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Approximate Solutions to the Discontinuous Riemann-Hilbert Problem of Elliptic Systems of First Order Complex Equations

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Abstract

Several approximate methods have been used to find approximate solutions of elliptic systems of first order equations. One common method is the Newton imbedding approach, *i.e.* the parameter extension method. In this article, we discuss approximate solutions to discontinuous Riemann-Hilbert boundary value problems, which have various applications in mechanics and physics. We first formulate the discontinuous Riemann-Hilbert problem for elliptic systems of first order complex equations in multiply connected domains and its modified well-posedness, then use the parameter extensional method to find approximate solutions to the modified boundary value problem for elliptic complex systems of first order equations, and then provide the error estimate of approximate solutions for the discontinuous boundary value problem.

Keywords

Discontinuous Riemann-Hilbert Problem, Elliptic Systems of First Order Complex Equations, Estimates and Existence of Solutions, Multiply Connected Domains

1. Introduction

Let D be an $N+1(N\ge 1)$-connected bounded domain in $\mathbb{C}$ with the boundary $\partial D=\Gamma=\bigcup_{j=0}^{N}\Gamma_j\in C_\mu^1\ (0<\mu<1)$. Without loss of generality, we assume that $D(0\in D)$ is a circular domain

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in $|z|<1$, bounded by the $(N+1)$-circles $\Gamma_j:|z-z_j|=r_j,\ j=0,1,\cdots,N$ and $\Gamma_0=\Gamma_{N+1}:|z|=1$. In this article, the notations are the same as in references [1]-[12]. If the first order elliptic system with $2n$ unknown real functions

$$\Phi_j\left(x,y,u_1,\cdots,u_{2n},u_{1x},\cdots,u_{2nx},u_{1y},\cdots,u_{2ny}\right)=0,\ j=1,\cdots,2n \text{ in } D \tag{1.1}$$

satisfies certain conditions, then (1.1) can be transformed into the complex form

$$w_{k\bar{z}}=F_k\left(z,w_1,\cdots,w_n,w_{1z},\cdots,w_{nz}\right),\ k=1,\cdots,n, \tag{1.2}$$

where $w_k(z)=u_k(z)+iu_{k+n}(z),\ k=1,\cdots,n$ (see Section 4, Chapter 2 in [5]). Its vector form is as follows:

$$w_{\bar{z}}=F(z,w,w_z),\ w=(w_1,\cdots,w_n)',\ F=(F_1,\cdots,F_n)', \tag{1.3}$$

where $(w_1,\cdots,w_n)'=(w_1,\cdots,w_n)^{\mathrm{T}}$ is the transposed matrix of $(w_1,\cdots,w_n)$. We discuss the first order complex system (1.3) in the form

$$w_{\bar{z}}=F(z,w,w_z),\ F=Q^1w_z+Q^2\overline{w}_{\bar{z}}+A^1w+A^2\overline{w}+A^3, \tag{1.4}$$

in which $Q^j=\left(Q^j_{km}\right)$ with $Q^j_{km}=Q^j_{km}(z,w,w_z)$, $A^j=\left(A^j_{km}\right)$ with $A^j_{km}=A^j_{km}(z,w), j=1,2,$

$A^3=\left(A^3_1,\cdots,A^3_n\right)'$ with $A^3_k=A^3_k(z,w),\quad k,m=1,\cdots,n.$

We assume (1.4) satisfies the following conditions:

Condition C 1) $Q^j_{km}(z,w,U), A^j_{km}(z,w), A^3_k(z,w)\ (j=1,2,k,m=1,\ldots,n)$ are continuous in $w\in\mathbb{C}^n$ for almost every point $z\in D, U\in\mathbb{C}^n$.

2) The above functions are measurable in $z\in D$ for all systems of continuous functions $w(z)$ in $D^*=\overline{D}\setminus Z$ and any systems of measurable functions $U(z)$ in D^* and satisfy

$$L_p\left[A^j_{km}(z,w),\overline{D}\right]\le k_0, 1\le j\le 2,\quad L_p\left[A^3_k(z,w),\overline{D}\right]\le k_2, 1\le k,m\le n, \tag{1.5}$$

$$L_p\left[A^j_{km}(z,w),\overline{D}\right]\le k_1, 1\le j\le 2, 1\le k<m\le n, \tag{1.6}$$

where Z is as stated in (1.8) below, $p(>2)$ and $k_j\ (j=0,1,2)$ are non-negative constants.

3) The complex system (1.4) satisfies the following ellipticity condition

$$\begin{cases}\left|F_k\left(z,w^1,U^1\right)-F_k\left(z,w^2,U^2\right)\right|\le\sum\limits_{m=1}^{n}\left[q_{km}\left|U^1_m-U^2_m\right|+\sum\limits_{m=1}^{n}\kappa_{km}\left|w^1_m-w^2_m\right|\right],\\ \sum\limits_{k=1}^{n}\sum\limits_{m=1}^{n}q_{km}=\sum\limits_{k=1}^{n}q_k\le q_0<1,\ q_{km}\le k_3\le k_0, 1\le k<m\le n,\\ \sum\limits_{k=1}^{n}\sum\limits_{m=1}^{n}\kappa_{km}=\sum\limits_{k=1}^{n}\kappa_k\le k_0,\ \kappa_{km}\le k_3\le k_0,\ 1\le k<m\le n,\end{cases} \tag{1.7}$$

where $q_{km},q_k,\kappa_{km},\kappa_k\ (k,m=1,\cdots,n), q_0,k_0,k_3$ are non-negative constants.

For convenience, $R(z)w(z)\in C_\beta\left(\overline{D}\right)$ and $R(z)\tilde{R}(z)U(z)\in L_{p_0}\left(\overline{D}\right)$ are used to indicate

$R(z)w_k(z)\in C_\beta\left(\overline{D}\right)$ and $R(z)\tilde{R}(z)U_k(z)\in L_{p_0}\left(\overline{D}\right)$, respectively, $k=1,\cdots,n,$ and we define the following:

$$C_\beta\left[Rw,\overline{D}\right]=\sum_{k=1}^{n}C_\beta\left[Rw_k,\overline{D}\right],\quad L_{p_0}\left[R\tilde{R}U,\overline{D}\right]=\sum_{k=1}^{n}L_{p_0}\left[R\tilde{R}U_k,\overline{D}\right],$$

in which $w=(w_1,\cdots,w_n)'$, $U=(U_1,\cdots,U_n)'$, and $R(Z),\tilde{R}(Z)$ are stated as in (1.12), (2.1) below, and $\beta(0<\beta<1)$ and $p_0(2<p_0\le p)$ are non-negative constants.

