

Positive Periodic Solution for a Two-Species Predator-Prey System

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Abstract

A two-species predator-prey system with time delay in a two-patch environment is investigated. By using a continuation theorem based on coincidence degree theory, we obtain some sufficient conditions for the existence of periodic solution for the system.

Keywords

Predator-Prey System, Diffusion, Periodic Solution, Coincidence Degree

1. Introduction

Dynamical systems generated by predator-prey models have long been the topic of research interest of many biomathematical scholars, and there have been vast studies to investigate the dynamics of predator-prey models, see e.g., Refs. [1]-[12] and references therein. In 1975, Beddington [13] and DeAngelis [14] proposed the predator-prey system with the Beddington-DeAngelis functional response as follows.

$$\begin{cases} x' = x \left(a - bx - \frac{cy}{m_1 + m_2x + m_3y} \right), \\ y' = y \left(-d + \frac{fx}{m_1 + m_2x + m_3y} \right), \end{cases} \quad (1.1)$$

In the last years, some experts have studied the system [15]-[21]. Recently, Li and Takeuchi [22] proposed the following model with both Beddington-DeAngelis functional response and density dependent predator

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$$\begin{cases} x' = x \left(a - bx - \frac{cy}{m_1 + m_2x + m_3y} \right), \\ y' = y \left(-d - ey + \frac{fx}{m_1 + m_2x + m_3y} \right), \end{cases} \tag{1.2}$$

and discussed the dynamic behaviors of the model. In this paper, we consider the following nonautonomous two-species predator-prey system with diffusion and time delays.

$$\begin{cases} x'_1(t) = x_1(t) \left[a_1(t) - b_1(t)x_1(t) - h_1(t) - \frac{c(t)x_3(t)}{m_1(t) + m_2(t)x_1(t) + m_3(t)x_3(t)} \right] + D_1(t)[x_2(t) - x_1(t)] \\ x'_2(t) = x_2(t) [a_2(t) - b_2(t)x_2(t)] + D_2(t)[x_1(t) - x_2(t)] \\ x'_3(t) = x_3(t) \left[-d(t) - e(t)x_3(t) - h_2(t) - \frac{f(t)x_1(t-\tau)}{m_1(t) + m_2(t)x_1(t-\tau) + m_3(t)x_3(t-\tau)} \right] \end{cases} \tag{1.3}$$

where $x_i(t)$ represents the prey population in the i th patch ($i = 1, 2$), and $x_3(t)$ represents the predator population. $D_i(t)$ denotes the dispersal rate of the prey in the i th patch ($i = 1, 2$). We always make the following fundamental assumptions for system (1.3): τ is positive constant and $a_i(t)$ ($i = 1, 2$), $b_i(t)$, $h_i(t)$, $D_i(t)$, $c(t)$, $d(t)$, $e(t)$, $f(t)$, $m_1(t)$, $m_2(t)$, $m_3(t)$ re positive continuous ω -periodic functions.

The main purpose of this paper is, by using the coincidence degree theory to derive the sufficient conditions for the existence of periodic solution of (1.3).

2. Preliminaries

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. we shall use some concepts and results from the book by Gaines and Mawhin [23].

Let X, Z be real Banach spaces, $L : DomL \subset X \rightarrow Z$ be a linear mapping, and $N : X \rightarrow Z$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if

$$\dim KerL = co \dim Im L < +\infty$$

and $Im L$ is closed in Z . If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \rightarrow X$ and $Q : Z \rightarrow Z$ such that $Im P = KerL$, $Im L = KerQ = Im(I - Q)$, then the restriction L_P of L to $DomL \cap KerP : (I - P)X \rightarrow Im L$ is invertible. Denote the inverse of L_P by K_P . If Ω is an open bounded subset of X , the mapping N will be called L -compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \rightarrow X$ is compact. Since $Im Q$ is isomorphic to $KerL$, there exists isomorphism $J : Im Q \rightarrow KerL$.

