

Dynamics Behaviors of a Reaction-Diffusion Predator-Prey System with Beddington-DeAngelis Functional Response and Delay

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Abstract

This paper is concerned with the existence of traveling wave solutions in a reaction-diffusion predator-prey system with Beddington-DeAngelis functional response and a discrete time delay. By introducing a partial quasi-monotonicity condition and constructing a pair of upper-lower solutions, we establish the existence of traveling wave solutions. Moreover, a numerical simulation is carried out to illustrate the theoretical results.

Keywords

Traveling Wave Solutions, Reaction-Diffusion System, Upper-Lower Solutions, Local Stability

1. Introduction

Recently, the dynamics of predator-prey systems is one of the fastest developing areas of modern mathematics due to their significant nature in biological fields and other practical fields. One significant component in these systems is the functional response describing the number of prey consumed per predator per unit time for given quantities of prey N and predators P . The traditional mathematical model describing the predator-prey interactions consists of the following system of differential equations.

In recent years, many authors have explored the dynamic relationship between predators and their preys. There is extensive literature related to these topics for ordinary differential equation models (see [1]-[6] and the references

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cited therein). We know that more realistic prey-predator models were introduced by Holling suggesting three kinds of functional responses for different species to model the phenomena of predation [3]. On the other hand, species have the natural tendency to move from areas of bigger population concentration to those of smaller population concentration. This kind of diffusion process is called free diffusion and it is not considered in the above mentioned references. In the literature, many researchers have directly introduced the free diffusion to ODEs and DDEs and have also explained why to do so. To name a few, see [7]-[14]. Moreover, such models or similar models with delays and free diffusion have also arisen from a variety of situations like infectious disease dynamics, porous medium, chemical reaction, engineering control theory. Taking into account the inhomogeneous distribution of the species in different spatial locations within a fixed bounded domain $\Omega \in R^n$, Peng and Wang looked at a diffusive Holling-Tanner prey-predator model in [7]

$$\begin{aligned}
 v_t - d_2 v_{xx} &= bv - \frac{v^2}{ru}, \text{ in } \Omega \times (0, \infty), \\
 \partial_\eta u = \partial_\eta v &= 0, \text{ on } \partial\Omega \times (0, \infty), \\
 u(x, 0) &= u_0(x, 0) > 0, \text{ on } \bar{\Omega}, \\
 v(x, 0) &= v_0(x, 0) > 0, \text{ on } \bar{\Omega},
 \end{aligned}
 \tag{1.1}$$

where $u(x, t)$ and $v(x, t)$ represent the species densities of the prey and predator, respectively. η is the outward unit normal vector on the smooth boundary $\partial\Omega$. The constants $d_i (i = 1, 2)$ are the diffusion coefficients corresponding to u and v , respectively, and all the parameters appearing in (1.1) are assumed to be positive. The admissible initial data $u_0(x)$ and $v_0(x)$ are continuous functions on $\bar{\Omega}$. The homogeneous Neumann boundary condition means that (1.1) is self-contained and has no population flux across the boundary $\partial\Omega$. For more detailed biological implications of the model, one may further refer to [7] and the references cited therein.

In recent years, great attention has been paid to the study of the existence of traveling wave solutions in reaction-diffusion system, since they determine the long term behavior of other solutions, and account for phase transitions between different states of physical systems, propagation of patterns, and domain invasion of species in population biology (see, for example, [10]-[12], and the references cited therein).

Motivated by the work of Peng and Wang [7] and Wu, Zou [12], in the present paper, we consider the existence of traveling waves of the following predator-prey model with Beddington-DeAngelis functional response and a discrete time delay due to gestation of predator

$$\begin{aligned}
 u_t - d_1 u_{xx} &= u(a_1 - u(x, t - \tau)) - \frac{a_2 uv}{1 + m_1 u + m_2 v}, \text{ in } \Omega \times (0, \infty), \\
 v_t - d_2 v_{xx} &= v\left(b - \frac{v}{ru}\right), \text{ in } \Omega \times (0, \infty), \\
 u(x, t) &= \phi_1(x, t) \geq 0, x \in R, -\tau \leq t \leq 0, \\
 v(x, 0) &= \phi_2(x, 0) > 0, x \in R, -\tau \leq t \leq 0.
 \end{aligned}
 \tag{1.2}$$

The main purpose of the paper is to consider the existence of traveling wavefronts for the delay model (1.2). In order to study traveling wavefronts, we need to analyze the stability of the positive constant equilibrium first. As a result, the remaining part of this paper is organized as follows. We first use linearized method to study the stability of the positive constant equilibrium of (1.2) in Section 2. Then, applying the method of upper and lower solutions, we establish the existence of traveling wavefronts of (1.2) in Section 3. At the same time, we give some suitable examples to illustrate our results.

