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Characterization of Self Dual Lattices in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$

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Abstract

This paper shows that the only self dual lattices in $\mathbb{R},\mathbb{R}^2,\mathbb{R}^3$ are rotations of $\mathbb{Z}$, $\mathbb{Z}\times\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$.

Keywords

Self Dual Lattice

1. Introduction

Let

$$A := [a_1,\cdots,a_n], B := [b_1,\cdots,b_n],$$

be nonsingular $n\times n$ real matrices with column vectors $a_1,\cdots,a_n$ and $b_1,\cdots,b_n$, respectively. Let

$$\mathcal{L}_A := \left\{\sum_{k=1}^{n} m_k a_k : m_1,\cdots,m_n \in \mathbb{Z}\right\},$$

$$\mathcal{L}_B := \left\{\sum_{k=1}^{n} m_k b_k : m_1,\cdots,m_n \in \mathbb{Z}\right\}.$$

be the lattices in $\mathbb{R}^n$ that are generated by the columns of A, B. The lattice $\mathcal{L}_A$ will be a subset of the lattice $\mathcal{L}_B$ if and only if the generators $a_1,\cdots,a_n$ of $\mathcal{L}_A$ all lie in $\mathcal{L}_B$, *i.e.*,

$$a_k = \sum_{l=1}^{n} m_{lk} b_l, k = 1,2,\cdots,n$$

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for suitably chosen integers m_{lk}. Equivalently,

$$[a_1,\cdots,a_n]=[b_1,\cdots,b_n]\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

i.e.,

$$M := B^{-1}A$$

is a matrix of integers. Analogously, the lattice $\mathcal{L}_B$ is a subset of $\mathcal{L}_A$ if $A^{-1}B$ is a matrix of integers. In this way we see that

$$\mathcal{L}_A = \mathcal{L}_B$$

if and only if both $M = B^{-1}A$ and

$$A^{-1}B = \left(B^{-1}A\right)^{-1} = M^{-1}$$

are matrices with integer elements. When this is the case, $\det M$ and $\det M^{-1}$ are both integers and since

$$\det M \det M^{-1} = \det MM^{-1} = \det I = 1,$$

this implies that

$$\det M = \det M^{-1} = \pm 1.$$

Such a matrix is said to be unimodular. The above analysis (that can be found in [1]) is summarized as follows.

Theorem 1 The lattices $\mathcal{L}_A, \mathcal{L}_B$ are identical if and only if

$$M := A^{-1}B$$

is a matrix of integers with

$$\det M = \pm 1$$

Corollary 1 Lattices are preserved under integer column operations.

Proof 1 Let $A=[a_1,\cdots,a_n]$ generate the lattice $\mathcal{L}_A$, and let

$$K = \begin{bmatrix} 0 & k_{12} & k_{13}\cdots & \cdots & k_{1n} \\ 0 & 0 & k_{23} & \cdots & k_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k_{n-1n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

be a strictly upper triangular matrix of integers. Then $I+K$ is an upper triangular matrix of integers with a unit diagonal, and we can write

$$(I+K)^{-1} = I + L$$

where

$$L := -K + K^2 - K^3 + \cdots + (-1)^{n-1} K^{n-1}$$

is a strictly upper triangular matrix of integers. The columns of

$$B := A(I+K)$$

i.e.,

$$a_1, k_{12}a_1 + a_2, k_{13}a_1 + k_{23}a_2 + a_3, \cdots$$

generate the same lattice as the columns of A. To see this we observe that

$$B^{-1}A = \left[A(I+K)\right]^{-1} A = (I+K)^{-1} = I + L$$

is a matrix of integers with unit determinant.