Lemma 2.1 (Continuation theorem [23]) Let $\Omega \subset X$ be an open bounded set, L be a Fredholm mapping of index zero and N be L -compact on $\bar{\Omega}$. Assume

- 1) for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap DomL$, $Lx \neq \lambda Nx$;
- 2) for each $x \in KerL \cap \partial\Omega$, $QNx \neq 0$;
- 3) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $DomL \cap \bar{\Omega}$.

Throughout this paper, we adopt the notations $\bar{f} = \frac{1}{\omega} \int_0^\omega f(t) dt$, $f^l = \min_{t \in [0, \omega]} |f(x)|$, $f^M = \max_{t \in [0, \omega]} |f(x)|$, where $f(t)$ is an ω -periodic continuous function.

3. Main Result

Theorem 3.1 Assume that

- 1) $a_1(t) - h_1(t) - D_1(t) > 0$;
- 2) $h_2(t) - D_2(t) > 0$;

$$3) \overline{a_1(t) - h_1(t) - D_1(t)} - \overline{\left(\frac{c}{m_3}\right)} > 0;$$

$$4) \overline{\left(\frac{f}{m_2}\right)} - \overline{d + h_2} > 0.$$

Then system (1.3) has at least one positive ω -periodic solution.

Proof. Let $x_i(t) = e^{u_i(t)}$, $u'_i(t) = Y_i(t)$ ($i = 1, 2, 3$), $u = (u_1, u_2, u_3)^T$, $u' = (u'_1, u'_2, u'_3)^T$, $Y = (Y_1, Y_2, Y_3)^T$, then (1.3) can be rewritten as follows:

$$\begin{cases} Y_1(t) = a_1(t) - h_1(t) - b_1(t)e^{u_1(t)} - \frac{c(t)e^{u_3(t)}}{m_1(t) + m_2(t)e^{u_1(t)} + m_3(t)e^{u_3(t)}} + D_1(t)e^{u_2(t) - u_1(t)} \\ Y_2(t) = a_2(t) - b_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t) - u_2(t)} \\ Y_3(t) = -d(t) - h_2(t) - e(t)e^{u_3(t)} - \frac{f(t)e^{u_1(t-\tau)}}{m_1(t) + m_2(t)e^{u_1(t-\tau)} + m_3(t)e^{u_3(t-\tau)}}, \end{cases} \tag{3.1}$$

where all function are defined as ones in system (1.3). It is easy to know that if (3.1) has one ω -periodic solution $(x_1^*(t), x_2^*(t), x_3^*(t))^T$, then $(e^{x_1^*(t)}, e^{x_2^*(t)}, e^{x_3^*(t)})^T$ is a positive ω -periodic solution of system (1.3). Therefore, to complete the proof, it suffices to show that system (3.1) has one ω -periodic solution.

Take $X = Z = \{u \in C(R, R^3) : u(t + \omega) = u(t)\}$ and $\|u\| = \sum_{i=1}^3 \max_{t \in (0, \omega)} |u_i(t)|$, then X and Z are Banach space with the norm $\|\cdot\|$.

Set $Lu = u'$, $Nu = Y$, $Pu = \frac{1}{\omega} \int_0^\omega u(t) dt$, $Qu = \frac{1}{\omega} \int_0^\omega u(t) dt$, $u \in X$. Obviously, $KerL = R^3$,

$ImL = \{u \in X : \int_0^\omega u(t) dt = 0\}$ is closed in Z and $\dim KerL = co\ dim\ ImL = 3$. Therefore, L is a Fredholm mapping of index zero. Through an easy computation we find that the inverse K_p of L_p has the form

$K_p : ImL \rightarrow DomL \cap KerP$, $K_p(u) = \int_0^\omega u(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^\eta u(t) dt d\eta$. Clearly, QN and $K_p(I - Q)N$ are continuous. By using Arzela-Ascoli theorem, it is not difficult to prove that $\overline{K_p(I - Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \in X$. Moreover, $QN(\overline{\Omega})$ is bounded. Therefore, N is L -compact on $\overline{\Omega}$ with any open bounded set $\Omega \in X$.

