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Harmonic Solutions of Duffing Equation with Singularity via Time Map

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Abstract

This paper is devoted to the study of second-order Duffing equation $x''+g(x)=p(t)$ with singularity at the origin, where $g(x)$ tends to positive infinity as $x\to+\infty$, and the primitive function $G(x)\left(=\int_1^x g(s)\mathrm{d}s\right)\to+\infty$ as $x\to 0^+$. By applying the phase-plane analysis methods and Poincaré-Bohl theorem, we obtain the existence of harmonic solutions of the given equation under a kind of nonresonance condition for the time map.

Keywords

Harmonic Solutions, Duffing Equation, Singularity, Time Map, Poincaré-Bohl Theorem

1. Introduction

We deal with the second-order Duffing equation

$$x''+g(x)=p(t), \tag{1}$$

where $g:\mathbb{R}^+\to\mathbb{R}$ is locally Lipschitzian and has singularity at the origin, $p:\mathbb{R}\to\mathbb{R}$ is continuous and 2π periodic. Our purpose is to establish existence result for harmonic solution of Equation (1). Arising from physical applications (see [1] for a discussion of the Brillouin electron beam focusing problem), the periodic solution for equations with singularity has been widely investigated, referring the readers to [2]-[6] and their extensive references.

As is well known, time map is the right tool to build an approach to the study of periodic solution of Equation (1) (see [7]-[9]). However, the work mainly focused on the equations without singularity. Our goal in this paper is to study the periodic solution of Equation (1) with singularity via time map. There is a little difference

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between our time map and the time map in [7] [9]. We now introduce the time map.

Consider the auxiliary autonomous system

$$x' = y, \quad y' = -g(x), \tag{2}$$

and suppose that

$$\lim_{x \to 0^+} g(x) = -\infty, \tag{g_0}$$

$$\lim_{x \to +\infty} g(x) = +\infty. \tag{g_1}$$

$$\lim_{x \to 0^+} G(x) = +\infty, \quad G(x) = \int_1^x g(s)\,\mathrm{d}s. \tag{G_0}$$

Obviously, the orbits of system (2) are curves Γ_c determined by the equation

$$\Gamma_c : \frac{1}{2} y^2 + G(x) = c,$$

where c is an arbitrary constant.

In view of the assumptions (g_0), (g_1) and (G_0), there exists a $c_0 > 0$, such that for $c \ge c_0$, Γ_c is a closed curve. Let $(x(t), y(t))$ be a solution of (2) whose orbit is Γ_c. Then this solution is periodic, denoting by $\tau(c)$ the least positive period of this solution. It is easy to see that

$$\tau(c) = \sqrt{2} \int_{h(c)}^{d(c)} \frac{\mathrm{d}x}{\sqrt{c - G(x)}}, \tag{3}$$

where $0 < h(c) < d(c)$, $G(h(c)) = G(d(c)) = c$, $\lim_{c \to +\infty} h(c) = 0$, $\lim_{c \to +\infty} d(c) = +\infty$.

We recall an interesting result in [7]. Ding and Zanolin [7] proved that Equation (1) without singularity possesses at least one *T*-periodic solution provided that

$$\lim_{|x| \to +\infty} \operatorname{sgn}(x) g(x) = +\infty,$$

and a kind of nonresonance condition for the time map

$$\left[\tau_- + \tau_+, \tau^- + \tau^+\right] \cap \left\{ \frac{T}{n} : n \in \mathbb{N} \right\} = \varnothing, \tag{4}$$

where

$$\tau_\pm = \liminf_{c \to \pm\infty} \tau(c), \quad \tau^\pm = \limsup_{c \to \pm\infty} \tau(c), \quad \tau(c) = \sqrt{2} \int_0^{d(c)} \frac{\mathrm{d}s}{\sqrt{c - G(s)}}.$$

