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Lattices Associated with a Finite Vector Space

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Abstract

Let F_q^n be a n-dimensional row vector space over a finite field F_q. For $1 \le d \le n-1$, let W_0 be a d-dimensional subspace of F_q^n. $L(n,d)$ denotes the set of all the spaces which are the subspaces of F_q^n and not the subspaces of W_0 except $\{0\}$. We define the partial order on $L(n,d)$ by ordinary inclusion (resp. reverse inclusion), and then $L(n,d)$ is a poset, denoted by $L_O(n,d)$ (resp. $L_R(n,d)$). In this paper we show that both $L_O(n,d)$ and $L_R(n,d)$ are finite atomic lattices. Further, we discuss the geometricity of $L_O(n,d)$ and $L_R(n,d)$, and obtain their characteristic polynomials.

Keywords

Vector Space; Geometric Lattice; Characteristic Polynomial

1. Introduction

Let P be a poset. For $a,b \in P$, we say a covers b, denoted by $b < \cdot a$; if $b < a$ and there doesn't exist $c \in P$ such that $b < c < a$. If P has the minimum (resp. maximum) element, then we denote it by 0 (resp. 1) and say that P is a poset with 0 (resp. 1). Let P be a finite poset with 0. By a rank function on P, we mean a function r from P to the set of all the integers such that $r(0)=0$ and $r(a)=r(b)+1$ whenever $b < \cdot a$. Observe the rank function is unique if it exists. P is said to be ranked whenever P has a rank function.

Let P be a finite ranked poset with 0 and 1. The polynomial $\chi(P,x)=\sum_{a\in P}\mu(0,a)x^{r(1)-r(a)}$ is called the characteristic polynomial of P, where μ is the Möbius function on P and r is the rank function of P. A poset P is said to be a lattice if both $a \vee b := \sup\{a,b\}$ and $a \wedge b := \inf\{a,b\}$ exist for any two elements $a,b \in P$. $a \vee b$ and $a \wedge b$ are called the join and meet of a and b, respectively. Let P be a finite lattice with 0. By an atom in P, we mean an element in P covering 0. We say P is atomic if any element in $P\setminus\{0\}$ is the join of atoms. A finite atomic lattice P is said to be a geometric lattice if P admits a rank function r satisfying $r(a\wedge b)+r(a\vee b)\le r(a)+r(b)$, $\forall a,b\in P$. Notations and terminologies about posets and lattices will be adopted from books [1] [2].

The special lattices of rough algebras were discussed in [3]. The lattices generated by orbits of subspaces under finite (singular) classical groups were discussed in [4] [5]. Wang *et al*. [6]-[8] constructed some sublattices of the lattices in [4]. The subspaces of a d-bounded distance-regular have similar properties to those of a vector space. Gao *et al*. [9]-[11] constructed some lattices and posets by subspaces in a d-bounded distance-regular graph. In this paper, we continue this research, and construct some new sublattices of the lattices in [4], discussing their geometricity and computing their characteristic polynomials.

Let F_q be a finite field with q elements, where q is a prime power. For a positive integer n, let F_q^n be the n-dimensional row vector space over F_q. Let $1\le d\le n-1$. For a fixed d-dimensional subspace W_0 of F_q^n, let $L(n,d)=\{P\mid P \text{ is a subspace of } F_q^n \text{ and is not of } W_0\}\cup\{\{0\}\}$.

If we define the partial order on $L(n,d)$ by ordinary inclusion (resp. reverse inclusion), then $L(n,d)$ is a poset, denoted by $L_O(n,d)$ (resp. $L_R(n,d)$). In the present paper we show that both $L_O(n,d)$ and $L_R(n,d)$ are finite atomic lattices, discuss their geometricity and compute their characteristic polynomials.

2. The Lattice $L_O(n,d)$

In this section we prove that the lattice $L_O(n,d)$ is a finite geometric lattice, and compute its characteristic polynomial. We begin with a useful proposition.