2. Dual Lattices

Definition 1 Two linearly independent sets of real n (column) vectors $a_1,\cdots,a_n$ and $b_1,\cdots,b_n$ are said to be biorthogonal if

$$\langle a_k, b_l \rangle := a_k^{\mathrm{T}} b_l = \delta_{kl}, k, l = 1, 2, \cdots, n$$

where δ_{kl} is the Kronecker's delta, T denotes the transpose and $\langle\ \rangle$ denotes the usual inner product. When the columns of

$$A := [a_1, \cdots, a_n]$$

and

$$B := [b_1, \cdots, b_n]$$

are biorthogonal, we find

$$A^{\mathrm{T}} B = I$$

so that

$$B = \left(A^{\mathrm{T}}\right)^{-1} =: A^{-\mathrm{T}}.$$

This being the case, given linearly independent vectors $a_1,\cdots,a_n$ we can form A and then obtain the biorthogonal vectors $b_1,\cdots,b_n$ as the columns of $A^{-\mathrm{T}}$.

The lattice $\mathcal{L}_{A^{-\mathrm{T}}}$ generated by vectors biorthogonal to $a_1,\cdots,a_n$ is said to be the dual of the lattice $\mathcal{L}_A$. More generally, $\mathcal{L}_B$ is dual to $\mathcal{L}_A$ if and only if B generates the same lattice as $A^{-\mathrm{T}}$, *i.e.*,

$$\left(A^{-\mathrm{T}}\right)^{-1} B = A^{\mathrm{T}} B$$

is a matrix of integers with determinant ± 1.

Suppose now that A_1, A_2 generate the same lattice, *i.e.*,

$$\mathcal{L}_{A_1} = \mathcal{L}_{A_2}.$$

Let

$$B_1 = A_1^{-\mathrm{T}}, B_2 = A_2^{-\mathrm{T}}$$

be the generators of lattices $\mathcal{L}_{B_1}, \mathcal{L}_{B_2}$ dual to $\mathcal{L}_{A_1}, \mathcal{L}_{A_2}$, respectively. Since

$$B_2^{-1} B_1 = \left(A_2^{-\mathrm{T}}\right)^{-1} A_1^{-\mathrm{T}} = A_2^{\mathrm{T}} A_1^{-\mathrm{T}} = \left(A_1^{-1} A_2\right)^{\mathrm{T}}$$

we see that $A_1^{-1} A_2$ will be a matrix of integers with determinant ± 1 if and only if the same is true of $B_2^{-1} B_1$. Thus $\mathcal{L}_{B_1} = \mathcal{L}_{B_2}$ if and only if $\mathcal{L}_{A_1} = \mathcal{L}_{A_2}$.

We are interested in characterizing those lattices $\mathcal{L}_A$ that are self dual, *i.e.*,

$$\mathcal{L}_A = \mathcal{L}_{A^{-\mathrm{T}}}.$$

This will be the case if and only if

$$\left(A^{-\mathrm{T}}\right)^{-1} A = A^{\mathrm{T}} A$$

is a matrix of integers with determinant ± 1. Since

$$\det A^{\mathrm{T}} A = (\det A)^2,$$

this will be the case only if

$$\det A^{\mathrm{T}} A = 1$$

or equivalently

$$\det A = \pm 1.$$

In this way we see that a lattice $\mathcal{L}_A$ is self dual if and only if $A^{\mathrm{T}} A$ is a matrix of integers with unit determinant. The parallelopiped in $\mathbb{R}^n$ with vertices $0, a_1, a_2 \cdots, a_n, a_1 + a_2, a_1 + a_3, \cdots, a_1 + a_2 + \cdots + a_n$, *i.e.*, the unit cell of the lattice has the volume

$$V(a_1, a_2, \cdots, a_n) = |\det A|,$$

[2] [3]. Thus a lattice can be self dual only if each of its primitive cells, has unit volume.