The so-called Riemann-Hilbert boundary value problem for the complex system (1.4) may be formulated as follows.

Problem A Find a system of continuous solutions $w(z)=\left(w_1(z),\cdots,w_n(z)\right)'$ in $D^*=\overline{D}\setminus Z$ of (1.4), which satisfies the boundary condition

$$\mathrm{Re}\left[\overline{\lambda(z)}w(z)\right]=r(z),\ z\in\Gamma^*=\Gamma\setminus Z, \tag{1.8}$$

in which $\lambda(z)=(\lambda_{km}(z))$ with $|\lambda_{km}|=1$ for $z\in\Gamma, 1\le k,m\le n$, $r(z)=(r_1(z),\cdots,r_n(z))'$, and $Z=\{t_1,t_2,\cdots,t_{m'}\}$ are the first kind of discontinuous points of $\lambda_{km}(z)$ on Γ.

Denote by $\lambda_{km}(t_j-0)$ and $\lambda_{km}(t_j+0)$ the left limit and right limit of $\lambda_{km}(z)$ as $z\to t_j\ (j=1,2,\cdots,m', k,m=1,\cdots,n)$ on Γ, and

$$\begin{gathered}\mathrm{e}^{i\phi_{kmj}}=\frac{\lambda_{km}(t_j-0)}{\lambda_{km}(t_j+0)},\ \gamma_{kmj}=\frac{1}{\pi i}\ln\left[\frac{\lambda_{km}(t_j-0)}{\lambda_{km}(t_j+0)}\right]=\frac{\phi_{kmj}}{\pi}-K_{kmj},\\ K_{kmj}=\left[\frac{\phi_{kmj}}{\pi}\right]+J_{kmj},\ J_{kmj}=0\text{ or }1,\ j=1,\cdots,m', 1\le k,m\le n,\end{gathered} \tag{1.9}$$

where $0\le\gamma_{kmj}<1$ when $J_{kmj}=0$, and $-1<\gamma_{kmj}<0$ when $J_{kmj}=1, k,m=1,\cdots,n, j=1,\cdots,m'$. There is no harm in assuming that the partial indexes K_k of $\lambda(z)$ on $\Gamma_k(k=1,\cdots,N_0(\le N))$ are not integers, and the partial indexes K_k of $\lambda(z)$ on $\Gamma_k(j=0,N_0+1,\cdots,N)$ are integers. Set

$$K_k=\frac{1}{2\pi}\Delta_\Gamma\arg\lambda_{kk}(z)=\sum_{j=1}^{m'}\frac{K_{kkj}}{2},\ k=1,\cdots,n, \tag{1.10}$$

and we call $K=(K_1,\cdots,K_n)'$ the index of Problem A.

For problem A, we will assume $\lambda_{km}(z), r_k(z)\ (k,m=1,\cdots,n)$ satisfy the conditions

$$\begin{gathered}C_\alpha\left[\lambda_{kkj}(z),\hat{\Gamma}_j\right]\le k_0,\ k=1,\cdots,n,\ C_\alpha\left[\lambda_{kmj},\hat{\Gamma}_j\right]\le k_4,\ 1\le k<m\le n,\\ C_\alpha\left[|z-t_j|^{\beta_{kkj}}|z-t_{j-1}|^{\beta_{kkj-1}}r_k(z),\hat{\Gamma}_j\right]\le k_5,\ 1\le k\le n,\ j=1,\cdots,m',\end{gathered} \tag{1.11}$$

in which Γ_j is an open arc from the point t_{j-1} to t_j on Γ $\alpha(1/2<\alpha<1)$, k_4,k_5 are non-negative constants, $\beta_{kmj}+|\gamma_{kmj}|<1, j=1,\cdots,m', k,m=1,\cdots,n$. Moreover, we require that the solution $w(z)$ possess the property

$$\begin{gathered}R(z)w(z)=C_\delta(\bar{D}),\ R(z)=\Pi_{j=1}^{m'}|z-t_j|^{-\eta_j/\tau^2},\ \eta_j=\max(\eta_{11j},\eta_{12j},\cdots,\eta_{n-1,nj},\eta_{n,nj}),\ j=1,\cdots,m',\\ \eta_{kmj}=\begin{cases}\beta_{kmj}+\tau, & \text{for }\gamma_{kmj}\ge0,\text{ and }\gamma_{kmj}<0,\ \beta_{kmj}\ge|\gamma_{kmj}|,\\ |\gamma_{kmj}|+\tau, & \text{for }\gamma_{kmj}<0,\ \beta_{kmj}<|\gamma_{kmj}|,\ j=1,\cdots,m',\end{cases}\quad k,m=1,\cdots,n,\end{gathered} \tag{1.12}$$

in D^*, where $|R(z)|\le1$ in $\bar{D}$, and $\delta,\tau(<\min(\alpha,1-2/p_0))$ are small positive constants.

In general, Problem A may not be solvable. Hence we propose a modified problem as follows.