Corresponding to the operator equation $Lu = \lambda Nu$, $\lambda \in (0, 1)$, we have

$$u'(t) = \lambda Y(t). \tag{3.2}$$

Suppose that $u \in X$ is a solution of (3.2) for an appropriate $\lambda \in (0, 1)$. Integrating (3.2) over the interval $[0, \omega]$ leads to

$$\begin{aligned} & \int_0^\omega (a_1(t) - h_1(t) - D_1(t)) dt + \int_0^\omega D_1(t) e^{u_2(t) - u_1(t)} dt \\ &= \int_0^\omega b_1(t) e^{u_1(t)} dt + \int_0^\omega \frac{c(t) e^{u_3(t)}}{m_1(t) + m_2(t) e^{u_1(t)} + m_3(t) e^{u_3(t)}} dt, \end{aligned} \tag{3.3}$$

$$\int_0^\omega (a_2(t) - D_2(t)) dt + \int_0^\omega D_2(t) e^{u_1(t) - u_2(t)} dt = \int_0^\omega b_2(t) e^{u_2(t)} dt, \tag{3.4}$$

$$\int_0^\omega (d(t) + h_2(t)) dt + \int_0^\omega e(t) e^{u_3(t)} dt = \int_0^\omega \frac{f(t) e^{u_1(t-\tau)}}{m_1(t) + m_2(t) e^{u_1(t-\tau)} + m_3(t) e^{u_3(t-\tau)}} dt. \tag{3.5}$$

From (3.2)-(3.5), we have

$$\begin{aligned}
 \int_0^\omega |u_1'(t)| dt &\leq \int_0^\omega (a_1(t) - h_1(t) - D_1(t)) dt + \int_0^\omega b_1(t) e^{u_1(t)} dt \\
 &\quad + \int_0^\omega \frac{c(t) e^{u_3(t)}}{m_1(t) + m_2(t) e^{u_1(t)} + m_3(t) e^{u_3(t)}} dt + \int_0^\omega D_1(t) e^{u_2(t) - u_1(t)} dt \\
 &= 2 \int_0^\omega b_1(t) e^{u_1(t)} dt + 2 \int_0^\omega \frac{c(t) e^{u_3(t)}}{m_1(t) + m_2(t) e^{u_1(t)} + m_3(t) e^{u_3(t)}} dt \\
 &< 2b_1^M \int_0^\omega e^{u_1(t)} dt + 2\omega \overline{\left(\frac{c}{m_3}\right)},
 \end{aligned} \tag{3.6}$$

$$\begin{aligned}
 \int_0^\omega |u_2'(t)| dt &\leq \int_0^\omega (a_2(t) - D_2(t)) dt + \int_0^\omega b_2(t) e^{u_2(t)} dt \\
 &= 2 \int_0^\omega b_2(t) e^{u_2(t)} dt < 2b_2^M \int_0^\omega e^{u_2(t)} dt,
 \end{aligned} \tag{3.7}$$

$$\begin{aligned}
 \int_0^\omega |u_3'(t)| dt &\leq \int_0^\omega (d(t) + h_2(t)) dt + \int_0^\omega e(t) e^{u_3(t)} dt \\
 &\quad + \int_0^\omega \frac{f(t) e^{u_1(t-\tau)}}{m_1(t) + m_2(t) e^{u_1(t-\tau)} + m_3(t) e^{u_3(t-\tau)}} dt \\
 &= 2 \int_0^\omega \frac{f(t) e^{u_1(t-\tau)}}{m_1(t) + m_2(t) e^{u_1(t-\tau)} + m_3(t) e^{u_3(t-\tau)}} dt \\
 &< 2\omega \overline{\left(\frac{c}{m_2}\right)} \stackrel{\text{def}}{=} d_3.
 \end{aligned} \tag{3.8}$$