Self dual lattices are preserved under orthogonal transformations. Indeed, let Q be an orthogonal transformation on $\mathbb{R}^n$, *i.e.*,

$$Q^{\mathrm{T}} Q = I,$$

and let $\mathcal{L}_A, \mathcal{L}_B$ be the lattices generated by the columns of a nonsingular $n \times n$ matrix A and $B := A^{-\mathrm{T}}$. The matrix

$$A' := QA$$

has columns

$$a_1' = Qa_1, a_2' = Qa_2, \cdots, a_n' = Qa_n$$

that generate the lattice $\mathcal{L}_{A'}$. We can use such a matrix Q to rotate $a_1, a_2 \cdots, a_n$, to reflect one or more vectors of the set $a_1, a_2 \cdots, a_n$, to permute $a_1, a_2 \cdots, a_n$, etc. The lattice $\mathcal{L}_{B'}$ which is dual to $\mathcal{L}_{A'}$ is generated by the columns of

$$B' = (A')^{-\mathrm{T}} = (QA)^{-\mathrm{T}} = Q^{-\mathrm{T}} A^{-\mathrm{T}} = QB,$$

i.e., by

$$b_1' = Qb_1, b_2' = Qb_2, \cdots, b_n' = Qb_n.$$

Thus the generators of the dual lattice $\mathcal{L}_B$ are transformed in the same way as the generators of the lattice $\mathcal{L}_A$. In this way we see that a lattice $\mathcal{L}_A$ is self dual if and only if the lattice $\mathcal{L}_{A'}$ is self dual. Indeed,

$$(A')^{\mathrm{T}} A' = (QA)^{\mathrm{T}} QA = A^{\mathrm{T}} A$$

so $A^{\mathrm{T}} A$ is a matrix of integers with unit determinant if and only if the same is true of $(A')^{\mathrm{T}} A'$. Moreover, since

$$\|Qx\|_2^2 = x^{\mathrm{T}} Q^{\mathrm{T}} Qx = x^{\mathrm{T}} x = \|x\|_2^2$$

we see that the orthogonal transformation Q preserves the Euclidean lengths of a set of generators for the lattice $\mathcal{L}_A$.

3. Main Results

We will now show that the only self dual lattices in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ are rotations of $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$, and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, respectively.

The case n = 1

Let $A = [a_1]$ be a vector in $\mathbb{R}$ that generates the lattice $\mathcal{L}_A$. We do not change the lattice if we assume that $a_1 > 0$. Let $b_1 = 1/a_1$ be biorthogonal to A. The lattice $\mathcal{L}_B$ generated by $B = [b_1]$ will be identical to the lattice $\mathcal{L}_A$ if and only if

$$a_1 = \frac{1}{a_1},$$

i.e., if and only if

$$a_1 = 1.$$

Thus the only self dual lattice in $\mathbb{R}$ is the lattice

$$\mathcal{L} = \mathbb{Z}.$$

The case *n* = 2

Theorem 2 Every self dual lattice in $\mathbb{R}^2$ is some rotation of $\mathbb{Z}\times\mathbb{Z}$.

Proof 2 Let $A = [a_1 \quad a_2]$ where a_1, a_2 are linearly independent vectors in $\mathbb{R}^2$ and assume that $\mathcal{L}_A$ is self dual. Fix the origin at some lattice point of $\mathcal{L}_A$ and rotate the axes, if necessary, so that the nearest nonzero lattice point of $\mathcal{L}_{A'}$ lies on the positive x-axis, *i.e.*

$$QA = A' = [a_1' \quad a_2'] = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}$$

where $\alpha > 0$ and

$$\alpha^2 \le \beta^2 + \gamma^2. \tag{1.1}$$

The lattice $\mathcal{L}_{A'}$ does not change if a_2' is replaced by $-a_2'$ so we can and do assume that $\gamma > 0$. Likewise the lattice $\mathcal{L}_{A'}$ does not change if a_2' is replaced by $a_2' - ka_1', k = 0, \pm 1, \pm 2, \cdots$ since this is the result of an integer column operation. Thus we can and do assume that

$$|\beta| \le \alpha/2. \tag{1.2}$$

By hypothesis the lattice $\mathcal{L}_A$ is self dual so the same is true of $\mathcal{L}_{A'}$. This implies that

$$\alpha\gamma = \det A' = 1,$$

and

$$(A')^{-\mathrm{T}} = \begin{bmatrix} \gamma & 0 \\ -\beta & \alpha \end{bmatrix}.$$

