v \in \mathcal{D}(F). \quad (3.3)$$

It can be proved as in Proposition 3.1 of [22], that:

Proposition 3.1. In the previous notations and assumptions the form E_F with domain $\mathcal{D}(F)$ is a regular Dirichlet form in $L^2(F)$ and the space $\mathcal{D}(F)$ is a Hilbert space under the intrinsic norm (3.2).

For the definition and properties of regular Dirichlet forms we refer to [27]. We now define the Laplace operator on F . As $(E_F, \mathcal{D}(F))$ is a regular Dirichlet form on $L^2(F)$, with domain $\mathcal{D}(F)$ dense in $L^2(F)$, there exists (see Chap. 6, Theorem 2.1 in [28]) a unique self-adjoint, non positive operator Δ_F on $L^2(F)$ —with domain $\mathcal{D}(\Delta_F) \subseteq \mathcal{D}(F)$ dense in $L^2(F)$ —such that

$$E_F(u, v) = - \int_F (\Delta_F u) v d\mu, \quad u \in \mathcal{D}(\Delta_F), v \in \mathcal{D}(F). \quad (3.4)$$

Let $(\mathcal{D}(F))'$ denote the dual of the space $\mathcal{D}(F)$. We now introduce the Laplace operator on the fractal F as a variational operator from $\mathcal{D}(F) \rightarrow (\mathcal{D}(F))'$ by

$$E_F(u, w) = -\langle \Delta_F z, w \rangle_{(\mathcal{D}(F))', \mathcal{D}(F)} \quad (3.5)$$

for $z \in \mathcal{D}(F)$ and for all $w \in \mathcal{D}(F)$, where $\langle \cdot, \cdot \rangle_{(\mathcal{D}(F))', \mathcal{D}(F)}$ is the duality pairing between $(\mathcal{D}(F))'$ and $\mathcal{D}(F)$. We use the same symbol Δ_F to define the Laplace operator both as a self-adjoint operator in (3.4) and as a variational operator in (3.5). It will be clear from the context to which case we refer.

In the following we denote by

$$\bar{E}_F[u] = E_F[u] + \int_F b|u|^2 d\mu \quad (3.6)$$

defined in $\mathcal{D}(F)$; where b denotes a strictly positive continuous function in $\bar{\Omega}$. \bar{E}_F is also a Dirichlet form in $L^2(F)$.

Consider now the space of functions $u : \Omega \rightarrow \mathbb{R}$

$$V(\Omega, F) = \{u \in H^1(\Omega) : u|_F \in \mathcal{D}(F)\}. \quad (3.7)$$

The space $V(\Omega, F)$ is non trivial. We now introduce the energy form

$$E[u] = \int_{\Omega} |Du|^2 d\mathcal{L}_2 + \bar{E}_F[u|_F] \quad (3.8)$$

defined on the domain $V(\Omega, F)$. In the following we denote by $L^2(\Omega, m)$ the Lesbegue space with respect to the measure m with

$$dm = d\mathcal{L}_2 + d\mu. \quad (3.9)$$

By $E(u, v)$, we will denote the corresponding bilinear form

$$E(u, v) = \int_{\Omega} Du Dv d\mathcal{L}_2 + \bar{E}_F(u|_F, v|_F) \quad (3.10)$$

defined on $V(\Omega, F) \times V(\Omega, F)$.

Proposition 3.2. *The form E defined in (3.8) is a Dirichlet form in $L^2(\Omega, m)$ and the space $V(\Omega, F)$ is a Hilbert space equipped with the scalar product*

$$(u, v)_{V(\Omega, F)} = (u, v)_{H^1(\Omega)} + E_F(u|_F, v|_F) + (u|_F, v|_F)_{L^2(F)} \quad (3.11)$$

We denote by $\|u\|_{V(\Omega, F)}$ the norm in $V(\Omega, F)$, associated with (3.11), that is

$$\|u\|_{V(\Omega, F)} = \left(\|u\|_{H^1(\Omega)}^2 + \|u_F\|_{\mathcal{D}(F)}^2 \right)^{1/2}. \quad (3.12)$$

Resolvents and Semigroups Associated to Energy Forms

As $(E, V(\Omega, F))$ is a closed bilinear form on $L^2(\Omega, m)$, with domain $V(\Omega, F)$ dense in $L^2(\Omega, m)$, there exists (see chap. 6 Theorem 2.1 in [28]) a unique self-adjoint non positive operator A on $L^2(\Omega, m)$, with domain $\mathcal{D}(A) \subseteq V(\Omega, F)$ dense in $L^2(\Omega, m)$, such that

$$E(u, v) = -\int_{\Omega} Auv dm, \quad u \in \mathcal{D}(A), v \in V(\Omega, F). \quad (3.13)$$

Moreover in Theorem 13.1 of [27] it is proved that to each closed symmetric form E a family of linear operators $\{G_{\alpha}, \alpha > 0\}$ can be associated with the property

$$E(G_{\alpha}u, v) + \alpha(G_{\alpha}u, v) = (u, v), \quad u \in L^2(\Omega, m) \text{ and } v \in V(\Omega, F)$$

and this family is a strongly continuous resolvent with generator A , which also generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$.