2. Asymptotical Stability of the Positive Constant Equilibrium

Set the right side of the system (1.2) to zero. It is easy to check that the system (1.2) has an axial equilibrium $E_1(a, 0)$ and a unique positive equilibrium $E_2(k_1, k_2)$, where

$$k_1 = \frac{1}{2} \left[an - m - br + \sqrt{(an - m - br)^2 + 4amn} \right]$$

And $k_2 = brk_1$. In this section, we discuss the locally asymptotical stability of the positive constant equilibrium by the linearized method. The linearized system of (1.2) about a positive constant equilibrium (k_1, k_2) is

$$\begin{aligned} u_t - d_1 u_{xx} &= \frac{nk_1 k_2}{(m + nk_1)^2} u - k_1 u(x, t - \tau) - \frac{k_1}{m + nk_1} v, \\ v_t - d_2 v_{xx} &= b^2 ru - bv, \\ u(x, t) &= \phi_1(x, t) - k_1, \\ v(x, 0) &= \phi_2(x, 0) - k_2, x \in R, -\tau \leq t \leq 0. \end{aligned} \tag{2.1}$$

System (2.1) admits nontrivial solutions of the form $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t + i\sigma x}$ if and only if the determinant

$$\begin{vmatrix} \lambda + d_1 \sigma^2 - \frac{nk_1 k_2}{(m + nk_1)^2} + k_1 e^{-\lambda \tau} & \frac{k_1}{m + nk_1} \\ -b^2 r & \lambda + d_2 \sigma^2 + b \end{vmatrix} = 0$$

which implies that

$$\left(\lambda + d_1 \sigma^2 - \frac{nk_1 k_2}{(m + nk_1)^2} + k_1 e^{-\lambda \tau} \right) \cdot (\lambda + d_2 \sigma^2 + b) + \frac{b^2 r k_1}{m + nk_1} = 0. \tag{2.2}$$

where λ is a complex number and σ is a real number (see, for example, Ge and He [11] and the references therein).

It is easy to check that $\lambda + d_2 \sigma^2 + b \neq 0$, so, we rewrite (2.2) as

$$\lambda = -d_1 \sigma^2 + \frac{nk_1 k_2}{(m + nk_1)^2} - k_1 e^{-\lambda \tau} - \frac{b^2 r k_1}{(m + nk_1)(\lambda + d_2 \sigma^2 + b)}. \tag{2.3}$$

We claim that $\mu < 0$, if $(\lambda, \sigma) = (\mu + iv, \sigma)$ satisfies (2.3) and $\tau \geq 0$ is sufficiently small. Otherwise, suppose that there exists a $(\mu_0 + iv_0, \sigma_0)$ satisfying (2.3) such that $\mu_0 \geq 0$. Then direct computation gives us

$$\mu_0 = -d_1 \sigma_0^2 + \frac{nk_1 k_2}{(m + nk_1)^2} - k_1 e^{-\mu_0 \tau} \cos(v_0 \tau) - \frac{b^2 r k_1 (\mu_0 + d_2 \sigma_0^2 + b)}{(m + nk_1) [(\mu_0 + d_2 \sigma_0^2 + b)^2 + v_0^2]} \leq \frac{nk_1 k_2}{(m + nk_1)^2} - k_1 e^{-\mu_0 \tau} \cos(v_0 \tau).$$

We can prove that $k_1 > \frac{nk_1 k_2}{(m + nk_1)^2}$. So, if $\tau \geq 0$ is sufficiently small, we have

$$\frac{nk_1 k_2}{(m + nk_1)^2} - k_1 e^{-\mu_0 \tau} \cos(v_0 \tau) < 0.$$

It is a contradiction. This proves the claim. As a result, we have proved.

Theorem 2.1. The positive equilibrium $E_2(k_1, k_2)$ of (1.2) is locally asymptotically stable for $\tau \geq 0$ sufficiently small.