Since $\mathcal{L}_{A'}$ is self dual, the first column of A' can be expressed as an integral linear combination of the columns of $(A')^{-\mathrm{T}}$, *i.e.*,

$$\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = n \begin{bmatrix} \gamma \\ -\beta \end{bmatrix} + m \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

where $n, m \in \mathbb{Z}$. In this way we see in turn that

$$\alpha = n\gamma, \alpha = n/\alpha, \alpha = \sqrt{n}, \tag{1.3}$$

for some $n = 1, 2, \cdots,$

$$n\beta = m\alpha, \beta = m/\sqrt{n}, \tag{1.4}$$

for some $m = 0, \pm 1, \pm 2, \cdots,$ and

$$\gamma = 1/\alpha = 1/\sqrt{n}. \tag{1.5}$$

Using these expressions with (1.2) we find

$$\frac{|m|}{\sqrt{n}} \le \frac{\sqrt{n}}{2}$$

so

$$|m| \le \frac{n}{2}.$$

Using these expressions with (1.1) we find

$$n = \alpha^2 \le \beta^2 + \gamma^2 = \frac{m^2}{n} + \frac{1}{n},$$

and since

$$|m| \le \frac{n}{2}.$$

this implies that

$$n^2 \le 4/3.$$

It follows that $n = 1$ and $m = 0$. In this way we prove that $A' = I$, *i.e.*, the columns of A' and thus those of A are orthonormal. Thus $\mathcal{L}_A$ is some rotation of $\mathbb{Z} \times \mathbb{Z}$.

A theorem of Minkowski [1] states that

$$\|a\|_2 \le \sqrt{N}\,|\det A|^{1/n}$$

where a is the shortest nonzero vector in a lattice $\mathcal{L}_A$ in $\mathbb{R}^n$. Within the present context, this leads to the bound

$$\sqrt{n} = \alpha \le \sqrt{2}$$

which implies that $n = 1, 2.$ Our argument gives $n^2 \le 4/3$ from which we immediatly obtain $n = 1$.

Another result in [4] states that if Λ is a self-dual lattice in $\mathbb{R}^n$ then

$$\|a\|_2^2 = \min\left\{\langle u, u\rangle \,\middle|\, u \in \Lambda, u \ne 0\right\} \le [n/8] + 1$$

which leads to

$$\alpha \le \sqrt{5/4}.$$

The case *n* = 3

Theorem 3 Every self dual lattice in $\mathbb{R}^3$ is some rotation of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Proof 3 Let the self dual lattice $\mathcal{L}_A$ in $\mathbb{R}^3$ be generated by the columns of $A = [a_1 \quad a_2 \quad a_3]$ chosen so that $\|a_1\|_2, \|a_2\|_2, \|a_3\|_2$ are as small as possible subject to the constraint

$$\|a_1\|_2 \le \|a_2\|_2 \le \|a_3\|_2.$$

Following the analysis from the previous section, we set

$$A' = QA,$$

where Q is an orthogonal matrix chosen so that

$$A' = [a_1' \quad a_2' \quad a_3'] = \begin{bmatrix} \alpha & \beta & \delta \\ 0 & \gamma & \varepsilon \\ 0 & 0 & \zeta \end{bmatrix}$$

with

$$\alpha > 0, \gamma > 0, \zeta > 0.$$

By hypothesis the lattice $\mathcal{L}_A$ is self dual, and since Q is orthogonal, the same is true of $\mathcal{L}_{A'}$. This being the case

$$\alpha\gamma\zeta = \det A' = |\det A| = 1.$$

Since the lengths of the generators of the lattice $\mathcal{L}_A$ are preserved under the orthogonal transformation Q, it follows that

$$\alpha^2 \le \beta^2 + \gamma^2 \le \delta^2 + \varepsilon^2 + \zeta^2. \tag{1.6}$$

The columns of A (and thus the columns of A') have been chosen to be as small as possible subject to the above constraints, so we must have

$$|\beta| \le \alpha/2, |\delta| \le \alpha/2, |\varepsilon| \le \gamma/2. \tag{1.7}$$