With similar arguments it can be proved that there exists a nonnegative self-adjoint operator A_F with domain $\mathcal{D}(A_F) \subset \mathcal{D}(F)$ such that $\bar{E}_F(u, v) = -\int_F A_F u v d\mu$, $u \in \mathcal{D}(A_F), v \in \mathcal{D}(F)$ we denote by $T_F(t)$ the strongly continuous semigroup associated to \bar{E}_F on $L^2(F)$.

Proposition 3.3. *Let $\{T(t)\}_{t \geq 0}$ and $\{T_F(t)\}_{t \geq 0}$ be the semigroups generated by the operator A and A_F respectively, associated to the energy form in (3.13) and in (3.6). Then $\{T(t)\}_{t \geq 0}$ and $\{T_F(t)\}_{t \geq 0}$ are analytic*

contraction positive preserving semigroups in $L^2(\Omega, m)$ and $L^2(F, \mu)$ respectively.

Proof. The contraction property follows from Lumer Phillips Theorem on dissipative operators (Chapter 1 Theorem 4.3 in [16]). In order to prove the analyticity it will be enough to prove that there exists a positive α such that $E[u] + \|u\|_{L^2(\Omega, m)}^2 \geq \alpha \|u\|_{V(\Omega, F)}^2$ (see Proposition 3 Section 6 in Chapter XVII in [29]). Moreover since the semigroup is Markovian it is positive preserving. \square

Remark 3.4. It is well known that the symmetric and contraction analytic semigroup $T(t)$ uniquely determines analytic semigroups on the space L^p , $1 \leq p < \infty$ see (Theorem 1.4.1 [30]) which we still denote by $T(t)$ and by A_p its infinitesimal generator.

Let d_s denote the spectral dimension of F [31] [32]. By Theorem B3.7 in [33] one can prove

Proposition 3.5. For any $t > 0$, $T_F(t): L^1 \rightarrow L^\infty$ is a bounded operator and

$$\|T_F(t)\|_{\mathcal{L}(L^1 \rightarrow L^\infty)} \leq ct^{\frac{d_s}{2}}, \text{ for every } t \in (0, 1]. \quad (3.14)$$

Proof. The result follows by using the equivalence between (3.14) and Nash inequality. Actually it holds that for any $f \in \mathcal{D}(F)$

$$\|f\|_{L^2(F)}^{2+4d_s} \leq c \left(\bar{E}_F[f] + \|f\|_{L^2(F)} \right) \|f\|_{L^1(F)}^{4d_s}$$

(see [34]). \square

From Theorem 2.11 in [19] the following estimate on the decay of the heat semigroup holds.

Proposition 3.6. There exists a positive constant M such that

$$\|T(t)\|_{\mathcal{L}(L^1 \rightarrow L^\infty)} \begin{cases} Mt^{-\frac{n}{2}}, & \text{for every } t \in (0, 1] \\ Mt^{-\frac{d_s}{2}}, & \text{for every } t \in [1, \infty). \end{cases}$$

We will consider the case $n = 2$ and $d_s = 1$.

We remark that this property is called supercontractivity (see e.g. [30]).

From now on we set $\|T(t)\|_{\mathcal{L}(L^p \rightarrow L^q)} := \|T(t)\|_{L^p \rightarrow L^q}$, for $1 \leq p \leq q \leq \infty$.

We recall that for every $q > 1$ $T(t): L^q \rightarrow L^q$, and

$$\|T(t)\|_{L^q \rightarrow L^q} \leq 1.$$

From interpolation result theory (see e.g. [35]), it can be proved that for every $1 < p < r$

$$T(t): L^p \rightarrow L^r$$

with

$$\|T(t)\|_{L^p \rightarrow L^r} \leq \|T(t)\|_{L^1 \rightarrow L^\infty}^{1-\alpha} \|T(t)\|_{L^q \rightarrow L^q}^\alpha, \quad (3.15)$$

where $\alpha = 1 + \frac{1}{r} - \frac{1}{p}$ and $q = \frac{r}{\alpha}$.

In particular we will often use that $T(t)$ is bounded from $L^2 \rightarrow L^{2p}$, $p \in (1, \infty)$ with

$$\|T(t)\|_{L^2 \rightarrow L^{2p}} \leq \|T(t)\|_{L^1 \rightarrow L^\infty}^{1-\alpha} \|T(t)\|_{L^q \rightarrow L^q}^\alpha,$$

with $\alpha = \frac{1}{2} + \frac{1}{2p}$ and $q = \frac{2p}{\alpha}$.