Remark 2.1. In [7], it was showed the following result: Suppose that $m^2 + 2(a + by)m + a^2 - 2aby \geq 0$. Then positive equilibrium of (1.1) is asymptotically stable. However, in this paper, we also obtain the asymptotical stability about the positive equilibrium without conditions in absence time delay τ .

In order to illustrate the validity of the theoretical result on the asymptotical stability, we perform numerical calculations using the software MATLAB.

Consider the following system:

$$\begin{aligned}
 u_t - u_{xx} &= u \left(1 - u - \frac{v}{0.4 + u} \right), \text{ in } \Omega \times (0, +\infty), \\
 v_t - v_{xx} &= v \left(1 - \frac{v}{u} \right), \text{ in } \Omega \times (0, +\infty), \\
 \partial_n u &= \partial_n v = 0, \text{ on } \partial\Omega \times (0, +\infty), \\
 u(x, 0) &= u_0(x) = 0.3, v(x, 0) = v_0(x) = 0.4, \text{ on } \bar{\Omega},
 \end{aligned}$$

the positive equilibrium $E(k_1, k_2)$ of system (1.2) is locally asymptotically stable (see **Figure 1** and **Figure 2**).

3. Existence of Traveling Wavefront

A traveling wave solution of (1.2) is a special translation invariant solution of the form $(u(x, t), v(x, t)) = (\phi_1(x + ct), \phi_2(x + ct))$ with wave speed c . Various methods including the monotone itera-

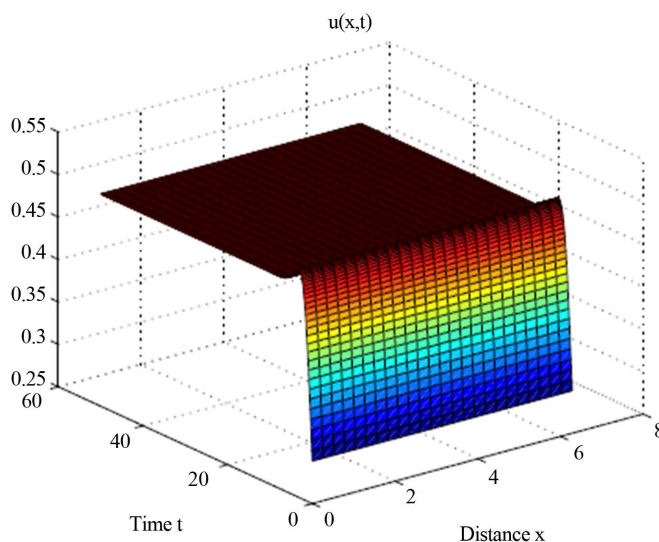


Figure 1. $k_1 = 0.4633$.

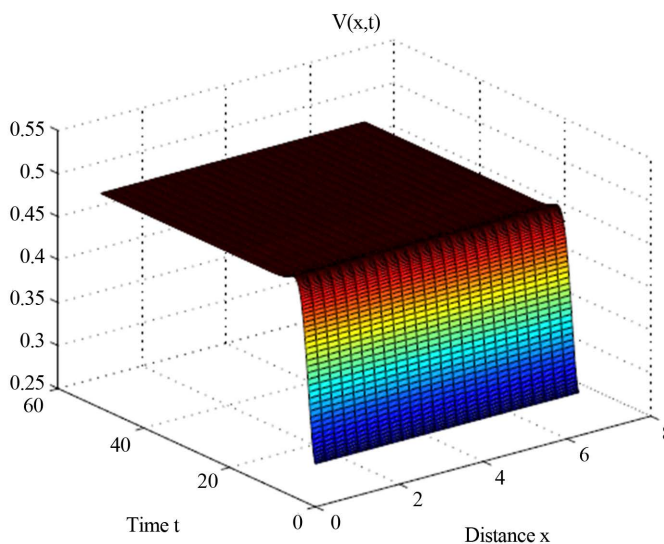


Figure 2. $k_2 = 0.4633$.