Problem B Find a system of continuous solutions $w(z)$ of the complex equation (1.4) in D^*, which satisfies the modified boundary condition

$$\mathrm{Re}\left[\overline{\lambda(z)}w(z)\right]=r(z)=h(z),\ z\in\Gamma^*=\Gamma\setminus Z. \tag{1.13}$$

Here

$$\frac{h_k(z)\lambda_k(z)}{X_k(z)}=\begin{cases}\left.\begin{matrix}0, & z\in\Gamma_0,\\ h_{kj}, & z\in\Gamma_j,\ j=1,\cdots,N,\end{matrix}\right\} & \text{if }K_k\ge0,\\ \left.\begin{matrix}h_{kj},\ z\in\Gamma_j,\ j=1,\cdots,N,\\ \left[1+(-1)^{2K_k}\right]h_{k0}+\mathrm{Re}\sum\limits_{m=1}^{[|K_k|+1/2]-1}(h_{km}^++ih_{km}^-)z^m,\ z\in\Gamma_0,\end{matrix}\right\} & \text{if }K_k<0,\end{cases}\quad k=1,\cdots,n, \tag{1.14}$$

in which $h_{kj}\left(j=\left[1-(-1)^{2K_k}\right]/2,\cdots,N\right)$, $h_{km}^+,h_{km}^-\left(m=1,\cdots,\left[|K_k|+1/2\right]-1,\ k=1,\cdots,n\right)$ are unknown real constants to be determined appropriately, and $h_{k,N+1}(=h_{k0})=0$, if $2|K_k|$ is an odd integer. More description on $\lambda_k(z)$ and $X_k(z)$ are given below. We begin with the following function

$$Y_k(z)=\prod_{j=1}^{m_0}(z-t_j)^{\gamma_{k,m_j}}\prod_{l=l}^{N}(z-z_l)^{-[\tilde{K}_{kl}]}\prod_{j=m_0+1}^{m_1}\left(\frac{z-t_j}{z-z_1}\right)^{\gamma_{k,m_j}}\left(\frac{z-t_1'}{z-z_1}\right)$$
$$\times\prod_{j=m_{N_0-1}+1}^{m_{N_0}}\left(\frac{z-t_j}{z-z_{N_0}}\right)^{\gamma_{k,m_j}}\left(\frac{z-t_{N_0}'}{z-z_{N_0}}\right)\prod_{j=m_{N_0}+1}^{m_{N_0+1}}\left(\frac{z-t_j}{z-z_{N_0+1}}\right)^{\gamma_{k,m_j}}\cdots\prod_{j=m_{N-1}+1}^{m}\left(\frac{z-t_j}{z-z_N}\right)^{\gamma_{k,m_j}},$$

where $\tilde{K}_{kl}=\sum_{j=m_{l-1}+1}^{m_l}K_{kj}$ denotes the partial index on $\Gamma_l(l=1,\cdots,N)$, $t_l'(\in\Gamma_l,l=1,\cdots,N_0)$ are fixed points, which are not the discontinuous points from Z. Note that the positive direction applies to the boundary circles $\Gamma_j(j=0,1,\cdots,N)$. Similarly to (1.7)-(1.12), Chapter V, [2], we see that

$$\frac{\lambda_{kk}(t_j-0)}{\lambda_{kk}(t_j+0)}\left[\overline{\frac{Y_k(t_j-0)}{Y_k(t_j+0)}}\right]=\frac{\lambda_{kk}(t_j-0)}{\lambda_{kk}(t_j+0)}\mathrm{e}^{-i\pi\gamma_{kj}}=\pm1,\quad k,j=1,\cdots,m.$$

Clearly, with certain modification on the symbols on some arcs on Γ, $\lambda_{kk}(z)\overline{Y_k(z)}/|Y_k(z)|$ on Γ is seen to be continuous. In this case, its index

$$\kappa_k=\frac{1}{2\pi}\Delta_\Gamma\left[\lambda_{kk}(z)\overline{Y_k(z)}\right]=K_k-\frac{N_0}{2},\quad k=1,\cdots,n$$

are integers. And we have the following:

$$X_k(z)=\begin{cases}iz^{[\kappa_k]}\mathrm{e}^{iS_k(z)}Y_k(z), & z\in\Gamma_0,\\ i\mathrm{e}^{i\theta_{kj}}\mathrm{e}^{iS_k(z)}Y_k(z), & z\in\Gamma_j,j=1,\cdots,N,\end{cases}\qquad \operatorname{Im}\left[\overline{\lambda_k(z)}X_k(z)\right]=0,\ z\in\Gamma,k=1,\cdots,n,$$

$$\operatorname{Re}S_k(z)=\begin{cases}\arg\lambda_{kk}(z)-[\kappa_k]\arg z-\arg Y_k(z), & z\in\Gamma_0,\\ \arg\lambda_{kk}(z)-\arg Y_k(z)-\theta_{kj}, & z\in\Gamma_j,j=1,\cdots,N,\end{cases}\qquad \operatorname{Im}[S_k(1)]=0,\ k=1,\cdots,n,$$

in which $S_k(z)(k=1,\cdots,n)$ are solutions of the modified Dirichlet problems with the above boundary conditions for analytic functions, $\theta_{kj}(j=1,\cdots,N,k=1,\cdots,n)$ are real constants, and $\kappa_k=K_k-N_0/2\,(k=1,\cdots,n)$.

In addition, we may assume that the solution $w(z)$ satisfies the following point conditions

$$\operatorname{Im}\left[\overline{\lambda(a_j)}w(a_j)\right]=b_{kj},\ j\in J_k=\{1,\cdots,2K_k+1\},\ \text{if } K_k\ge0,k=1,\cdots,n,\tag{1.15}$$

where $a_j\in\Gamma_0\,(j\in J_k)$ are distinct points, and $b_{kj}(j\in J_k,k=1,\cdots,n)$ are all real constants satisfying the conditions

$$|b_{kj}|\le k_6,\ j\in J_k,k=1,\cdots,n,\tag{1.16}$$

for a positive constant k_6. Problem B with $A_3(z,w)=0$ in D, $c(z)=0$ on Γ and $b_{kj}=0\,(j\in J_k,k=1,\cdots,n)$ is called Problem B_0.

If $\lambda_{kk}(z)=1,\lambda_{km}=0,k\ne m=1,\cdots,n,$ then Problem B for (1.4) is the modified Dirichlet boundary value problem for (1.4). It is easy to see that the solutions of (1.4) include the generalized hyperanalytic functions as special cases. In fact, if (1.4) is linear, and $Q_{km}^2=0,1\le k,m\le n$, $Q_{km}^1=A_{km}^1=A_{km}^2=0,1\le k<m\le n$ and $Q_{k+1,k}^1=Q_{k+2,k+1}^1$, $A_{k+1,k}^j=A_{k+2,k+1}^j,1\le k\le N-1$, then the solutions of (1.4) are called generalized hyperanalytic functions.