Taking into account 2.6 and $\|T(t)\|_{L^q \rightarrow L^q} \leq 1$ we obtain

$$\|T(t)\|_{L^2 \rightarrow L^{2p}} \leq \begin{cases} M^{\frac{1}{2} - \frac{1}{2p}} t^{-\frac{n}{4}(1-1/p)}, & \text{for every } t \in (0, 1] \\ M^{\frac{1}{2} - \frac{1}{2p}} t^{-\frac{d_s}{4}(1-1/p)}, & \text{for every } t \in [1, \infty). \end{cases}$$

4. The Abstract Cauchy Problem: Local and Global Existence

We study the solvability of the Cauchy problem:

$$(P) \quad \begin{cases} \frac{du(t)}{dt} = Au(t) + J(u), & 0 \leq t \leq T \\ u(0) = \phi \end{cases} \quad (4.1)$$

where $A: \mathcal{D}(A) \subset L^2(\Omega, m) \rightarrow L^2(\Omega, m)$ is the generator associated to the energy form E introduced in (3.8), T is a fixed positive real number, ϕ is a given function in $L^2(\Omega, m)$. We assume that J is a mapping from $L^{2p}(\Omega, m) \rightarrow L^2(\Omega, m)$, $p > 1$ locally Lipschitz i.e. Lipschitz on bounded sets in $L^{2p}(\Omega, m)$; we let $l(r)$ denote the Lipschitz constant of J :

$$\|J(u) - J(v)\|_{L^2(\Omega, m)} \leq l(r) \|u - v\|_{L^{2p}(\Omega, m)} \quad (4.2)$$

whenever $\|u\|_{L^{2p}(\Omega, m)} \leq r$, $\|v\|_{L^{2p}(\Omega, m)} \leq r$. We also assume that $J(0) = 0$. This assumption is not necessary in all that follows but it simplifies the calculations (see [18]). In order to prove the local existence theorem we make the following assumption on the growth of $l(r)$ when $r \rightarrow \infty$:

$$(g) \text{ Let } a := \frac{n}{4} \left(1 - \frac{1}{p}\right) \text{ there exists } 0 < b < a : l(r) = \mathcal{O}\left(r^{\frac{1-a}{b}}\right), r \rightarrow \infty$$

we note that $0 < a < 1$, for $n \leq 4$ and $p > 1$.

Let $p > 1$. Following the approach in Theorem 2 in [18] and adapting the proof of Theorem 5.1 in [8] we have:

Theorem 4.1. *Let condition (g) hold. Let $K > 0$ be sufficiently small, if $\phi \in L^2(\Omega, m)$ and*

$$\limsup_{t \rightarrow 0} \|t^b T(t) \phi\|_{L^{2p}} < K. \quad (4.3)$$

There is a $T > 0$ and a unique

$$u \in C([0, T], L^2(\Omega, m)) \cap C((0, T], L^{2p}(\Omega, m))$$

with $u(0) = \phi$ and $\|t^b u(t)\|_{L^{2p}} < 2K$ satisfying for every $t \in [0, T]$:

$$u(t) = T(t) \phi + \int_0^t T(t-s) J(u(s)) ds \quad (4.4)$$

with the integral being both an L^2 -valued and L^{2p} -valued Bochner integral.

The claim of the Theorem is proved by a contraction mapping argument on suitable spaces of continuous functions with values in Banach spaces. We adapt the proof of Theorem 5.1 in [8] to the new functional setting and for the reader's convenience we recall it.

Proof. Let Y be the complete metric space defined as follows

$$Y = \left\{ u \in C([0, T], L^2(\Omega, m)) \cap C((0, T], L^{2p}(\Omega, m)) : u(0) = \phi; \|t^b u(t)\|_{L^{2p}(\Omega, m)} < 2K \text{ for all } t \in [0, T] \right\} \quad (4.5)$$

equipped with the metric

$$d(u, v) = \max \left\{ \|u - v\|_{C([0, T], L^2(\Omega, m))}, \sup_{(0, T]} t^b \|u(t) - v(t)\|_{L^{2p}(\Omega, m)} \right\}.$$

Since condition (g) holds we choose N such that $l(r) \leq Nr^{\frac{1-a}{b}}$ for $r \geq 1$.