tion technique [11] [12] and the degree theory [10] have been adopted to study the existence of traveling wave solutions to reaction-diffusion systems with delays. In this section, we use the approach introduced by Canosa [14] to establish the existence of traveling wave solutions connecting the axial equilibrium $E_1(a, 0)$ to the positive equilibrium $E_2(k_1, k_2)$. To seek such a pair of traveling wavefronts of (1.2), we substitute $u(x, t) = \phi_1(s)$ and $v(x, t) = \phi_2(s)$, where $s = x + ct$, into (1.2) to obtain

$$\begin{aligned} d_1\phi_1''(s) - c\phi_1'(s) + \phi_1(s)(a - \phi_1(s - ct)) - \frac{\phi_1(s)\phi_2(s)}{m + n\phi_1(s)} &= 0, \\ d_2\phi_2''(s) - c\phi_2'(s) + \phi_2(s)\left(b - \frac{\phi_2(s)}{r\phi_1(s)}\right) &= 0, \\ \phi_1(-\infty) = a, \phi_2(-\infty) = 0, \phi_i(\infty) = k_i, i = 1, 2. \end{aligned} \tag{3.1}$$

Now, we follow the approach of Canosa [14] to construct a uniformly valid asymptotic approximation to the wavefronts for large values of the wave speed c . Suppose that c is large enough. Then $\varepsilon = 1/c^2$ is a small parameter. We aim to seek a pair of solutions to (3.1) of the form

$$\begin{pmatrix} \phi_1(s) \\ \phi_2(s) \end{pmatrix} = \begin{pmatrix} \psi_1(\eta) \\ \psi_2(\eta) \end{pmatrix} \text{ with } \eta = \sqrt{\varepsilon}s = s/c.$$

Then (3.1) becomes

$$\begin{aligned} \varepsilon d_1\psi_1''(\eta) - \psi_1'(\eta) + \psi_1(\eta)(a - \psi_1(\eta - \tau)) - \frac{\psi_1(\eta)\psi_2(\eta)}{m + n\phi_1(\eta)} &= 0, \\ \varepsilon d_2\psi_2''(\eta) - \psi_2'(\eta) + \psi_2(\eta)\left(b - \frac{\psi_2(\eta)}{r\psi_1(\eta)}\right) &= 0, \\ \psi_1(-\infty) = a, \psi_2(-\infty) = 0, \psi_i(+\infty) = k_i, i = 1, 2. \end{aligned} \tag{3.2}$$

Denote

$$\psi_i(\eta, \varepsilon) = \psi_{i0}(\eta) + \varepsilon\psi_{i1}(\eta) + \varepsilon^2\psi_{i2}(\eta) + \dots, i = 1, 2$$

and substitute them into (3.2). It turns out that $\psi_{10}(\eta)$ and $\psi_{20}(\eta)$ satisfy

$$\begin{aligned} \psi_{10}'(\eta) &= \psi_{10}(\eta)(a - \psi_{10}(\eta - \tau)) - \frac{\psi_{10}(\eta)\psi_{20}(\eta)}{m + n\phi_{10}(\eta)}, \\ \psi_{20}'(\eta) &= \psi_{20}(\eta)\left(b - \frac{\psi_{20}(\eta)}{r\psi_{10}(\eta)}\right), \\ \psi_{10}(-\infty) = a, \psi_{20}(-\infty) = 0, \psi_{i0}(+\infty) = k_i, i = 1, 2. \end{aligned} \tag{3.3}$$

For simplicity of notation, we still denote $\psi_{10}(\eta)$, $\psi_{20}(\eta)$ by $\phi_1(s)$, $\phi_2(s)$, respectively. Then (3.3) becomes

$$\begin{aligned} \phi_1'(s) &= \phi_1(s)(a - \phi_1(s - \tau)) - \frac{\phi_1(s)\phi_2(s)}{m + n\phi_1(s)}, \\ \phi_2'(s) &= \phi_2(s)\left(b - \frac{\phi_2(s)}{r\phi_1(s)}\right), \\ \phi_1(-\infty) = a, \phi_2(-\infty) = 0, \phi_i(+\infty) = k_i, i = 1, 2. \end{aligned} \tag{3.4}$$

Now, we are ready to state and prove the following result by the upper and lower solution technique developed by Wu and Zou [12].

Theorem 3.1. System (1.2) has a traveling wavefront connecting $(a, 0)$ to (k_1, k_2) for $\tau \geq 0$ sufficiently small.

Proof. The proof is divided into the following two steps.