Now naturally, we consider the question whether Equation (1) has harmonic solution when we permit $\dfrac{g(x)}{x}$ cross resonance points and use a kind of nonresonance condition for time map. In the following we will give a positive answer. In order to state the main result of this paper, set

$$\hat{\tau}_+ = \liminf_{c \to +\infty} \sqrt{2} \int_1^{d(c)} \frac{\mathrm{d}x}{\sqrt{c - G(x)}}; \quad \hat{\tau}^+ = \limsup_{c \to +\infty} \sqrt{2} \int_1^{d(c)} \frac{\mathrm{d}x}{\sqrt{c - G(x)}}, \tag{5}$$

and assume that

$$\frac{2\pi}{n+1} < \hat{\tau}_+ \le \hat{\tau}^+ < \frac{2\pi}{n}, \quad n \in \mathbb{Z}. \tag{τ}$$

Our main result is following.

Theorem 1.1 *Assume that* (g_0), (g_1) (G_0) *and* (τ) *hold, then Equation* (1) *has at least one* 2π-*periodic solution.*

In this case, we generalize the result in [7] to Equations (1) with singularity.

The remainer of the paper is organized as follows. In Section 2, we introduce some technical tools and present all the auxiliary results. In Section 3, we will give the proof of Theorem 1.1 by applying the phase-plane analysis methods and Poincaré-Bohl fixed point theorem.

2. Some Lemmas

we assume throughout the paper that $g(x)$ is locally Lipschitz continuous. In order to apply the phase-plane analysis methods conveniently, we study the equation

$$x'' + g(x) = p(t), \tag{6}$$

where $g:(-1,+\infty)\to\mathbb{R}$ is continuous and has a singularity at $x=-1$. In fact, we can take a parallel translation $x = 1+u$ to achieve the aim. Then the conditions (g_0) and (G_0) become

$$\lim_{u\to -1^+}\hat{g}(u) = -\infty,\quad \lim_{u\to -1^+}\hat{G}(u) = +\infty.$$

Dropping the hats for simplification of notations, we assume that

$$\lim_{x\to -1^+} g(x) = -\infty, \tag{g_0'}$$

and

$$\lim_{x\to -1^+} G(x) = +\infty,\quad G(x) = \int_0^x g(s)\,\mathrm{d}s. \tag{G_0'}$$

Thus,

$$\hat{\tau}_+ = \liminf_{c\to+\infty}\sqrt{2}\int_0^{d(c)}\frac{\mathrm{d}x}{\sqrt{c-G(x)}};\quad \hat{\tau}^+ = \limsup_{c\to+\infty}\sqrt{2}\int_0^{d(c)}\frac{\mathrm{d}x}{\sqrt{c-G(x)}}, \tag{7}$$

and $h(c)$ and $d(c)$ in (3) satisfy

$$\lim_{c\to+\infty} h(c) = -1,\quad \lim_{c\to+\infty} d(c) = +\infty.$$

We will prove Theorem 1.1 under conditions (g_0'), (g_1) (G_0') and (τ) instead of conditions (g_0), (g_1) (G_0) and (τ).

Consider the equivalent system of (6):

$$x' = y,\quad y' = -g(x) + p(t). \tag{8}$$

Let $(x(t), y(t)) = (x(t;x_0,y_0), y(t;x_0,y_0))$ be the solution of (8) satisfying the initial condition

$$x(0;x_0,y_0) = x_0,\quad y(0;x_0,y_0) = y_0.$$

We now follow a method which was used by [4] [6] and shall need the following result.

Lemma 2.1 *Assume that conditions* (G_0') *and* (g_1) *hold. They every solution of system* (8) *exists uniquely on the whole t-axis.*

By Lemma 2.1, we can define Poincaré map $P:(-1,+\infty)\times\mathbb{R}\to\mathbb{R}^2$ as follows

$$P:(x_0,y_0)\mapsto(x_1,y_1) = (x(2\pi;x_0,y_0), y(2\pi;x_0,y_0)).$$

It is obvious that the fixed points of the Poincaré map P correspond to 2π-periodic solutions of system (8). We will try to find a fixed point of P. To this end, we introduce a function $l:(-1,+\infty)\times\mathbb{R}\to\mathbb{R}^+$,

$$l(x,y) = x^2 + y^2 + \frac{1}{(1+x)^2}.$$

Lemma 2.2 *Assume that* (G_0') *and* (g_1) *hold. Then, for any* $r>0$, *there exists* $\rho>0$ *sufficiently large that, for* $l(x_0,y_0)\ge\rho^2$,

$$l(x(t),y(t))\ge r^2,\quad t\in[0,2\pi],$$

where $(x(t), y(t))$ is the solution of system (8) through the initial point (x_0, y_0).