For $u \in Y$, let $\mathcal{F}u = T(t) \phi + \int_0^t T(t-s) J(u(s)) ds$. By using arguments similar to those used in the proof of Lemma 2.1 of [36] we can prove that $\mathcal{F}u \in C([0, T], L^2(\Omega, m)) \cap C((0, T], L^{2p}(\Omega, m))$ and of course $\mathcal{F}u(0) = \phi$. We now prove that

$$\limsup_{t \rightarrow 0} \|t^b \mathcal{F}u(t)\|_{L^{2p}} < 2K \text{ for all } t \in [0, T]. \quad (4.6)$$

Taking into account (4.3) there exists $T > 0$ such that $\|t^b \mathcal{F}u(t)\|_{L^{2p}(\Omega, m)} \leq 2K$ for all $t \in [0, T]$.

$$\begin{aligned}
\|t^b \mathcal{F}u(t)\|_{L^{2p}(\Omega, m)} &\leq K + t^b \left\| \int_0^t T(t-s) J(u(s)) ds \right\|_{L^{2p}(\Omega, m)} \\
&\leq K + t^b \int_0^t \|T(t-s)\|_{L^2 \rightarrow L^{2p}} \|J(u(s))\|_{L^2(\Omega, m)} ds \\
&\leq K + t^b \int_0^t (t-s)^{-a} M^{\frac{1}{2} - \frac{1}{2p}} l \left(\|u(s)\|_{L^{2p}(\Omega, m)} \right) \|u(s)\|_{L^{2p}(\Omega, m)} ds
\end{aligned}$$

from (4.5) we have that

$$\|t^b \mathcal{F}u(t)\|_{L^{2p}(\Omega, m)} \leq K + M^{\frac{1}{2} - \frac{1}{2p}} t^b N (2K)^{\frac{1-a+b}{b}} \int_0^t (t-s)^{-a} s^{a-1-b} ds \leq K + NM^{\frac{1}{2} - \frac{1}{2p}} (2K)^{\frac{1-a+b}{b}} B,$$

where $B := \int_0^1 (1-s)^{-a} s^{a-1-b} ds$; thus choosing $K \leq \left(NBM^{\frac{1}{2} - \frac{1}{2p}} 2^{\frac{1-a+b}{b}} \right)^{\frac{-b}{1-a}}$ (4.6) is proved. It remains to prove

that, for a suitable choice of K , \mathcal{F} is a contraction.

$$\begin{aligned}
\|\mathcal{F}u(t) - \mathcal{F}v(t)\|_{L^2(\Omega, m)} &\leq \int_0^t \|T(t-s) [J(u(s)) - J(v(s))]\|_{L^2(\Omega, m)} ds \\
&\leq \int_0^t \|J(u(s)) - J(v(s))\|_{L^2(\Omega, m)} ds \\
&\leq N (2K)^{\frac{1-a}{b}} \int_0^t s^{a-1} \|u(s) - v(s)\|_{L^{2p}(\Omega, m)} ds \\
&\leq (2K)^{\frac{1-a}{b}} \|u - v\|_Y N \int_0^t s^{a-1-b} ds.
\end{aligned}$$

Therefore we have

$$\|\mathcal{F}u - \mathcal{F}v\|_{C([0, T], L^2(\Omega, m))} \leq (2K)^{\frac{1-a}{b}} NT^{a-b} / (a-b) \|u - v\|_Y.$$

We consider now $\|t^b (\mathcal{F}u(t) - \mathcal{F}v(t))\|_{L^{2p}(\Omega, m)}$. It holds

$$\begin{aligned}
&\|t^b (\mathcal{F}u(t) - \mathcal{F}v(t))\|_{L^{2p}(\Omega, m)} \\
&\leq t^b \int_0^t \|T(t-s) [J(u(s)) - J(v(s))]\|_{L^{2p}(\Omega, m)} ds \\
&\leq t^b \int_0^t \|T(t-s)\|_{L^2 \rightarrow L^{2p}} \|J(u(s)) - J(v(s))\|_{L^2(\Omega, m)} ds \\
&\leq M^{\frac{1}{2} - \frac{1}{2p}} t^b N (2K)^{\frac{1-a}{b}} \int_0^t (t-s)^{-a} s^{a-1} \|u(s) - v(s)\|_{L^{2p}(\Omega, m)} ds \\
&\leq M^{\frac{1}{2} - \frac{1}{2p}} t^b N (2K)^{\frac{1-a}{b}} \int_0^t (t-s)^{-a} s^{a-1-b} \|s^b (u(s) - v(s))\|_{L^{2p}(\Omega, m)} ds \\
&\leq M^{\frac{1}{2} - \frac{1}{2p}} t^b N (2K)^{\frac{1-a}{b}} \|u - v\|_Y \int_0^t (t-s)^{-a} s^{a-1-b} ds \\
&\sup_{t \in (0, T]} \|t^b (\mathcal{F}u(t) - \mathcal{F}v(t))\|_{L^{2p}(\Omega, m)} \leq M^{\frac{1}{2} - \frac{1}{2p}} N (2K)^{\frac{1-a}{b}} \|u - v\|_Y B.
\end{aligned}$$

In order to prove that it is a contraction it's enough to choose K such that $BM^{\frac{1}{2} - \frac{1}{2p}} N (2K)^{\frac{1-a}{b}} < 1$ and $(2K)^{\frac{1-a}{b}} NT^{a-b} / (a-b) < 1$. \square

Remark 4.2. If $J(u) = |u|^{p-1} u$ then $l(r) = \mathcal{O}(r^{p-1})$, $r \rightarrow \infty$. Thus condition (g) is satisfied for $b = \frac{1}{p-1} - \frac{n}{4p}$ with $p > 1 + \frac{4}{n}$.