Step I: Verify a quasi-monotonicity condition. For this purpose, we define the functional

$$f_c(\phi) = (f_{c1}(\phi), f_{c2}(\phi))^T$$

by

$$\begin{aligned} f_{c1}(\phi) &= \phi_1(0)(a - \phi_1(-\tau)) - \frac{\phi_1(0)\phi_2(0)}{m + n\phi_1(0)}, \\ f_{c2}(\phi) &= \phi_2(0)\left(b - \frac{\phi_2(0)}{\phi_1(0)}\right). \end{aligned} \tag{3.5}$$

For arbitrary $(\phi_1, \phi_2)^T$ and $(\psi_1, \psi_2)^T \in C([- \tau, 0], \mathbb{R}^2)$ such that

$$\begin{aligned} 0 < k_1 \leq \psi_1(s) \leq \phi_1(s) \leq a, \\ 0 \leq \psi_2(s) \leq \phi_2(s) \leq k_2, \text{ for } s \in [-\tau, 0] \end{aligned}$$

and $\phi_2(0) - \psi_2(0) < \theta(\phi_1(0) - \psi_1(0))$, for some $\theta > 0$, and $e^{\beta s}[\phi(s) - \psi(s)]$ nondecreasing in $s \in [-\tau, 0]$. we have

$$\begin{aligned} f_{c1}(\phi) - f_{c2}(\phi) &= a(\phi_1(0) - \psi_1(0)) - [\phi_1(0)\phi_1(-\tau) - \psi_1(0)\psi_1(-\tau)] - \left[\frac{\phi_1(0)\phi_2(0)}{m + n\phi_1(0)} - \frac{\psi_1(0)\psi_2(0)}{m + n\psi_1(0)} \right] \\ &\geq -\left(ae^{\beta\tau} + \frac{mk_2 + ma\theta + na^2\theta}{m^2} \right) (\phi_1(0) - \psi_1(0)), \end{aligned}$$

which implies

$$f_{c1}(\phi) - f_{c2}(\phi) + \beta_1(\phi_1(0) - \psi_1(0)) \geq \left(\beta_1 - ae^{\beta\tau} - \frac{mk_2 + ma\theta + na^2\theta}{m^2} \right) (\phi_1(0) - \psi_1(0)).$$

Similarly, we can get

$$f_{c2}(\phi) - f_{c2}(\psi) + \beta_2(\phi_2(0) - \psi_2(0)) \geq \left(\beta_2 + b - \frac{2ak_2}{rk_1^2} \right) (\phi_2(0) - \psi_2(0)).$$

Therefore, if we choose $\beta_1 > a + \frac{mk_2 + ma\theta + na^2\theta}{m^2}$, $\beta_2 > -b + \frac{2ak_2}{rk_1^2}$, then by continuity we know that, for τ sufficiently small, $\beta_1 - ae^{\beta\tau} - \frac{mk_2 + ma\theta + na^2\theta}{m^2} \geq 0$, $\beta_2 + b - \frac{2ak_2}{rk_1^2} \geq 0$. This proves the quasi-monotonicity condition.

Step II: Establish the existence of a pair of upper and lower solutions. To achieve this, we look for wave front solutions of (3.1) in the following profile set

$$\Gamma = \left\{ \phi = \begin{pmatrix} \phi_1(s) \\ \phi_2(s) \end{pmatrix} \in C(\mathbb{R}, \mathbb{R}^2) : \right.$$

- (i) ϕ is componentwise nondecreasing in \mathbb{R}
- (ii) $\begin{pmatrix} f_1(-\infty) \\ f_2(-\infty) \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} f_1(+\infty) \\ f_2(+\infty) \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$

Define

$$\phi_1(s) = \begin{cases} a, & s < 0, \\ k_1, & s \geq 0. \end{cases}$$

And $\phi_2(s) = \min\{k_2 e^{\lambda s}, k_2\}$ where $\lambda > b$.

Then $\phi = (\phi_1(s), \phi_2(s))^T \in \Gamma$. We distinguish two cases to show that $(\phi_1(s), \phi_2(s))^T$ is a pair of upper solutions to (3.4).