Multiplying the first equation of (3.2) by $e^{u_1(t)}$ and integrating over $[0, \omega]$ gives.

$$\int_0^\omega b_1(t) e^{2u_1(t)} dt < \int_0^\omega (a_1(t) - h_1(t) - D_1(t)) e^{u_1(t)} dt + \int_0^\omega D_1(t) e^{u_2(t)} dt,$$

which implies

$$b_1^l \int_0^\omega e^{2u_1(t)} dt < (a_1 - h_1 - D_1)^M \int_0^\omega e^{u_1(t)} dt + D_1^M \int_0^\omega e^{u_2(t)} dt \tag{3.9}$$

By using the inequalities

$$\left(\int_0^\omega e^{u_1(t)} dt \right)^2 \leq \omega \int_0^\omega e^{2u_1(t)} dt.$$

It follows from (3.9) that

$$\frac{b_1^l}{\omega} \left(\int_0^\omega e^{u_1(t)} dt \right)^2 < (a_1 - h_1 - D_1)^M \int_0^\omega e^{u_1(t)} dt + D_1^M \int_0^\omega e^{u_2(t)} dt, \tag{3.10}$$

This yields

$$\frac{2b_1^l}{\omega} \int_0^\omega e^{u_1(t)} dt < (a_1 - h_1 - D_1)^M + \left\{ \left[(a_1 - h_1 - D_1)^M \right]^2 + \frac{4b_1^l D_1^M}{\omega} \int_0^\omega e^{u_2(t)} dt \right\}^{\frac{1}{2}}. \tag{3.11}$$

By using the inequalities

$$(a + b)^{\frac{1}{2}} < a^{\frac{1}{2}} + b^{\frac{1}{2}}, \quad a > 0, b > 0.$$

It follows from (3.10) that

$$\frac{b_1^l}{\omega} \int_0^\omega e^{u_1(t)} dt < (a_1 - h_1 - D_1)^M + \sqrt{\frac{b_1^l D_1^M}{\omega} \left(\int_0^\omega e^{u_2(t)} dt \right)^{\frac{1}{2}}}. \tag{3.12}$$

Multiplying the second equation of (3.2) by $e^{u_2(t)}$ and integrating over $[0, \omega]$, similarly, we can obtain

$$\frac{b_2^l}{\omega} \int_0^\omega e^{u_2(t)} dt < (a_2 - D_2)^M + \sqrt{\frac{b_2^l D_2^M}{\omega}} \left(\int_0^\omega e^{u_1(t)} dt \right)^{\frac{1}{2}}. \tag{3.13}$$

Substitute (3.13) to (3.12), which leads to

$$\sqrt{b_2^l b_1^l} \int_0^\omega e^{u_1(t)} dt < \omega \sqrt{b_2^l} (a_1 - h_1 - D_1)^M + \sqrt{\omega b_1^l b_2^l D_1^M} \left[\sqrt{\omega (a_2 - D_2)^M} + \sqrt{\omega b_2^l D_2^M} \int_0^\omega e^{u_1(t)} dt \right]$$

So, there exist a positive constant ρ_1 such that

$$\int_0^\omega e^{u_1(t)} dt < \rho_1. \tag{3.14}$$

It follows from (3.13) and (3.14) that there exist a positive constant ρ_2 such that

$$\int_0^\omega e^{u_2(t)} dt < \rho_2. \tag{3.15}$$

Substitute (3.14), (3.15) to (3.6) and (3.7), which leads to

$$\int_0^\omega |u_1'| dt < 2b_1^M + 2\omega \overline{\left(\frac{c}{m_3} \right)} = d_1, \tag{3.16}$$

$$\int_0^\omega |u_2'| dt < 2b_2^M \rho_2 = d_1. \tag{3.17}$$

From (3.3) we have

$$\begin{aligned} b_1^M \int_0^\omega e^{u_1(t)} dt &> \int_0^\omega b_1(t) e^{u_1(t)} dt > \omega \overline{(a_1 - h_1 - D_1)} - \int_0^\omega \frac{c(t) e^{u_3(t)}}{m_1(t) + m_2(t) e^{u_1(t)} + m_3(t) e^{u_3(t)}} dt \\ &> \omega \overline{(a_1 - h_1 - D_1)} - \omega \overline{\left(\frac{c}{m_3} \right)} \end{aligned} \tag{3.18}$$