Since $T(t)$ is an analytic semigroup on both $L^2(\Omega, m)$ and $L^{2p}(\Omega, m)$ from Corollary 2.1 in [18], the following regularity result holds (see also Theorem 5.3 in [8]).

Theorem 4.3. *Under the assumptions of Theorem 4.1 we have.*

- a) *The solution $u(t)$ can be continuously extended to a maximal interval $(0, T_\phi)$ as a solution of (4.4), until $\|u(t)\|_{L^{2p}(\Omega, m)} \rightarrow \infty$.*
b)

$$u \in C^1((0, T_\phi); L^2(\Omega, m)) \cap C((0, T_\phi); \mathcal{D}(A))$$

and satisfies

$$\frac{du(t)}{dt} = Au(t) + J(u), \text{ for every } t \in (0, T_\phi)$$

i.e. it is a classical solution.

Proof. As to the proof of condition a), we follow Theorem 4.2 in [18]. From the proof of Theorem 4.1 it turns out that the minimum existence time for the solution to the integral equation is as long as $\|t^\beta T(t)\phi\|_{L^{2p}(\Omega, m)} \leq K$, (see also Corollary 2.1. in [18]).

To prove that the mild solution is classical we use the classical regularity results for linear equations (see e.g. Theorem 4.3.4. in [13]) by proving that $J(u)$ is Hölder continuous on $(0, T]$ into L^2 for any fixed $T < T_\phi$. Taking into account the local Lipschitz continuity of $J(u)$ it is enough to show that $u(t)$ is Hölder continuous on $(\epsilon, T) \forall \epsilon > 0$ into L^{2p} . Let $\psi = u(\epsilon)$, we set $w(t) = T(t)\psi + \int_0^t T(t-s)J(w(s))ds$, if we prove that

$$w(t) \in C^1((0, T); L^2(\Omega)) \cap C((0, T); \mathcal{D}(A))$$

then, as $u(t+\epsilon) = w(t)$ due to the uniqueness of the solution of (4), then

$$u(t) \in C^1((\epsilon, T-\epsilon); L^2(\Omega)) \cap C((\epsilon, T-\epsilon); \mathcal{D}(A)),$$

for every $\epsilon > 0$, hence $u(t)$ is a classical solution (see claim b). Let $\sup_{t \in (0, T)} \|w\|_{L^{2p}} \leq R$. Since $T(t)$ is an analytic semigroup, $T(t)\phi$ is continuously differentiable on $(\epsilon, T) \forall \epsilon > 0$, hence Hölder continuous with any exponent $\beta \in (0, 1)$. It is enough to show that $v(t) = \int_0^t T(t-s)J(w(s))ds$ is Hölder continuous.

For $\lambda > \beta > 0$, $(\lambda - A)^{-\beta}$ is a bounded operator in L^2 and from Theorem 11.3 and 12.1 in [37] there exists a constant c such that

$$\|(\lambda - A_{2p})^\beta T(t)\|_{L^{2p}} \leq ct^{-\beta}, \quad t \in (0, T]$$

$$\|(T(t) - I)(\lambda - A_{2p})^{-\beta}\|_{L^{2p}} \leq ct^\beta, \quad t \in (0, T].$$

Now let $0 \leq t \leq t+\tau \leq T$, then

$$\begin{aligned} v(t+\tau) - v(t) &= \int_0^{t+\tau} T(t+\tau-s)J(w(s))ds - \int_0^t T(t-s)J(w(s))ds \\ &= \int_0^t (T(t+\tau-s) - T(t-s))J(w(s))ds + \int_t^{t+\tau} T(t+\tau-s)J(w(s))ds \\ &= \int_0^t (T(\tau) - I)T(t-s)J(w(s))ds + \int_0^\tau T(\sigma)J(w(t+\tau-\sigma))d\sigma = v_1(t) + v_2(t). \\ v_1(t) &= (T(\tau) - I)(\lambda - A_{2p})^{-\beta} \int_0^t (\lambda - A_{2p})^\beta T(t-s)J(w(s))ds \\ &= (T(\tau) - I)(\lambda - A_{2p})^{-\beta} \int_0^t (\lambda - A_{2p})^\beta T(\sigma/2)T(\sigma/2)(Jw(t-\sigma))d\sigma. \end{aligned}$$