This result has been proved in [6] and we omit it.

Using Lemma 2.2, we see that $x^2(t)+y^2(t)>0$ for $t\in[0,2\pi]$ if $l(x_0,y_0)$ is large enough. Therefore, transforming to polar coordinates $x=r\cos\theta$, $y=r\sin\theta$, system (8) becomes

$$\begin{cases}\dfrac{\mathrm{d}r}{\mathrm{d}t}=r\sin\theta\cos\theta-g(r\cos\theta)\sin\theta+p(t)\sin\theta,\\[2mm] \dfrac{\mathrm{d}\theta}{\mathrm{d}t}=-\sin^2\theta-\dfrac{1}{r}g(r\cos\theta)\cos\theta+\dfrac{1}{r}p(t)\cos\theta.\end{cases} \tag{9}$$

Denote by $(r(t),\theta(t))=(r(t;r_0,\theta_0),\theta(t;r_0,\theta_0))$ the solution of (9) with

$$r(0)=0,\quad \theta(0)=\theta_0.$$

Thus, we can rewrite the Poincaré map in the form

$$P:(r_0,\theta_0)\mapsto(r_1,\theta_1)=(x(2\pi;r_0,\theta_0),\theta(2\pi;r_0,\theta_0)),$$

where $r_0\cos\theta_0=x_0>-1, r_0\sin\theta_0=y_0$.

For the convenience, two lemmas in [6] will be written and the proof can be found in [6].

Lemma 2.3 *Assume that* (g_0') *and* (g_1) *hold. Then there exists a* $l_0>0$ *such that, for* $l(x,y)\ge l_0$, $\theta'(t)<0, t\in[0,2\pi]$.

Lemma 2.4 *Assume that* (g_0'), (G_0') *and* (g_1) *hold. Then there exists a* $c_0>0$ *such that, for* $c\ge c_0$, $\Gamma_c:\frac{1}{2}y^2+G(x)=c$ *is a star-shaped closed curve about the origin* O.

Lemma 2.5 *Assume that* (g_0'), (g_1) *and* (G_0') *hold. Denote by* $\Delta(r_0,\theta_0)$ *the time for the solution* $(r(t),\theta(t))$ *to make one turn around the origin. Then* $\hat{\tau}_+ +o(1)\le\Delta(r_0,\theta_0)\le\hat{\tau}^+ +o(1)$ *as* $l(x_0,y_0)\to+\infty$, *where* $\hat{\tau}_+$ *and* $\hat{\tau}^+$ *are given in* (7).

Proof. Without loss of generality, we may assume that $x_0=0$. From Lemma 2.3, we have $\frac{\mathrm{d}\theta}{\mathrm{d}t}<0$ for sufficiently large $l(x_0,y_0)$ and $t\in[0,2\pi]$. Hence, there exist $0=t_0<t_1<t_2<t_3<t_4$ such that $y(t_1)=x(t_2)=y(t_3)=x(t_4)=0$, and

$$x(t)>0, y(t)>0, t\in(t_0,t_1);\quad x(t)>0, y(t)<0, t\in(t_1,t_2);$$
$$x(t)<0, y(t)<0, t\in(t_2,t_3);\quad x(t)<0, y(t)>0, t\in(t_3,t_4).$$

Throughout the lemma, we always assume that $l(x_0,y_0)$ is large enough.

(1) We shall first estimate t_3-t_2 and t_4-t_3. We can refer to Lemma 2.6 in [6] and obtain $t_3-t_2=o(1)$, $t_4-t_3=o(1)$ as $l(x_0,y_0)\to+\infty$.