It can be verified that A' has the inverse

$$(A')^{-1} = \begin{bmatrix} 1/\alpha & -\beta/(\alpha\gamma) & -\delta/(\alpha\zeta)+\beta\varepsilon/(\alpha\gamma\zeta) \\ 0 & 1/\gamma & -\varepsilon/(\gamma\zeta) \\ 0 & 0 & 1/\zeta \end{bmatrix},$$

and after using $\alpha\gamma\zeta = 1$ to simplify the components we obtain

$$(A')^{-\mathrm{T}} = \begin{bmatrix} 1/\alpha & 0 & 0 \\ -\beta\zeta & 1/\gamma & 0 \\ -\delta\gamma+\beta\varepsilon & -\alpha\varepsilon & 1/\zeta \end{bmatrix}.$$

Since $\mathcal{L}_{A'}$ is self dual, the columns of $(A')^{-\mathrm{T}}$ generate the same lattice as the columns of A' so we can write

$$\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = n\begin{bmatrix} 1/\alpha \\ -\beta\zeta \\ -\delta\gamma+\beta\varepsilon \end{bmatrix} + m\begin{bmatrix} 0 \\ 1/\gamma \\ -\alpha\varepsilon \end{bmatrix} + l\begin{bmatrix} 0 \\ 0 \\ 1/\zeta \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 1/\zeta \end{bmatrix} = p\begin{bmatrix} \delta \\ \varepsilon \\ \zeta \end{bmatrix} + q\begin{bmatrix} \beta \\ \gamma \\ 0 \end{bmatrix} + r\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

for suitably chosen $n, m, l, p, q, r \in \mathbb{Z}$. In this way we see in turn that

$$\alpha^2 = n \text{ so that } \alpha = \sqrt{n} \tag{1.8}$$

$$\frac{1}{\zeta^2} = p \text{ so that } \zeta = \frac{1}{\sqrt{p}} \tag{1.9}$$

for some $n = 1, 2, \cdots, p = 1, 2, \cdots,$ and

$$1 = \alpha\gamma\zeta = \sqrt{n}\gamma\frac{1}{\sqrt{p}} \text{ so that } \gamma = \frac{\sqrt{p}}{\sqrt{n}}. \tag{1.10}$$

We also have

$$0 = -n\beta\zeta + \frac{m}{\gamma} \text{ so that } \beta = \frac{m}{\sqrt{n}} \tag{1.11}$$

$$0 = p\varepsilon + q\gamma \text{ so that } \varepsilon = \frac{-q}{\sqrt{pn}}, \tag{1.12}$$

for some $m = 0, \pm1, \pm2, \cdots, q = 0, \pm1, \pm2, \cdots,$ and

$$0 = n(-\delta\gamma + \beta\varepsilon) - m\varepsilon\alpha + \frac{l}{\zeta} = -\delta\sqrt{np} + l\sqrt{p}$$

so that

$$\delta = \frac{l}{\sqrt{n}} \text{ for some } l = 0, \pm1, \pm2, \cdots. \tag{1.13}$$

Using (1.7) and (1.8)-(1.12) we find

$$2|m| \le n, 2|q| \le p, 2|l| \le n. \tag{1.14}$$

Using (1.6) and (1.7) we see that,

$$\alpha^2 \le \beta^2 + \gamma^2 \le \left(\frac{\alpha}{2}\right)^2 + \gamma^2$$

which implies that

$$\gamma \ge \frac{\sqrt{3}}{2}\alpha.$$

Again using (1.6) and (1.7) we see that,

$$\gamma^2 \le \beta^2 + \gamma^2 \le \delta^2 + \varepsilon^2 + \zeta^2 \le \left(\frac{\alpha}{2}\right)^2 + \left(\frac{\gamma}{2}\right)^2 + \zeta^2$$

which implies that

$$\zeta^2 \ge \frac{3}{4}\gamma^2 - \frac{1}{4}\alpha^2 \ge \frac{9}{16}\alpha^2 - \frac{1}{4}\alpha^2 = \frac{5}{16}\alpha^2$$