Hence,

$$\begin{aligned} \|v_1(t)\|_{L^{2p}} &\leq \|(T(\tau) - I)(\lambda - A_{2p})^{-\beta}\|_{L^{2p}} \int_0^t \|(\lambda - A_{2p})^\beta T(\sigma/2)T(\sigma/2)J(w(t-\sigma))\|_{L^{2p}} d\sigma \\ &\leq c\tau^\beta \int_0^t \|(\lambda - A_{2p})^\beta T(\sigma/2)\|_{L^{2p}} \|T(\sigma/2)\|_{L^2 \rightarrow L^{2p}} \|J(w(t-\sigma))\|_{L^2(\Omega, m)} d\sigma \\ &\leq c\tau^\beta \int_0^t cM^{\frac{1}{2} - \frac{1}{2p}} (\sigma/2)^{-\beta} (\sigma/2)^{-a} l(R) R d\sigma. \end{aligned}$$

If we choose $\beta < 1 - a$, it follows $\|v_1(t)\|_{L^{2p}(\Omega, m)} \leq c\tau^\beta$. As to the function v_2 it holds

$$\begin{aligned} \|v_2(t)\|_{L^{2p}(\Omega, m)} &\leq \int_0^\tau \|T(\sigma)J(w(t+\tau-\sigma))\|_{L^{2p}} d\sigma \\ &\leq \int_0^\tau \|T(\sigma)\|_{L^2 \rightarrow L^{2p}} \|J(w(t+\tau-\sigma))\|_{L^2} d\sigma \\ &\leq M^{\frac{1}{2} - \frac{1}{2p}} \int_0^\tau (\sigma)^{-a} l(R)^{p-1} R d\sigma. \end{aligned}$$

Hence $\|v_2(t)\|_{L^{2p}} \leq c\tau^{(1-a)}$. Therefore if $\beta < 1 - a$ $v(t)$ is Hölder continuous on $[0, T]$ with exponent β . \square

We now give a sufficient condition on the initial datum in order to obtain a global solution adapting Theorem 3 (b) in [38] see also Theorem 5.4 in [8].

Theorem 4.4. *Let condition (g) hold. Let $q = \frac{2np}{n+4pb}$, $\phi \in L^q(\Omega, m)$, $\phi \geq 0$ a.e. and $\|\phi\|_{L^q(\Omega, m)}$ is sufficiently small, then there exists a nonnegative $u \in C([0, \infty), L^q(\Omega, m))$ which is a global solution of (4.4).*

Proof. Since $q < 2p$, from (3.15) it follows that $T(t)$ is a bounded operator from L^q into L^{2p} with

$$\|T(t)\|_{L^q \rightarrow L^{2p}} \leq M^{\frac{1}{q} - \frac{1}{2p}} t^{\frac{n}{2}(\frac{1}{q} - \frac{1}{2p})} \equiv M^{\frac{1}{q} - \frac{1}{2p}} t^{-b}$$

hence

$$\|t^b T(t)\phi\|_{L^{2p}} \leq M^{\frac{1}{q} - \frac{1}{2p}} \|\phi\|_{L^q(\Omega, m)};$$

by choosing $\|\phi\|_{L^q(\Omega, m)}$ sufficiently small from Theorem 4.1 there exists a local solution of (4.4),

$u \in C([0, T], L^q(\Omega, m))$. Furthermore from Theorem 4.1 $u \in C((0, T], L^{2p}(\Omega, m))$ and

$\|t^b u(t)\|_{L^{2p}} \leq 2M \|\phi\|_{L^q(\Omega, m)}$. From Theorem 4.3 (a) to show that $u(t)$ is a global solution it is enough to show that $\|u(t)\|_{L^{2p}(\Omega, m)}$ is bounded for every $t > 0$. We will prove that $\|t^b u(t)\|_{L^{2p}(\Omega, m)}$ is bounded for every $t > 0$, and we will use the notations of the proof in Theorem 4.1.

$$\begin{aligned} \|t^b u(t)\|_{L^{2p}(\Omega, m)} &\leq M^{\frac{1}{q} - \frac{1}{2p}} \|\phi\|_{L^q(\Omega, m)} + t^b \int_0^t \|T(t-s)\|_{L^2 \rightarrow L^{2p}} \|J(u(s))\|_{L^2(\Omega, m)} ds \\ &\leq M^{\frac{1}{q} - \frac{1}{2p}} \|\phi\|_{L^q(\Omega, m)} + M^{\frac{1}{2} - \frac{1}{2p}} N \left(2M^{\frac{1}{q} - \frac{1}{2p}} \|\phi\|_{L^q(\Omega, m)} \right)^{\frac{1-a+b}{b}} t^b \int_0^t (t-s)^{-a} s^{a-1-b} \|s^b u(s)\|_{L^{2p}(\Omega, m)} ds \\ &\leq M^{\frac{1}{q} - \frac{1}{2p}} \|\phi\|_{L^q(\Omega, m)} + NM^{\frac{1}{2} - \frac{1}{2p}} \left(2M^{\frac{1}{q} - \frac{1}{2p}} \|\phi\|_{L^q(\Omega, m)} \right)^{\frac{1-a+b}{b}} \sup_{t \in [0, T]} \|t^b u(t)\|_{L^{2p}(\Omega, m)} \int_0^1 (1-s)^{-a} s^{a-1-b} ds \end{aligned}$$