Case I: $s < 0$. It is easy to see that

$$\phi_1(s) = \phi_1(s - \tau) = a, \phi_2(s) = k_2 e^{\lambda s}.$$

Thus

$$\phi_1'(s) - \phi_1(s)(a - \phi_1(s - \tau)) + \frac{\phi_1(s)\phi_2(s)}{m + n\phi_1(s)} = \frac{ak_2 e^{\lambda s}}{m + na} > 0.$$

Similarly,

$$\phi_2'(s) - \phi_2(s) \left(b - \frac{\phi_2(s)}{r\phi_1(s)} \right) \geq \lambda k_2 e^{\lambda s} - k_2 e^{\lambda s} \left(b - \frac{k_2 e^{\lambda s}}{ra} \right) \geq (\lambda - b)k_2 e^{\lambda s} + \frac{k_2^2 e^{2\lambda s}}{ra} > 0.$$

Case II: $s \geq 0$. We have

$$\phi_1(s) = k_1, \phi_2(s) = k_2, \phi_1(s - \tau) = \begin{cases} k_1, & \text{if } s \geq \tau, \\ a, & \text{if } s < \tau, \end{cases}$$

which implies that

$$\phi_1(s - \tau) \leq k_1 \text{ for } s \geq 0.$$

Therefore,

$$\phi_1'(s) - \phi_1(s)(a - \phi_1(s - \tau)) + \frac{\phi_1(s)\phi_2(s)}{m + n\phi_1(s)} \geq -k_1(a - k_1) + \frac{k_1 k_2}{m + nk_1} = 0$$

and

$$\phi_2'(s) - \phi_2(s) \left(b - \frac{\phi_2(s)}{r\phi_1(s)} \right) = -bk_2 + \frac{k_2^2}{rk_1} = 0.$$

The above discussion tells us that $(\phi_1(s), \phi_2(s))^T$ is an upper solution to (3.4). Now, define

$$\psi_1(s) = \begin{cases} a - \varepsilon e^{\lambda_1 s}, & \text{if } s < 0 \\ \varepsilon, & \text{if } s \geq \tau \end{cases} \text{ and } \psi_2(s) = 0, \tag{3.6}$$

where

$$0 < \varepsilon < \min \left\{ k_1, \frac{a}{2} \right\}, \lambda_1 > 0. \tag{3.7}$$

Using (3.6)-(3.7) we have

$$\phi_1'(s) - \phi_1(s)(a - \phi_1(s - \tau)) + \frac{\phi_1(s)\phi_2(s)}{m + n\phi_1(s)} \leq -\varepsilon(a - \varepsilon) < 0$$

if $s \geq 0$, and

$$\phi_1'(s) - \phi_1(s)(a - \phi_1(s - \tau)) + \frac{\phi_1(s)\phi_2(s)}{m + n\phi_1(s)} = -\lambda \varepsilon e^{\lambda_1 s} - (a - \varepsilon e^{\lambda_1 s}) \varepsilon e^{\lambda_1 s} \leq -\varepsilon e^{\lambda_1 s} (\lambda_1 + a - \varepsilon) < 0$$

if $s \leq 0$, This proves that $(\psi_1(s), \psi_2(s))^T$ is a pair of lower solutions to (3.4).

So far, we have verified all the assumptions in the theory developed by Wu and Zou [12]. Therefore, there exists at least one solution in the set Γ , that is, system (1.2) has a traveling wavefront solution connecting $(a, 0)$ to (k_1, k_2) . This completes the proof.

Remark 3.1. We study the existence of traveling wave for a reaction-diffusion Holling-Tanner model with

delay. In order to illustrate the validity of the theoretical result obtained in this section, we also perform numerical calculations using the software MATLAB.

Consider the following Holling-Tanner system:

$$\begin{aligned} u_t - d_1 u_{xx} &= au(1-u(x,t-\tau)) - \frac{uv}{m+nu}, \\ v_t - d_2 v_{xx} &= bv\left(1 - \frac{v}{ru}\right). \end{aligned} \tag{3.8}$$

It should satisfy the following boundary conditions:

$$\begin{cases} u(-\infty) = 1, \\ v(-\infty) = 0, \end{cases} \text{ and } \begin{cases} u(+\infty) = k_1, \\ v(+\infty) = k_2, \end{cases}$$

where $k_1 = \frac{n-m-r + \sqrt{(n-m-r)^2 + 4mn}}{2n}$, $k_2 = rk_1$. Fix $d_1 = d_2 = 1, \tau = 0, a = 0.6, b = 0.2$,

$m = 1, n = 0, r = 1$. Then system (3.8) has a positive equilibrium $E(0.5, 0.5)$. By Theorem 3.1, we see that system (3.8) has a traveling wave solution connecting the axial solution $(1, 0)$ with the positive equilibrium $(0.5, 0.5)$ when the wave speed c is larger. Numerical simulation can be carried out by using MATLAB7.01. See **Figure 3**. However, when the wave speed c is smaller, the system (3.8) doesn't exist the traveling wave solution. Following we also give a simulation to illustrate our result. See **Figure 4**.