Proposition 2.1. ([12], Lemma 9.3.2 and [13], Corollaries 1.8 and 1.9). For $0\le k\le m\le n$, the following hold:

1) The number of k-dimensional subspaces contained in a given m-dimensional subspace of F_q^n is

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \prod_{i=m-k-1}^{m}\left(q^i-1\right)\Big/\prod_{i=1}^{k}\left(q^i-1\right).$$

2) The number of m-dimensional subspaces containing a given k-dimensional subspace of F_q^n is

$$\begin{bmatrix} n-k \\ m-k \end{bmatrix}_q.$$

3) Let P be a fixed m-dimensional subspaces of F_q^n. Then the number of k-dimensional subspaces Q of F_q^n satisfying $\dim(P\cap Q)=t$ is

$$q^{(m-t)(k-t)}\begin{bmatrix} n-m \\ k-t \end{bmatrix}_q\begin{bmatrix} m \\ t \end{bmatrix}_q.$$

Theorem 2.2. $L_O(n,d)$ is a geometric lattice.

Proof. For any two elements $P,Q\in L_O(n,d)$,

$$P\vee Q=P+Q, P\wedge Q=\begin{cases} P\cap Q & \text{if } P\cap Q\not\subset W; \\ \{0\} & \text{otherwise.}\end{cases}$$

Therefore $L_O(n,d)$ is a finite lattice. Note that $\{0\}$ is the unique minimum element. Let $P(n,d;j)$ be the set of all the j-dimensional subspaces of $L_O(n,d)$, where $1\le j\le n$. Then $P(n,d;1)$ is the set of all the atoms in $L_O(n,d)$. In order to prove $L_O(n,d)$ is atomic, it suffices to show that every element of $P(n,d;j)(1\le j\le n)$ is a join of some atoms. The result is trivial for $j=1$. Suppose that the result is true for $j=l>1$. Let $U\in P(n,d;l+1)$. By Proposition 2.1 and $\dim(W_0\cap U)\le l$, the number of l-dimensional subspaces of $L_O(n,d)$ contained in U at least is

$$\begin{bmatrix} l+1 \\ l \end{bmatrix}_q-1=\frac{q\left(q^l-1\right)}{q-1}\ge 2.$$

Therefore there exist two different l-dimensional subspaces $U',U''\subseteq U$ of $L_O(n,d)$ such that $U=U'\vee U''$.

By induction U is a join of some atoms. Hence $L_O(n,d)$ is a finite atomic lattice. For any $U \in L_O(n,d)$, define $r_O(U) = \dim U$. It is routine to check that r_O is the rank function on $L_O(n,d)$. For any $U, V \in L_O(n,d)$, we have

$$\begin{aligned} r_O(U \vee V) + r_O(U \wedge V) &= \dim(U+V) + \dim(U \wedge V) \\ &\leq \dim(U+V) + \dim(U \cap V) \\ &= \dim U + \dim V = r_O(U) + r_O(V). \end{aligned}$$

Hence $L_O(n,d)$ is a geometric lattice. □

Lemma 2.3. For any $P, Q \in L_O(n,d)$, suppose that $\dim P = t$, $\dim Q = t+s$ and $\dim(W_0 \cap Q) = m$. Then the Möbius function of $L_O(n,d)$ is

$$\mu(P,Q) = \begin{cases} (-1)^s q^{\binom{s}{2}} & \text{if } \{0\} \neq P \leq Q \text{ or } P = Q = \{0\}; \\ \sum_{l=1}^{s} (-1)^{s-l+1} \left(\begin{bmatrix} s \\ l \end{bmatrix}_q - \begin{bmatrix} m \\ l \end{bmatrix}_q \right) q^{\binom{s-l}{2}} & \text{if } \{0\} = P < Q; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The Möbius function of $L_O(n,d)$ is

$$\mu(P,Q) = \begin{cases} (-1)^s q^{\binom{s}{2}} & \text{if } \{0\} \neq P \leq Q \text{ or } P = Q = \{0\}; \\ \sum_{\{0\} < U \leq Q} -\mu(U,Q) & \text{if } \{0\} = P < Q; \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.1, we have

$$\sum_{\{0\} < U \leq Q} -\mu(U,Q) = \sum_{l=1}^{s} (-1)^{s-l+1} \left(\begin{bmatrix} s \\ l \end{bmatrix}_q - \begin{bmatrix} m \\ l \end{bmatrix}_q \right) q^{\binom{s-l}{2}}.$$