2. Parameter Extension Method of the Discontinuous Riemann-Hilbert Problem for Elliptic Systems of First Order Complex Equations

We begin with the following estimates of the solution for problem B.

Theorem 2.1 Suppose that the complex system (1.4) satisfies Condition C and the constants k_1,k_3,k_4 in

(1.6), (1.7), (1.11) are small enough. Then any solution $w(z)=(w_1(z),\cdots,w_n(z))'$ of Problem B for (1.4) satisfies the estimate

$$\begin{aligned}&C_\beta\left[R(z)w(z),\bar{D}\right]+L_{p_0}\left[R(z)\tilde{R}(z)\left(|w_{\bar{z}}|+|w_z|\right),\bar{D}\right]\\&=\sum_{k=1}^{n}\left\{C_\beta\left[R(z)w_k(z),\bar{D}\right]+L_{p_0}\left[R(z)\tilde{R}(z)\left(|w_{k\bar{z}}|+|w_{kz}|\right),\bar{D}\right]\right\}\le M_1k_*,\end{aligned}\tag{2.1}$$

where $\tilde{R}(z)=\Pi_{j=1}^{m'}\left|z-t_j\right|^{1/\tau^2}$ with $\left|\tilde{R}(z)\right|\le 1$ in $\bar{D}$, $\beta=\min(\alpha,1-2/p_0)$, $p_0\,(2<p_0\le p)$, $k_*=k_2+k_5+k_6$, $M_1=M_1(q_0,p_0,k_0,\beta,K,D)$ with $K=(K_1,\cdots,K_n)'$ are non-negative constants.

Proof There is no harm in assuming that $k_*=k_2+k_5+k_6\ne 0$. Let $W(z)=w(z)/k_*$. It can be seen that $W(z)$ is a solution of the following boundary value problem

$$W_{\bar{z}}=Q^1W_z+Q^2\overline{W_z}+A^1W+A^2\bar{W}+A^3/k_*,\ z\in D,\tag{2.2}$$

$$\operatorname{Re}\left[\overline{\lambda(z)}W(z)\right]=\left[r(z)+h(z)\right]/k_*,\quad z\in\Gamma,\tag{2.3}$$

$$\operatorname{Im}\left[\overline{\lambda(a_j)}W(a_j)\right]=b_j/k_*,\quad j\in J_k,k=1,\cdots,n,\tag{2.4}$$

in which

$$L_p\left[A_k^3/k_*,\bar{D}\right]\le 1,\ C_\alpha\left[r_k(z)/k_*,\Gamma\right]\le 1,\ \left|b_{kj}/k_*\right|\le 1,\ j\in J_k,k=1,\cdots,n.\tag{2.5}$$

Following the proof of the Theorem 2.1 of Chapter VI in [1], we can derive the estimate

$$\begin{aligned}&C_\beta\left[R(z)W(z),\bar{D}\right]+L_{p_0}\left[R(z)\tilde{R}(z)\left(|W_{\bar{z}}|+|W_z|\right)\bar{D}\right]\le M_1\\&=M_1(q_0,p_0,k_0,\beta,K,D).\end{aligned}\tag{2.6}$$

From the above estimate, it immediately follows that the estimate (2.1) is true.

In addition, we assume that (1.4) satisfies the following condition: For any continuous vectors $w^1(z),w^2(z)$ and any measurable vector $U(z)\in L_{p_0}(\bar{D})$,

$$\begin{aligned}&F(z,w^1,U)-F(z,w^2,U)\\&=\tilde{Q}(z,w^1,w^2,U)U+\tilde{A}(z,w^1,w^2,U)(w^1-w^2),\end{aligned}\tag{2.7}$$

where $\tilde{Q}=(\tilde{Q}_{km})$, $\tilde{A}=(\tilde{A}_{km})$ satisfy the condition

$$\begin{aligned}&\left|\tilde{Q}_{km}\right|\le q_{km},\ \sum_{k=1}^{n}\sum_{m=1}^{n}q_{km}=\sum_{k=1}^{n}q_k\le q_0<1,\ q_{km}\le k_3\le k_0,1\le k<m\le n,\lim_{x\to\infty}\\&L_{p_0}\left[\tilde{A}_{km},\bar{D}\right]\le k_0,1\le k,m\le n,\ L_{p_0}\left[\tilde{A}_{km},\bar{D}\right]\le k_1\le k_0,1\le k<m\le n,\end{aligned}\tag{2.8}$$

in which $p_0\,(2\le p_0\le p),k_0,k_1$ are non-negative constants.

Now, we prove that there exists a unique solution of the modified Riemann-Hilbert problem (Problem B) for analytic vectors by the parameter extensional method.

Theorem 2.2 Let k_4 in (1.11) be a sufficiently small positive constant. Then Problem B for analytic vectors has a solution.

Proof We consider the modified Riemann-Hilbert problem (Problem B$'$) for analytic vectors with the boundary conditions

$$\operatorname{Re}\left[\overline{\Lambda(\zeta)}w(z)\right]+t\operatorname{Re}\left[\overline{\Delta(z)}w(z)\right]=\hat{r}(z)+h(z),\ z\in\Gamma,\tag{2.9}$$

$$\operatorname{Im}\left[\overline{\Lambda(a_j)}w(a_j)\right]+t\operatorname{Im}\left[\overline{\Delta(a_j)}w(a_j)\right]=B_j,\ j\in J_k,1\le k\le n,\tag{2.10}$$

where

$$\Lambda(z)=(\Lambda_{km}(z)),\ \Lambda_{km}(z)=\begin{cases}\lambda_{km}(z), & k\ge m,\\ 0, & k<m,\end{cases}\ \text{on }\Gamma,$$

$$\Delta(z)=(\Delta_{km}(z)),\ \Delta_{km}(z)=\begin{cases}0, & k\ge m,\\ \lambda_{km}(z), & k<m,\end{cases}\ \text{on }\Gamma,$$