Let $f(T) = \sup_{t \in (0, T]} \|t^b u(t)\|_{L^{2p}(\Omega, m)}$, $f(T)$ is a continuous non decreasing function with $f(0) = 0$, which satisfies

$$f(T) \leq M^{\frac{1}{q} - \frac{1}{2p}} \|\phi\|_{L^q(\Omega, m)} + \left(2M^{\frac{1}{q} - \frac{1}{2p}} \|\phi\|_{L^q(\Omega, m)} \right)^{\frac{1-a+b}{b}} NM^{\frac{1}{2} - \frac{1}{2p}} Bf(T),$$

if $M^{\frac{1}{q} - \frac{1}{2p}} \|\phi\|_{L^q(\Omega, m)} \leq \alpha$ and $2^{\frac{1-a+b}{b}} NM^{\frac{1}{2} - \frac{1}{2p}} B\alpha^{\frac{1-a+b}{b}} < 1$ then $f(T)$ can never equal 2α . If it did we would have $2\alpha \leq \alpha + (2\alpha)^{\frac{1-a+b}{b}} NBM^{\frac{1}{2} - \frac{1}{2p}}$ i.e. $\alpha \leq (2\alpha)^{\frac{1-a+b}{b}} NBM^{\frac{1}{2} - \frac{1}{2p}}$ which is false. This proves that for $\|\phi\|_{L^q(\Omega, m)}$ sufficiently small $\|t^b u(t)\|_{L^{2p}(\Omega, m)}$ must remain bounded. \square

5. Strong Interpretation and Regularity Results

Theorem 5.1. Let u be the solution of problem (P). Then we have for every fixed $t \in [0, T]$

$$\begin{cases} u_t(t, P) - \Delta u(t, P) = J(u(t, P)) & \text{for a.e. } P \in \Omega \\ \frac{\partial u}{\partial n} \in (B_{\beta}^{2,2}(F))', & \beta = \frac{d_f}{2}, \\ u(0, P) = \phi(P) & \text{for a.e. } P \in \Omega \end{cases}$$

and for every $z \in \mathcal{D}(F)$

$$\langle u_t, z \rangle_{(\mathcal{D}(F))', \mathcal{D}(F)} = -c_0 E_F(u|_F, z) - \left\langle \frac{\partial u}{\partial n}, z \right\rangle_{(\mathcal{D}(F))', \mathcal{D}(F)} + \langle J(u), z \rangle_{(\mathcal{D}(F))', \mathcal{D}(F)} - \int_F bu|_F z d\mu \quad (5.7)$$

where $\frac{\partial u}{\partial n}$, is the inward “normal derivative”, to be defined in a suitable sense. Moreover

$$\frac{\partial u}{\partial n} \in C\left([0, T]; (B_{\beta}^{2,2}(F))'\right).$$

Proof. By proceeding as in Theorem 6.1 of [39] and taking into account that $J(u(t, \cdot)) \in L^2(\Omega, m)$ we obtain for each $t \in [0, T]$

$$u_t(t, P) = \Delta u(t, P) + J(u(t, P)) \text{ in } L^2(\Omega) \quad (5.8)$$

from this we deduce $\Delta u(t, P) = u_t(t, P) - J(u(t, P))$ and, since the right hand-side belongs to $C([0, T]; L^2(\Omega))$, we deduce that $\Delta u \in C([0, T]; L^2(\Omega))$ hence

$$u \in C([0, T]; V(\Omega))$$

where

$$V(\Omega) = \{u \in H^1(\Omega); \Delta u \in L^2(\Omega)\},$$

here the Laplacian is intended in the distributional sense. By proceeding as in (3.26) of [40] [41] we prove that, for every fixed t , the normal derivative $\frac{\partial u}{\partial n}$ is in $(B_{\beta}^{2,2}(F))'$, the dual of the space $B_{\beta}^{2,2}(F)$, where

$$\beta = \frac{d_f}{2} \text{ and}$$

$$\left\langle \frac{\partial u}{\partial n}, v|_F \right\rangle_{(B_{\beta}^{2,2}(F))', B_{\beta}^{2,2}(F)} = \int_{\Omega} Du(t, P) Dv(P) d\mathcal{L}_2 + \int_{\Omega} v(P) \Delta u(t, P) d\mathcal{L}_2 \quad (5.9)$$

for every $t \in [0, T]$ and every $v \in H^1(\Omega)$ and by proceeding as in 6.1 of [39] we prove that

$$\frac{\partial u}{\partial n} \in C\left([0, T]; (B_{\beta}^{2,2}(F))'\right).$$