From (3.4) we have

$$b_2^M \int_0^\omega e^{u_2(t)} dt > \omega \overline{(a_2 - D_2)}. \tag{3.19}$$

It follows from (3.14), (3.15), (3.18) and (3.19) that there exist $t_i, t_i^* (i = 1, 2)$ such that

$$e^{u_i(t_i)} < \frac{\rho_i}{\omega}, \quad i = 1, 2, \tag{3.20}$$

$$e^{u_1(t_1^*)} > \frac{\omega \overline{(a_1 - h_1 - D_1)} - \omega \overline{(c/m_3)}}{b_1^M}, \tag{3.21}$$

$$e^{u_2(t_2^*)} > \frac{\omega \overline{(a_2 - D_2)}}{b_2^M}. \tag{3.22}$$

From (3.16), (3.17) and (3.20)-(3.22) we have

$$\begin{aligned} u_i(t) &< \ln \frac{\rho_i}{\omega} + \int_0^\omega |u_i'(s)| ds < \ln \frac{\rho_i}{\omega} + d_i, \quad i = 1, 2, \\ u_1(t) &> \ln \frac{\omega \overline{(a_1 - h_1 - D_1)} - \omega \overline{(c/m_3)}}{b_1^M} - \int_0^\omega |u_1'(s)| ds \\ &> \ln \frac{\omega \overline{(a_1 - h_1 - D_1)} - \omega \overline{(c/m_3)}}{b_1^M} - d_1, \end{aligned}$$

$$u_2(t) > \ln \frac{\overline{\omega(a_2 - D_2)}}{b_2^M} - d_2.$$

So, for $t \in [0, \omega]$ we have

$$|u_1(t)| < \max \left\{ \left| \ln \frac{\rho_1}{\omega} + d_1 \right|, \left| \ln \frac{\overline{\omega(a_1 - h_1 - D_1)} - \overline{\omega(c/m_3)}}{b_1^M} - d_1 \right| \right\} \stackrel{\text{def}}{=} R_1, \tag{3.23}$$

$$|u_2(t)| < \max \left\{ \left| \ln \frac{\rho_2}{\omega} + d_2 \right|, \left| \ln \frac{\overline{\omega(a_2 - D_2)}}{b_2^M} - d_2 \right| \right\} \stackrel{\text{def}}{=} R_2. \tag{3.24}$$

From (3.5) we have

$$\begin{aligned} e^l \int_0^\omega e^{u_3(t)} dt &< \int_0^\omega e(t) e^{u_3(t)} dt = \int_0^\omega \frac{f(t) e^{u_1(t-\tau)}}{m_1(t) + m_2(t) e^{u_1(t-\tau)} + m_3(t) e^{u_3(t-\tau)}} dt - \int_0^\omega [d(t) + h_2(t)] dt \\ &< \overline{\omega(c/m_2)} - \overline{\omega(d + h_2)}. \end{aligned}$$

So, there exist $\eta_1 \in (0, \omega)$ such that

$$e^{u_3(\eta_1)} < \frac{\overline{(f/m_2)} - \overline{(d + h_2)}}{e^l}. \tag{3.25}$$

From (3.5) we also have

$$\int_0^\omega \frac{f(t) e^{u_1(t-\tau)}}{m_1(t) + m_2(t) e^{u_1(t-\tau)} + m_3(t) e^{u_3(t-\tau)}} dt < \overline{\omega(d + h_2)} + e^M \int_0^\omega e^{u_3(t)} dt. \tag{3.26}$$