(2) We now estimate t_1-t_0 and t_2-t_1. According to conditions (G_0') and (g_1), we can choose a constant $M>0$ such that $2G(x)+M>0$ for $x\in(-1,+\infty)$. Set

$$u(t)=\sqrt{y^2(t)+2G(x(t))+M}. \tag{10}$$

Then,

$$|u'(t)|=\frac{|y(t)y'(t)+g(x(t))x'(t)|}{\sqrt{y^2(t)+2G(x(t))+M}}=\frac{|y(t)p(t)|}{\sqrt{y^2(t)+2G(x(t))+M}}\le|p(t)|.$$

Therefore, for $t,s\in\left[0,\frac{4\pi}{n}\right]$,

$$|u(t)-u(s)|\le e,\quad e=\int_0^{\frac{4\pi}{n}}|p(t)|\mathrm{d}t.$$

Note that $u(0)=\sqrt{2c+M}$, we get

$$\sqrt{2c+M}-e\le u(t)\le\sqrt{2c+M}+e,\quad t\in\left[0,\frac{4\pi}{n}\right].$$

Since $G(d(c)) = c$, we have

$$\sqrt{2G(d)+M} - e \le u(t) \le \sqrt{2G(d)+M} + e,$$

where $d = d(c)$. By condition (g_1), we know that $\sqrt{2G(x)+M}$ increases for *x* sufficiently large, and tends to $+\infty$ as $x \to +\infty$. Therefore, there exist constants $a > d > b > 0$ such that

$$\sqrt{2G(a)+M} = \sqrt{2G(d)+M} + e, \quad \sqrt{2G(b)+M} = \sqrt{2G(d)+M} - e. \tag{11}$$

By (10) and (11), we have

$$2\big(G(b) - G(x(t))\big) \le y^2(t) \le 2\big(G(a) - G(x(t))\big). \tag{12}$$

Let $t_b \in (t_0, t_1)$ be such that $x(t_b) = b$, and $0 \le x(t) \le b, t \in [0, t_b]$. Following (12), we derive

$$\sqrt{2\big(G(b) - G(x(t))\big)} \le y(t) \le \sqrt{2\big(G(a) - G(x(t))\big)}, \tag{13}$$

that is,

$$\sqrt{2\big(G(b) - G(x(t))\big)} \le x'(t) \le \sqrt{2\big(G(a) - G(x(t))\big)}.$$

Consequently,

$$\frac{x'(t)}{\sqrt{2\big(G(a) - G(x(t))\big)}} \le 1 \le \frac{x'(t)}{\sqrt{2\big(G(b) - G(x(t))\big)}}.$$

Integrating both sides of the above inequality from t_0 to t_b, we obtain

$$\int_0^b \frac{\mathrm{d}x}{\sqrt{2\big(G(a) - G(x)\big)}} \le t_b - t_0 \le \int_0^b \frac{\mathrm{d}x}{\sqrt{2\big(G(b) - G(x)\big)}}. \tag{14}$$

Recalling the conditions (g_1) and (11), we know that there is $\varsigma > 0$, such that $|a - b| \le \varsigma$. Applying Lemma 2.8 in [6], we can derive

$$\int_0^a \frac{\mathrm{d}x}{\sqrt{2\big(G(a) - G(x)\big)}} = \int_0^b \frac{\mathrm{d}x}{\sqrt{2\big(G(a) - G(x)\big)}} + o(1) \tag{15}$$

for $c \to +\infty$. Combining (14) and (15), we have

$$\int_0^a \frac{\mathrm{d}x}{\sqrt{2\big(G(a) - G(x)\big)}} + o(1) \le t_b - t_0 \le \int_0^b \frac{\mathrm{d}x}{\sqrt{2\big(G(b) - G(x)\big)}}.$$