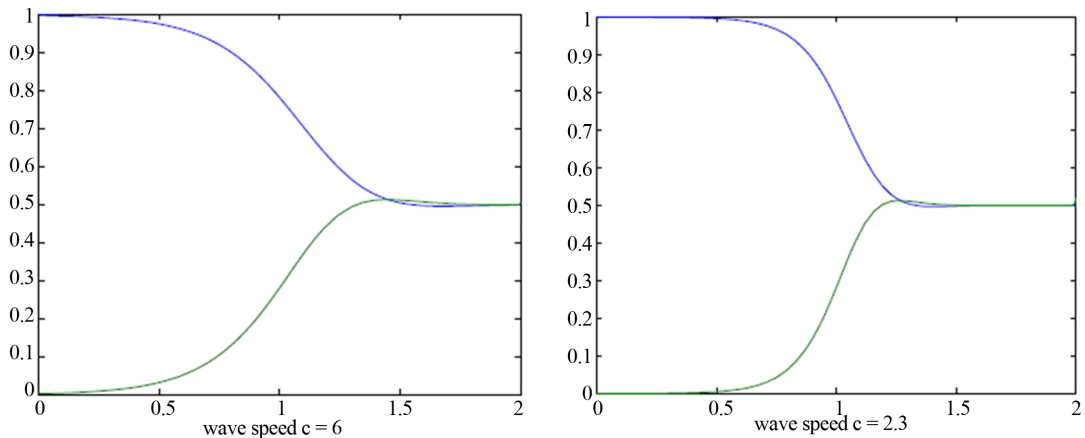


Figure 3. The wave speed $c = 6, c = 2.3$.

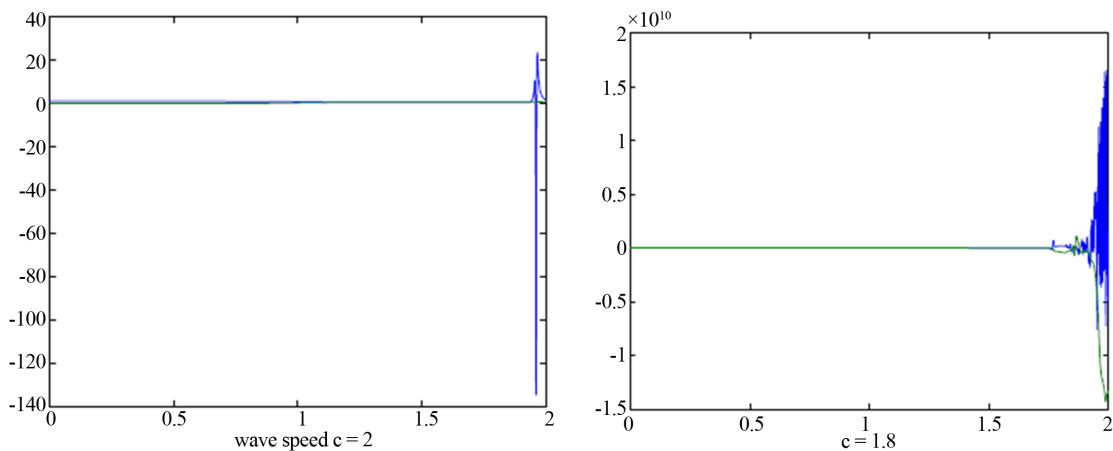


Figure 4. The wave speed $c = 2, c = 1.8$.

Of course, we mention that the smallest wave speed should exist in between $c = 2$ and $c = 2.3$ by numerical simulation, theoretical proof of the existence of such wave speed seems extraordinarily difficult in this paper and this remains as our future work.

4. Conclusion

In summary, when $\tau \geq 0$ sufficiently small, we have obtained the positive equilibrium $E_2(k_1, k_2)$ of system (1.2) which is locally asymptotically stable. At the time, we establish the existence of traveling wavefronts of (1.2) connecting $(a, 0)$ to (k_1, k_2) .

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