It follows from (3.26) that there exist $\eta_2 \in (0, \omega)$ such that

$$\frac{f(\eta_2 + \tau) e^{u_1(\eta_2)}}{m_1(\eta_2 + \tau) + m_2(\eta_2 + \tau) e^{u_1(\eta_2)} + m_3(\eta_2 + \tau) e^{u_3(\eta_2)}} < \overline{\omega(d + h_2)} + e^M e^{u_3(\eta_2 + \tau)}.$$

So

$$e^{u_3(\eta_2)} > \frac{f(\eta_2 + \tau) e^{u_1(\eta_2)}}{\left\{ \overline{(d + h_2)} + \frac{e^M}{e^l} \left[\overline{\left(\frac{f}{m_2} \right)} - \overline{(d + h_2)} \right] \right\} m_3(\eta_2 + \tau)} - \frac{m_1(\eta_2 + \tau) + m_2(\eta_2 + \tau) e^{u_1(\eta_2)}}{m_3(\eta_2 + \tau)} \stackrel{\text{def}}{=} p. \tag{3.27}$$

It follows from (3.8), (3.25) and (3.27) that for $t \in [0, \omega]$ we have

$$u_3(t) < \ln \frac{\overline{(f/m_2)} - \overline{(d + h_2)}}{e^l} + d_3, \quad u_3(t) > \ln p - d_3.$$

So $t \in [0, \omega]$ we have

$$|u_3(t)| < \max \left\{ \left| \ln \frac{\overline{(f/m_2)} - \overline{(d + h_2)}}{e^l} + d_3 \right|, \left| \ln p - d_3 \right| \right\} \stackrel{\text{def}}{=} R_3.$$

Clearly, $R_i (i = 1, 2, 3)$ are independent of λ . On other hand, we consider the following algebraic equation

$$\begin{cases} \left(\overline{a_1 - h_1 - D_1}\right) - \overline{b_1}e^{u_1} - \frac{ce^{u_3}}{\overline{m_1} + \overline{m_2}e^{u_1} + \overline{m_3}e^{u_3}} + \overline{D_1}e^{u_2 - u_1} = 0 \\ \left(\overline{a_2 - D_2}\right) - \overline{b_2}e^{u_2} + \overline{D_2}e^{u_1 - u_2} = 0 \\ \left(\overline{-d - h_2}\right) - \overline{e}e^{u_3} + \frac{\overline{f}e^{u_1}}{\overline{m_1} + \overline{m_2}e^{u_1} + \overline{m_3}e^{u_3}} = 0, \end{cases} \quad (3.28)$$

Take $\rho = R_1 + R_2 + R_3 + R_0$, where R_0 is large enough such that the solution $u^* = (u_1^*, u_2^*, u_3^*)^T$ of (3.28) satisfies

$$\|u^*\| = \left\| (u_1^*, u_2^*, u_3^*)^T \right\| < \rho.$$

Let $\Omega = \{u \in X : \|u\| < \rho\}$, then Ω satisfies the condition (1) in Lemma 2.1. When $u \in \partial\Omega \cap \text{Ker}L = \partial\Omega \cap R^3$, u is a constant vector in R^3 and $\|u\| = \rho$. It follows from the definition of ρ that $QN(u) \neq 0$, so the condition (2) in Lemma 2.1 is satisfied. In order to verify the condition (3) in Lemma 2.1, we define $\phi : \text{Dom}L \times [0, 1] \rightarrow X$ by