in which $t(0\le t\le 1)$ is a real parameter, and $\hat{r}(z)=(\hat{r}_1(z),\cdots,\hat{r}_n(z))'$ is any vector of real functions, $|z-t_{j-1}|^{\beta_{k(j-1)-1}}|z-t_j|^{\beta_{kj-1}}\hat{r}_k(z)\in C_\alpha(\Gamma_j), 0<\alpha<1, j=1,\cdots,m', k=1,\cdots,n,$ and $B_j=(B_{j1},\cdots,B_{jn})'$ is any vector of constants. When $t=0$, it is clear that Problem B′ for analytic vectors has a unique solution (see [1]). If Problem B′ with $t=t_0\ (0\le t_0<1)$ for analytic vectors is solvable, we shall prove that there exists a positive number δ independent of t_0, such that Problem B′ for every $t\in E=\{|t-t_0|\le\delta, 0\le t\le 1, \delta>0\}$ has a unique solution. In fact, the boundary conditions (2.9), (2.10) can be rewritten in the form

$$\begin{aligned}&\operatorname{Re}\left[\overline{\Lambda(z)}w(z)\right]+t_0\operatorname{Re}\left[\overline{\Delta(z)}w(z)\right]\\&=(t_0-t)\operatorname{Re}\left[\overline{\Delta(z)}w(z)\right]+\hat{r}(z)+h(z),\ z\in\Gamma,\end{aligned}\tag{2.11}$$

$$\begin{aligned}&\operatorname{Im}\left[\overline{\Lambda(a_j)}w(a_j)\right]+t_0\operatorname{Im}\left[\overline{\Delta(a_j)}w(a_j)\right]\\&=(t_0-t)\operatorname{Im}\left[\overline{\Delta(a_j)}w(a_j)\right]+B_j,\ j\in J_k, 1\le k\le n.\end{aligned}\tag{2.12}$$

Substituting the zero vector $w^0(z)=(0,\cdots,0)'$ into the position of $w(z)$ on the right hand side of (2.11) and (2.12), by the hypothesis, the boundary value problem (2.11), (2.12) for analytic vectors has a unique solution $w^1(z)=(w_1^1(z),\cdots,w_n^1(z))$ and $R(z)w_k^1(z)\in C_\alpha(\bar{D}), k=1,\cdots,n.$ Using the successive iteration, we can find a sequence $\{w^n(z)\}$ of analytic vectors, which satisfies the boundary conditions

$$\begin{aligned}&\operatorname{Re}\left[\overline{\Lambda(z)}w^{n+1}(z)\right]+t_0\operatorname{Re}\left[\overline{\Delta(z)}w^{n+1}(z)\right]\\&=(t_0-t)\operatorname{Re}\left[\overline{\Delta(z)}w^n(z)\right]+\hat{r}(z)+h(z),\ z\in\Gamma,\end{aligned}\tag{2.13}$$

$$\begin{aligned}&\operatorname{Im}\left[\overline{\Lambda(a_j)}w^{n+1}(a_j)\right]+t_0\operatorname{Im}\left[\overline{\Delta(a_j)}w^{n+1}(a_j)\right]\\&=(t_0-t)\operatorname{Im}\left[\overline{\Delta(a_j)}w^n(a_j)\right]+B_j,\ j\in J_k, 1\le k\le n.\end{aligned}\tag{2.14}$$

From (2.13) and (2.14), we have

$$\begin{aligned}&\operatorname{Re}\left[\overline{\Lambda(z)}(w^{n+1}-w^n)\right]+t_0\operatorname{Re}\left[\overline{\Delta(z)}(w^{n+1}-w^n)\right]\\&=(t_0-t)\operatorname{Re}\left[\overline{\Delta(z)}(w^n-w^{n-1})\right]+h(z),\ z\in\Gamma,\end{aligned}\tag{2.15}$$

$$\begin{aligned}&\left\{\operatorname{Im}\left[\overline{\Lambda(z)}(w^{n+1}-w^n)\right]+t_0\operatorname{Im}\left[\overline{\Delta(z)}(w^{n+1}-w^n)\right]\right\}\Big|_{z=a_j}\\&=(t_0-t)\operatorname{Im}\left[\overline{\Delta(z)}(w^n-w^{n-1})\right]\Big|_{z=a_j}+B_j,\ j\in J_k, 1\le k\le n, n=1,2,\cdots.\end{aligned}\tag{2.16}$$

In accordance with Theorem 2.1, we can conclude

$$\begin{aligned}C_\alpha\left[R(w^{n+1}-w^n),\bar{D}\right]&\le M_2C_\alpha\left[(t-t_0)\overline{\Delta(z)}R(w_n-w_{n-1}),\Gamma\right]\\&\le M_2|t-t_0|k_0C_\alpha\left[R(w^n-w^{n-1}),\Gamma\right]\\&\le M_2k_0|t-t_0|C_\alpha\left[R(w^n-w^{n-1}),\bar{D}\right],\end{aligned}\tag{2.17}$$

where $M_2 = M_2(k', \beta, K, D)$ with $k' = (k_0, k_1, \cdots, k_4)$, and $K = (K_1, \cdots, K_n)'$. Choosing a positive constant δ, such that $\delta M_2 k_0 < 1$, it is not difficult to see that

$$C_\alpha\left[R\left(w^{n+1} - w^n\right), \bar{D}\right] \le \frac{1}{2} C_\alpha\left[R\left(w^n - w^{n-1}\right), \bar{D}\right] \le \frac{1}{2^N} C_\alpha\left[Rw^1, \bar{D}\right],$$

and

$$C_\alpha\left[R\left(w^n - w^m\right), \bar{D}\right] \le \frac{1}{2^N} C_\alpha\left[Rw^1, \bar{D}\right]$$

for $n \ge m > N$, where N is a positive integer. This shows that

$$C_\alpha\left(R\left(w^n - w^m\right)\right) \to 0 \text{ as } n, m \to \infty.$$

Hence, there exists an analytic vector $w^*(z) = \left(w_1^*(z), \cdots, w_n^*(z)\right)$, such that

$$C_\alpha\left[R\left(w^n - w^*\right), \bar{D}\right] \to 0 \text{ as } n \to \infty. \tag{2.18}$$

Thus $w_*(z)$ is a solution of Problem B′ with $t \in E$. From this we can derive that Problem B′ with $t = 1, R(z) = r(z)3011, B_j = b_j \ (j \in J_k, 1 \le k \le n)$, *i.e.* Problem B for analytic vectors is solvable.