Let ϕ be an arbitrary function in $V(\Omega, F)$, for every fixed $t \in [0, T]$ we multiply Equation (4.1) in (P) and we integrate over Ω

$$(u_t(t), \phi(t))_{L^2(\Omega, m)} = (Au(t), \phi(t))_{L^2(\Omega, m)} + (J(u(t)), \phi(t))_{L^2(\Omega, m)} \quad (5.10)$$

the left hand-side of (5.10) can be written as:

$$(u_t(t), \phi(t))_{L^2(\Omega)} + (u_t(t), \phi(t))_{L^2(F)}$$

from (3.13) we deduce

$$(u_t, \phi)_{L^2(\Omega)} + (u_t, \phi)_{L^2(F)} \quad (5.11)$$

$$= -\int_{\Omega} Du D\varphi d\mathcal{L}_2 - c_0 E_F(u|_F, \varphi) - (bu|_F, \varphi)_{L^2(F)} + (J(u), \varphi)_{L^2(\Omega)} + (J(u), \varphi)_{L^2(F)} \quad (5.12)$$

taking into account that $u \in V(\Omega)$ from (5.9), we have

$$\int_{\Omega} Du D\varphi d\mathcal{L}_2 = -\left\langle \frac{\partial u}{\partial n}, \varphi \right\rangle_{(B_{\beta}^{2,2}(F))', B_{\beta}^{2,2}(F)} + \int_{\Omega} \varphi \Delta u d\mathcal{L}_2$$

from (5.11) we have

$$(u_t, \varphi)_{L^2(F)} = c_0 \langle \Delta_F u, \varphi \rangle_{(\mathcal{D}(F))', \mathcal{D}(F)} - (bu|_F, \varphi)_{L^2(F)} - \left\langle \frac{\partial u}{\partial n}, \varphi \right\rangle_{(B_{\beta}^{2,2}(F))', B_{\beta}^{2,2}(F)} + (J(u), \varphi)_{L^2(F)}$$

by proceeding as in Section 6.1 of [39] it can be proved that $\forall t \in [0, T]$, $\frac{\partial u}{\partial n} \in (\mathcal{D}(F))'$ and the boundary condition holds in $(\mathcal{D}(F))'$ that is

$$u_t - c_0 \Delta_F u + bu = -\frac{\partial u}{\partial n} + J(u) \quad \text{in } (\mathcal{D}(F))'. \quad (5.13)$$

As a consequence of Theorem (5.1) the solution of problem (P) is the solution of the following problem. For every $t \in [0, T]$,

$$\begin{cases} u_t - \Delta u = J(u) & \text{in } L^2(\Omega), & j) \\ u_t - c_0 \Delta_F u + bu = -\frac{\partial u}{\partial n} + J(u) & \text{in } (\mathcal{D}(F))', & jj) \end{cases}$$

Theorem 5.2. Let u be the strict solution of problem (P). Then for every $t \in [0, T]$, $u(t, \cdot) \in C(\bar{\Omega})$.

Proof. For every $t \in [0, T]$ we consider the weak solutions w and $\hat{w} \in H^1(\Omega)$ of the following auxiliary problems

$$\begin{cases} \Delta \hat{w} = 0 & \text{in } \Omega \\ \hat{w} = u & \text{on } F \end{cases} \quad (5.14)$$

$$\begin{cases} -\Delta w = -(u)_t + J(u) & \text{in } \Omega, \\ w = 0 & \text{on } F, \end{cases} \quad (5.15)$$

The regularity of u follows from the regularity of w and \hat{w} since

$$u = w + \hat{w}. \quad (5.16)$$

We note that for every $t \in [0, T]$, $u \in C^{\beta}(F)$ $\beta = \log 4 / \log 9$ (see Corollary 3.3 in [42]) thus in particular $u \in C(F)$. Since Ω is a quasicircle from Theorem 2.7 in [43] it is also a non-tangentially accessible domain (N.T.A.), this implies that it is regular for the Dirichlet problem (5.14) in the sense of Jerison and Kenig (see Definition 2.12 in [43]); this yields in particular that $\hat{w} \in C(\bar{\Omega})$. As to the regularity of w , taking into account that $J(u(\cdot, t)) \in L^2(\Omega)$ from Theorem 1.3 in [44] part B, it follows that $w \in C^{\frac{1}{3}}(\bar{\Omega})$, this concludes the proof.

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