so that

$$\zeta \ge \frac{\sqrt{5}}{4}\alpha.$$

Since $\alpha\gamma\zeta = 1$ we must have

$$1 = \alpha\gamma\zeta \ge \alpha\left(\frac{\sqrt{3}}{2}\alpha\right)\left(\frac{\sqrt{5}}{4}\alpha\right) = \frac{\sqrt{15}}{8}\alpha^3$$

or

$$\sqrt{n} = \alpha \le \left(\frac{8}{\sqrt{15}}\right)^{1/3} = 1.2735\ldots.$$

In this way we see in turn that $n = 1$ and $m = l = 0$ so that $\alpha = 1, \beta = 0, \delta = 0$. Finally, we again use (1.6) with (1.13), (1.12), (1.9) to write

$$p = \gamma^2 \le \delta^2 + \varepsilon^2 + \zeta^2 = \frac{l^2}{n} + \frac{q^2}{pn} + \frac{1}{p} = \frac{q^2}{p} + \frac{1}{p} \le \frac{p^2}{4p} + \frac{1}{p}.$$

It follows that $p \le \sqrt{4/3}$ so we must have $p = 1, q = 0$ and $\varepsilon = 0, \gamma = \zeta = 1$. In this way we see that the columns of A' (and thus those of A) must be orthonormal. Thus $\mathcal{L}_A$ is some rotation of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Suppose now that a_1, a_2 are linearly independent vectors in $\mathbb{R}^2$ and that

$$\text{grid}_{a_1,a_2}(x) := \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(x - ma_1 - na_2) = \sum_{a \in \mathcal{L}_{\mathcal{A}}} \delta(x - a)$$

where $\mathcal{A} := [a_1 \quad a_2]$. We know that

$$\text{grid}_{\hat{a}_1,a_2}(s) = \left|\det[A_1 \quad A_2]\right| \text{grid}_{A_1,A_2}(s) = \left|\det[A_1 \quad A_2]\right| \sum_{a \in \mathcal{L}_{\mathcal{A}} - T} \delta(s - a)$$

where the biorthogonal vectors A_1, A_2 are the columns of $\mathcal{A}^{-T}$. In this way we see that

$$\text{grid}_{\hat{a}_1,a_2} = \text{grid}_{a_1,a_2}$$

if and only if $\mathcal{L}_A$ is self dual, where $4pt\mathcal{A} = [a_1 \quad a_2]$. This proves the following.

Theorem 4 Let a_1, a_2 be linearly independent vectors in $\mathbb{R}^2$. Then

$$\text{grid}_{\hat{a}_1,a_2} = \text{grid}_{a_1,a_2}$$

if and only if

$$\text{grid}_{a_1,a_2} = \text{grid}_{a'_1,a'_2}$$

for some orthonormal choice of the vectors a_1', a_2'.

Analogously, we can prove the following 3-dimensional generalization.

Theorem 5 Let a_1, a_2, a_3 be linearly independent vectors in $\mathbb{R}^3$. Then

$$\text{grid}_{\hat{a}_1, a_2, a_3} = \text{grid}_{a_1, a_2, a_3}$$

if and only if

$$\text{grid}_{a_1, a_2} = \text{grid}_{a_1', a_2', a_3'}$$

for some orthonormal choice of the vectors a_1', a_2', a_3'.

These results correspond to the familiar identity

$$\text{III}^{\wedge} = \text{III}$$

from univariate Fourier analysis. The possibility of rotations (other than reflections) in $\mathbb{R}^2, \mathbb{R}^3$ slightly complicates the generalization of this result.

References

[1] Kannan, R. (1987) Minkowski's Convex Body Theorem and Integer Programming. *Mathematics of Operations Research*, **12**, 415-440.

[2] Strang, G. (1976) Linear Algebra and Its Applications. Academic Press, Inc., New York.

[3] Senechal, M. (1995) Quasicrystals and Geometry. Cambridge University Press, New York.

[4] Conway, J.H., Odlyzko, A.M. and Sloane, N.J.A. (1978) Extremal Self-Dual Lattices Exit Only in Dimensions 1 to 8, 12, 14, 15, 23, and 24.