$$\phi(u_1, u_2, u_3, \mu) = \begin{pmatrix} \left(\overline{a_1 - h_1 - D_1}\right) - \overline{b_1}e^{u_1(t)} \\ \left(\overline{a_2 - D_2}\right) - \overline{b_2}e^{u_2(t)} \\ \left(\overline{-d - h_2}\right) + \frac{\overline{f}e^{u_1(t)}}{\overline{m_1} + \overline{m_2}e^{u_1(t)} + \overline{m_3}e^{u_3(t)}} \end{pmatrix} + \mu \begin{pmatrix} \frac{\overline{c}e^{u_3(t)}}{\overline{m_1} + \overline{m_2}e^{u_1(t)} + \overline{m_3}e^{u_3(t)}} + \overline{D_1}e^{u_2(t) - u_1(t)} \\ \overline{D_2}e^{u_1(t) - u_2(t)} \\ -\overline{e}e^{u_3(t)} \end{pmatrix},$$

where $u_i \in R, (i=1, 2, 3), \mu \in [0, 1]$ is a parameter. When $u \in \partial\Omega \cap \text{Ker}L = \partial\Omega \cap R^3$, u is a constant vector in R^3 and $\|u\| = \rho$. It is easy to obtain that $u \in \partial\Omega \cap \text{Ker}L$, then $\phi(u_1, u_2, u_3, \mu) \neq 0$. So, $\phi(u_1, u_2, u_3, \mu)$ is a Homotopy mapping, due to homotopy invariance theorem of topology degree, we have

$$\begin{aligned} & \deg\{QN(u), \Omega \cap \text{Ker}L, (0, 0, 0)^T\} \\ &= \deg\{\phi(u_1, u_2, u_3, 1), \Omega \cap \text{Ker}L, (0, 0, 0)^T\} \\ &= \deg\{\phi(u_1, u_2, u_3, 0), \Omega \cap \text{Ker}L, (0, 0, 0)^T\} \\ &= \deg\left\{\left(\overline{a_1 - h_1 - D_1}\right) - \overline{b_1}e^{u_1(t)}, \left(\overline{a_2 - D_2}\right) - \overline{b_2}e^{u_2(t)}, \left(\overline{-d - h_2}\right) + \frac{\overline{f}e^{u_1(t)}}{\overline{m_1} + \overline{m_2}e^{u_1(t)} + \overline{m_3}e^{u_3(t)}}, \Omega \cap \text{Ker}L, (0, 0, 0)^T\right\}. \end{aligned}$$

It is not difficult to see that the following algebraic equation

$$\begin{cases} \left(\overline{a_1 - h_1 - D_1}\right) - \overline{b_1}e^{u_1} = 0 \\ \left(\overline{a_2 - D_2}\right) - \overline{b_2}e^{u_2} = 0 \\ \left(\overline{-d - h_2}\right) + \frac{\overline{f}e^{u_1}}{\overline{m_1} + \overline{m_2}e^{u_1} + \overline{m_3}e^{u_3}} = 0 \end{cases}$$

has a unique solution

$$y_1^* = \frac{\left(\overline{a_1 - h_1 - D_1}\right)}{\overline{b_1}}, \quad y_2^* = \frac{\left(\overline{a_2 - D_2}\right)}{\overline{b_2}}, \quad y_3^* = \frac{\overline{f}y_1^* - (\overline{m_1} + \overline{m_2}y_1^*)(\overline{d + h_2})}{(\overline{d + h_2})\overline{m_3}}.$$

Thus

$$\deg\{QN(u), \Omega \cap \text{Ker}L, (0, 0, 0)^T\} = \begin{vmatrix} -\bar{b}_1 & 0 & 0 \\ 0 & -\bar{b}_2 & 0 \\ \frac{\overline{fm}_1 + \overline{fm}_3 y_3^*}{(\bar{m}_1 + \bar{m}_2 y_1^* + \bar{m}_3 y_3^*)^2} & 0 & \frac{-\overline{fm}_3 y_1^*}{(\bar{m}_1 + \bar{m}_2 y_1^* + \bar{m}_3 y_3^*)^2} \end{vmatrix} = -1 \neq 0.$$

By now we have proved the condition (3) in Lemma 2.1. This completes the proof of Theorem 3.1.

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