Next we prove the solvability of Problem B for the system (1.4).

Theorem 2.3 Let the nonlinear elliptic system (1.4) satisfy Condition C, and k_1, k_3, k_4 in (1.6), (1.7), (1.11) be sufficiently small positive constants. Then Problem B for the complex system (1.4) is solvable.

Proof We consider the nonlinear elliptic complex system with the parameter $t \in [0,1]$:

$$w_{\bar{z}} - tF(z, w, w_z) = A(z), \tag{2.19}$$

where $A(z) = \left(A_1(z), \cdots, A_n(z)\right)'$ is any measurable vector in D and $R(z)\tilde{R}(z)A_j(z) \in L_{p_0}(\bar{D}), j = 1, \cdots, n.$ Applying Theorem 2.2, we see that Problem B for (2.19) with $t = 0$ is solvable, and the solution $w(z)$ can be expressed as

$$w(z) = w_0(z) + \Psi(z), \ \Psi(z) = TA - \frac{1}{\pi}\iint_D \frac{A(\zeta)}{\zeta - z} \mathrm{d}\sigma_\zeta, \tag{2.20}$$

where $w_0(z)$ is an analytic vector satisfying the boundary conditions

$$\operatorname{Re}\left[\overline{\lambda(z)}\left(w_0(z) + \Psi(z)\right)\right] = r(z) + h(z), \ z \in \Gamma, \tag{2.21}$$

$$\operatorname{Im}\left[\overline{\lambda(a_j)}\left(w_0(a_j) + \Psi(a_j)\right)\right] = b_j, \ j \in J_k, 1 \le k \le n. \tag{2.22}$$

Suppose that when $t = t_0 \ (0 \le t_0 < 1)$, Problem B for the system (2.19) has a unique solution. Then we shall prove that there exists a neighborhood of $t_0 : E = \{|t - t_0| \le \delta, 0 \le t \le 1, \delta > 0\}$, so that for every $t \in E$ and any function $R(z)\tilde{R}(z)A(z) \in L_{p_0}(\bar{D})$, Problem B for (2.19) is solvable. In fact, the complex system (2.19) can be written in the form

$$w_{\bar{z}} - t_0 F(z, w, w_z) = (t - t_0)F(z, w, w_z) + A(z). \tag{2.23}$$

Suppose that Problem B for (2.13) with $t = t_0 \ (0 \le t_0 < 1)$ is solvable, by using the similar method as in the proof of Theorem 2.2, we can find a positive constant δ, so that for every $t \in E = \{|t - t_0| \le \delta, 0 \le t \le 1\}$, there exists a sequence $\{w^n(z)\}$ of solutions satisfying

$$w_{n+1\bar{z}} - t_0 F(z, w_{n+1}, w_{n+1z}) = (t - t_0)F(z, w_n, w_{nz}) + A(z), \ n = 1, 2, \cdots. \tag{2.24}$$

The difference of the above equations for $n+1$ and n is as follows:

$$\begin{aligned}&(w_{n+1} - w_n)_{\bar{z}} - t_0\left[F(z, w_{n+1}, w_{n+1z}) - F(z, w_n, w_{nz})\right] \\ &= (t - t_0)\left[F(z, w_n, w_{nz}) - F(z, w_{n-1}, w_{n-1z})\right], \ n = 1, 2, \cdots.\end{aligned} \tag{2.25}$$

From Condition C, we can derive that

$$
\begin{aligned}
& F\left(z, w_{n+1}, w_{n+1z}\right)-F\left(z, w_n, w_{nz}\right) \\
& =F\left(z, w_{n+1}, w_{n+1z}\right)-F\left(z, w_{n+1}, w_{nz}\right)+F\left(z, w_{n+1}, w_{nz}\right)-F\left(z, w_n, w_{nz}\right) \\
& =\tilde{Q}_{n+1}(z)\left(w_{n+1}-w_n\right)_z+\tilde{A}_{n+1}(z)\left(w_{n+1}-w_n\right), \\
& \left|\tilde{Q}_{n+1}(z)\right| \leq q_0<1,\ \tilde{A}_{n+1}(z) \in L_{p_0}(\bar{D}),\quad n=1,2,\cdots,
\end{aligned}
$$

and

$$
\begin{aligned}
& L_{p_0}\left[R\tilde{R}\left(F\left(z, w_n, w_{nz}\right)-F\left(z, w_{n-1}, w_{n-1z}\right)\right), \bar{D}\right] \\
& \leq q_0 L_{p_0}\left[R\tilde{R}\left(w_n-w_{n-1}\right)_z, \bar{D}\right]+n k_0 C\left[R\left(w_n-w_{n-1}\right), \bar{D}\right] \\
& \leq\left(q_0+n k_0\right)\left[C_\beta\left[R\left(w_n-w_{n-1}\right), \bar{D}\right]+L_{p_0}\left[R\tilde{R}\left(\left|\left(w_n-w_{n-1}\right)_{\bar{z}}\right|+\left|\left(w_n-w_{n-1}\right)_z\right|\right), \bar{D}\right]\right] \\
& =\left(q_0+n k_0\right) L_n .
\end{aligned}
$$

Moreover, $w_{n+1}(z)-w_n(z)$ satisfies the homogeneous boundary conditions

$$
\operatorname{Re}\left[\overline{\lambda(z)}\left[w_{n+1}(z)-w_n(z)\right]\right]=h(z),\quad z \in \Gamma, \tag{2.26}
$$

$$
\operatorname{Im}\left[\overline{\lambda\left(a_j\right)}\left(w_{n+1}\left(a_j\right)-w_n\left(a_j\right)\right)\right]=0,\quad j \in J_k, 1 \leq k \leq n. \tag{2.27}
$$