From [10], we know that

$$\int_0^b \frac{\mathrm{d}x}{\sqrt{2\big(G(b) - G(x)\big)}} = \int_0^a \frac{\mathrm{d}x}{\sqrt{2\big(G(a) - G(x)\big)}} + o(1) = \int_0^d \frac{\mathrm{d}x}{\sqrt{2\big(G(d) - G(x)\big)}} + o(1)$$

for $c \to +\infty$. Hence,

$$t_b - t_0 = \int_0^d \frac{\mathrm{d}x}{\sqrt{2\big(G(d) - G(x)\big)}} + o(1).$$

In the following, we deal with $t_1 - t_b$. Integrating $y'(t) = -g(x(t)) + p(t)$ from t_b to t_1, we get

$$-y(t_b) = y(t_1) - y(t_b) = -\int_{t_b}^{t_1} g(x(t))\mathrm{d}t + \int_{t_b}^{t_1} p(t)\mathrm{d}t. \tag{16}$$

By (13), we derive

$$y(t_b) \le \sqrt{2\big(G(a) - G(b)\big)}. \tag{17}$$

On the other hand, from (11) we have

$$\sqrt{2G(a)+M}-\sqrt{2G(b)+M}=2e.$$

As a result,

$$\frac{\left(\sqrt{G(a)}-\sqrt{G(b)}\right)\left(\sqrt{G(a)}+\sqrt{G(b)}\right)}{\sqrt{2G(a)+M}+\sqrt{2G(b)+M}}=e.$$

Accordingly,

$$\left|\sqrt{G(a)}-\sqrt{G(b)}\right|<2e. \tag{18}$$

Meanwhile, following (g_1), for any given $A>0$ sufficiently large, there exist $N\geq b$ large enough, such that

$$g(x)>A, \quad \text{for } x>N. \tag{19}$$

Combining (16)-(19), we get

$$t_1-t_b\leq\frac{2\left(\sqrt{G(a)}-\sqrt{G(b)}\right)}{A-E}=o(1)$$

for $b\to+\infty$, where $E=\max_{t\in\mathbb{R}}|p(t)|$. Thus,

$$t_1-t_0=\int_0^{d(c)}\frac{\mathrm{d}x}{\sqrt{2(c-G(x))}}+o(1). \tag{20}$$

Using the same arguments as above, we can get

$$t_2-t_1=\int_0^{d(c)}\frac{\mathrm{d}x}{\sqrt{2(c-G(x))}}+o(1). \tag{21}$$

By the conditions (20), (21), we have

$$\frac{\hat{\tau}_+}{2}+o(1)\leq t_1-t_0\leq\frac{\hat{\tau}^+}{2}+o(1),$$
$$\frac{\hat{\tau}_+}{2}+o(1)\leq t_2-t_1\leq\frac{\hat{\tau}^+}{2}+o(1).$$

Recalling $t_3-t_2=o(1)$, $t_4-t_3=o(1)$, we have

$$\hat{\tau}_-+o(1)\leq\Delta(r_0,\theta_0)=(t_1-t_0)+(t_2-t_1)+(t_3-t_2)+(t_4-t_3)\leq\hat{\tau}^++o(1).$$

The proof is complete.

3. Proof of Theorem 1.1

In this section, we establish the existence of harmonic solutions for Equation (1) by appealing to Poincaré-Bohl theorem [11]. We consider the Poincaré map

$$P:(r_0,\theta_0)\mapsto(r_1,\theta_1)=\left(r(2\pi;r_0,\theta_0),\theta(2\pi;r_0,\theta_0)\right), \quad r_0\cos\theta_0>-1.$$

From Lemma 2.5 and condition (τ), we obtain

$$\frac{2\pi}{n+1}<\Delta(r_0,\theta_0)=(t_1-t_0)+(t_2-t_1)+(t_3-t_2)+(t_4-t_3)<\frac{2\pi}{n},$$

which implies

$$-2(n+1)\pi<\theta(2\pi;r_0,\theta_0)-\theta_0<-2n\pi.$$

Thus, the image $(r_1,\theta_1) = P(r_0,\theta_0)$ cannot lie on the line $\theta = \theta_0$. Therefore, the Poincaré-Bohl theorem guarantees that the map P has at least one fixed point, *i.e.* Equation (6) has at least one 2π-periodic solution.

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