Similarly to Theorem 3.3, Chapter I, [1], we have

$$
\begin{aligned}
L_{n+1} & =C_\beta\left[R\left(w_{n+1}-w_n\right), \bar{D}\right]+L_{p_0}\left[R\tilde{R}\left(\left|\left(w_{n+1}-w_n\right)_{\bar{z}}\right|+\left|\left(w_{n+1}-w_n\right)_z\right|\right), \bar{D}\right] \\
& \leq M_3\left|t-t_0\right|\left(q_0+n k_0\right) L_n,
\end{aligned} \tag{2.28}
$$

where $M_3=M_3\left(q_0, p_0, k^{\prime}, \alpha, K, D\right)\left(k^{\prime}=\left(k_0, k_2, k_5, k_6\right)\right)$ are positive constants. Provided $\delta(>0)$ is small enough, so that $\eta=\delta M_3\left(q_0+n k_0\right)<1$, we can obtain

$$
L_{n+1} \leq \eta L_n \leq \eta^n L_1=\eta^n\left[C_\beta\left(R w_1, \bar{D}\right)+L_{p_0}\left(R\tilde{R}\left(\left|w_{1\bar{z}}\right|+\left|w_{1z}\right|\right), \bar{D}\right)\right] \tag{2.29}
$$

for every $t \in E$. Thus

$$
\begin{aligned}
S\left(w_n-w_m\right) & =C_\beta\left[R\left(w_n-w_m\right), \bar{D}\right]+L_{p_0}\left[R\tilde{R}\left(\left|\left(w_n-w_m\right)_{\bar{z}}\right|+\left|\left(w_n-w_m\right)_z\right|\right), \bar{D}\right] \\
& \leq L_n+L_{n-1}+\cdots+L_{m+1} \leq\left(\eta^{n-1}+\eta^{n-2}+\cdots+\eta^m\right) L_1 \\
& =\eta^m\left(1+\eta+\cdots+\eta^{n-m-1}\right) L_1 \leq \eta^{N+1} \frac{1-\eta^{n-m}}{1-\eta} L_1 \leq \frac{\eta^{N+1}}{1-\eta} L_1
\end{aligned}
$$

for $n \geq m>N$, where N is a positive integer. This shows that $S\left(w_n-w_m\right) \rightarrow 0$ as $n, m \rightarrow \infty$. Thus there exists a system of continuous functions $w_*(z)$ in D^*, such that

$$
S\left(w-w_*\right)=C_\beta\left[R\left(w_n-w_*\right), \bar{D}\right]+L_{p_0}\left[R\tilde{R}\left(\left|\left(w_n-w_*\right)_{\bar{z}}\right|+\left|\left(w_n-w_*\right)_z\right|\right), D\right] \rightarrow 0 \text{ as } n \rightarrow \infty .
$$

By Condition C, it follows that $w_*(z)$ is a solution of Problem B for the system (2.23), *i.e.* (2.19) for $t \in E$. It is easy to see that the positive constant δ is independent of $t_0\left(0 \leq t_0<1\right)$. Hence Problem B for the system (2.19) with $t=t_0=0$ is solvable. Correspondingly we can derive that when $t=\delta, 2\delta, \cdots,[1/\delta]\delta, 1$, Problem B for (2.19) is solvable. Especially Problem B for (2.19) with $t=1$ and $A(z)=(1-t) F(z, 0,0)$, namely Problem B for the system (1.4) has a solution.

3. Error Estimates of Approximate Solutions of the Discontinuous Riemann Hilbert Problem for Elliptic Systems of First Order Complex Equations

In this section, we shall introduce an error estimate of the above approximate solutions.

Theorem 3.1 Under the same conditions as in Theorem 2.3, let $w=w(z)$ be a solution of Problem B for the complex system (1.4) satisfying Condition C in $\bar{D}$, and $w_n^t = w_n(z,t)$ be its approximation as stated in the proof of Theorem 2.3 with $A(z)=(1-t)F(z,0,0)$. Then we have the following error estimate

$$\begin{aligned} S\left(w-w_n^t\right) &= C_\beta\left[R\left(w-w_n^t\right),\bar{D}\right]+L_{p_0}\left[R\tilde{R}\left(\left|\left(w-w_n^t\right)_{\bar{z}}\right|+\left|\left(w-w_n^t\right)_z\right|\right),\bar{D}\right] \\ &\le \gamma k\left[\frac{1-\gamma\left|t-t_0\right|^n}{1-\gamma\left|t-t_0\right|}(1-t)+\left(\gamma\left|t-t_0\right|\right)^n(1-t_0)\right], \end{aligned} \tag{3.1}$$

where $\gamma = M_3(q_0+nk_0), k = M_3(k_2+k_5+k_6)$ with M_3, q_0 as in (2.28), and $k_j(j=2,5,6)$ as in (1.6),(1.7), (1.11) and (1.16).

Proof From (1.4) and (2.24) with $A(z)=(1-t)F(z,0,0)$, we have

$$\begin{aligned} \left(w-w_{n+1}^t\right)_{\bar{z}} &= F(z,w,w_z)-t_0F\left(z,w_{n+1}^t,w_{n+1z}^t\right)-(t-t_0)F\left(z,w_n^t,w_{nz}^t\right)-(1-t)F(z,0,0) \\ &=(1-t)\left[F(z,w,w_z)-F(z,0,0)\right]+t_0\left[F(z,w,w_z)-F\left(z,w_{n+1}^t,w_{n+1z}^t\right)\right] \\ &\quad+(t-t_0)\times\left[F(z,w,w_z)-F\left(z,w_n^t,w_{nz}^t\right)\right] \\ &=t_0\left[\tilde{Q}\left(z,w,w_z,w_{n+1}^t\right)\left(w-w_{n+1}^t\right)_z+\tilde{A}\left(z,w,w_{n+1}^t,w_{n+1z}^t\right)\left(w-w_{n+1}^t\right)\right] \\ &\quad+(1-t)\left[F(z,w,w_z)-F(z,0,0)\right]+(t-t_0)\left[F(z,w,w_z)-F\left(z,w_n^t,w_{nz}^t\right)\right]. \end{aligned} \tag{3.2}$$

It is clear that $w-w_{n+1}^t$ satisfies the homogeneous boundary conditions

$$\begin{aligned} &\mathrm{Re}\left[\overline{\lambda(z)}\left(w(z)-w_{n+1}^t(z)\right)\right]=h(z),\ z\in\Gamma, \\ &\mathrm{Im}\left[\overline{\lambda(a_j)}\left(w(a_j)-w_{n+1}^t(a_j)\right)\right]=0,\ j\in J_k, k=1,\cdots,n. \end{aligned} \tag{3.3}$$

Noting that $\tilde{Q}=\tilde{Q}\left(z,w,w_z,w_{n+1}^t\right), \tilde{A}=\tilde{A}\left(z,w,w_{n+1}^t,w_{n+1z}^t\right)$ satisfy $\left|\tilde{Q}\right|\le q_0<1, L_{p_0}\left[\tilde{A},\bar{D}\right]\le k_0$, and

$$\begin{aligned} &L_{p_0}\left[R\tilde{R}\left(F(z,w,w_z)-F(z,0,0)\right),\bar{D}\right] \\ &\le q_0L_{p_0}\left[R\tilde{R}w_z,\bar{D}\right]+nk_0C\left[Rw,\bar{D}\right] \\ &\le (q_0+nk_0)\left[L_{p_0}\left(R\tilde{R}w_z,\bar{D}\right)+C\left(Rw,\bar{D}\right)\right] \\ &\le (q_0+nk_0)S(w), \\ &L_{p_0}\left[F(z,w,w_z)-F\left(z,w_n^t,w_{nz}^t\right),\bar{D}\right] \\ &\le q_0L_{p_0}\left[R\tilde{R}\left(w-w_n^t\right)_z,\bar{D}\right]+nk_0C\left[R\left(w-w_n^t\right),\bar{D}\right] \\ &\le (q_0+nk_0)\times\left[L_{p_0}\left(R\tilde{R}\left(w-w_n^t\right)_z,\bar{D}\right)+C\left(R\left(w-w_n^t\right),\bar{D}\right)\right] \\ &\le (q_0+nk_0)S\left(w-w_n^t\right), \end{aligned}$$

and then $w-w_0^t$ is a solution of Problem B for the complex equation

$$\begin{aligned} \left(w-w_0^t\right)_{\bar{z}} &= F(z,w,w_z)-t_0F\left(z,w_0^t,w_{0z}^t\right)-(1-t_0)F(z,0,0) \\ &=t_0\left[F(z,w,w_z)-F\left(z,w_0^t,w_{0z}^t\right)\right]+(1-t_0)\left[F(z,w,w_z)-F(z,0,0)\right] \\ &=t_0Q\left(w-w_0^t\right)_z+A\left(w-w_0^t\right)+(1-t_0)\left[F(z,w,w_z)-F(z,0,0)\right], \end{aligned} \tag{3.4}$$

hence we have

$$\begin{aligned} S\left(w-w_0^t\right) &\le M_3(1-t_0)\left[q_0L_{p_0}\left(R\tilde{R}w_z,\bar{D}\right)+nk_0C\left(Rw,\bar{D}\right)\right] \\ &\le M_3(q_0+nk_0)(1-t_0)S(w)\le\gamma(1-t_0)k, \end{aligned} \tag{3.5}$$

in which

$$S(w) \le M_3(k_2+k_5+k_6)=k, \tag{3.6}$$

where the non-negative constants M_3, k_2, k_5, k_6 are as stated in (2.28), (1.5), (1.11) and (1.12). Moreover according to the proof of Theorem 2.3, we can derive

$$\begin{aligned} S\left(w-w_{n+1}^t\right) &\le M_3\left[(1-t)(q_0+nk_0)S(w)+|t-t_0|(q_0+nk_0)S\left(w-w_n^t\right)\right] \\ &= M_3(q_0+nk_0)\left[(1-t)S(w)+|t-t_0|S\left(w-w_n^t\right)\right]. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), it follows that

$$\begin{aligned} S\left(w-w_{n+1}^t\right) &\le \gamma\left[(1-t)S(w)+|t-t_0|S\left(w-w_n^t\right)\right] \\ &\le \gamma(1-t)S(w)\left(1+\gamma|t-t_0|+\gamma^2|t-t_0|^2+\cdots+\gamma^n|t-t_0|^n\right) \\ &\quad +\gamma^{n+1}|t-t_0|^{n+1}S\left(w-w_0^t\right) \\ &\le \gamma(1-t)S(w)\times\frac{1-\left(\gamma|t-t_0|\right)^{n+1}}{1-\gamma|t-t_0|}+\gamma^{n+1}|t-t_0|^{n+1}S\left(w-w_0^t\right), \end{aligned}$$

where $\gamma = M_3(q_0+nk_0)$, and $w_0^t = w(z,t_0)$ is the solution of Problem B for (2.24) with $t=t_0$ and $A(z)=(1-t_0)F(z,0,0)$. Finally, we obtain

$$\begin{aligned} S\left(w-w_{n+1}^t\right) &\le \gamma k\left[\frac{1-\gamma|t-t_0|^{n+1}}{1-\gamma|t-t_0|}(1-t)+\gamma k\gamma^{n+1}\left(|t-t_0|\right)^{n+1}(1-t_0)\right] \\ &= \gamma k\left[\frac{1-\gamma|t-t_0|^{n+1}}{1-\gamma|t-t_0|}(1-t)+\left(\gamma|t-t_0|\right)^{n+1}(1-t_0)\right], \end{aligned} \tag{3.8}$$

This shows that (3.1) holds. If the positive constant δ is small enough, so that when $|t-t_0| \le \delta, \gamma|t-t_0|<1$, n is sufficiently large and t is close to 1, then the right hand side becomes very small.

Note: The opinions expressed herein are those of the authors and do not necessarily represent those of the Uniformed Services University of the Health Sciences and the Department of Defense.

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