Thus, the assertion follows. □

Theorem 2.4. The characteristic polynomial of $L_O(n,d)$ is

$$\begin{aligned} \chi(L_O(n,d),x) &= x^n + \sum_{l=1}^{n} (-1)^{n-l+1} \left(\begin{bmatrix} n \\ l \end{bmatrix}_q - \begin{bmatrix} d \\ l \end{bmatrix}_q \right) q^{\binom{n-l}{2}} \\ &+ \sum_{j=1}^{n-1} \sum_{t=\max\{0,d+j-n\}}^{\min\{d,j-1\}} \sum_{l=1}^{j} (-1)^{j-l+1} q^{(d-t)(j-t)+\binom{j-l}{2}} \begin{bmatrix} d \\ t \end{bmatrix}_q \begin{bmatrix} n-d \\ j-t \end{bmatrix}_q \left(\begin{bmatrix} j \\ l \end{bmatrix}_q - \begin{bmatrix} t \\ l \end{bmatrix}_q \right) x^{n-j}. \end{aligned}$$

Proof. By Proposition 2.1 and Lemma 2.3, we have

$$\begin{aligned} \chi(L_O(n,d),x) &= \sum_{P \in L_O(n,d)} \mu(\{0\},P) x^{r_O(F_q^n) - r_O(P)} \\ &= x^n + \sum_{\{0\} \neq P \in L_O(n,d)} \mu(\{0\},P) x^{n-\dim(P)} \\ &= x^n + \sum_{l=1}^{n} (-1)^{n-l+1} \left(\begin{bmatrix} n \\ l \end{bmatrix}_q - \begin{bmatrix} d \\ l \end{bmatrix}_q \right) q^{\binom{n-l}{2}} \\ &+ \sum_{j=1}^{n-1} \sum_{t=\max\{0,d+j-n\}}^{\min\{d,j-1\}} \sum_{l=1}^{j} (-1)^{j-l+1} q^{(d-t)(j-t)+\binom{j-l}{2}} \begin{bmatrix} d \\ t \end{bmatrix}_q \begin{bmatrix} n-d \\ j-t \end{bmatrix}_q \left(\begin{bmatrix} j \\ l \end{bmatrix}_q - \begin{bmatrix} t \\ l \end{bmatrix}_q \right) x^{n-j}. \end{aligned}$$

3. The Lattice $L_R(n,d)$

In this section we prove that the lattice $L_R(n,d)$ is a finite atomic lattice, classify its geometricity and compute its characteristic polynomial.

Theorem 3.1. The following hold:

1) $L_R(n,d)$ is a finite atomic lattice.

2) $L_R(n,d)$ is geometric if and only if $n=2$.

Proof. 1) For any two elements $P,Q\in L_R(n,d)$, $P\wedge Q=P+Q$ and

$$P\vee Q=\begin{cases}P\bigcap Q & \text{if } P\bigcap Q\not\subset W;\\ \{0\} & \text{otherwise.}\end{cases}$$

Therefore $L_R(n,d)$ is a finite lattice. Note that $\{0\}$ is the unique minimum element. Let $P(n,d;j)$ be the set of all the *j*-dimensional subspaces of $L_R(n,d)$, where $0\le j\le n-1$. Then $P(n,d;n-1)$ is the set of all the atoms in $L_R(n,d)$. In order to prove $L_R(n,d)$ is atomic, it suffices to show that every element of $P(n,d;j)(0\le j\le n-1)$ is a join of some atoms. The result is trivial for $j=n-1$. Suppose that the result is true for $j=n-l\le n-1$. Let $U\in P(n,d;n-l-1)$. By Proposition 2.1, the number of $n-l$-dimensional subspaces of $L_R(n,d)$ containing U is equal to

$$\begin{bmatrix}l+1\\1\end{bmatrix}_q=\frac{q^{l+1}-1}{q-1}\ge 2.$$

Then there exist two different $(n-l)$-dimensional subspaces $U\subseteq U',U''\in L_R(n,d)$ such that $U=U'\vee U''$. By induction U is a join of some atoms. Therefore $L_R(n,d)$ is a finite atomic lattice.

2) For any $U\in L_R(n,d)$, we define $r_R(U)=n-\dim U$. It is routine to check that r_R is the rank function on $L_R(n,d)$. It is obvious that $L_R(2,1)$ is a geometric lattice. Now assume that $n\ge 3$. Let *P* be a 1-dimensional subspace of F_q^n and $P\subseteq W_0$. By Proposition 2.1, the number of 2-dimensional subspaces of $L_R(n,d)$ containing *P* is equal to

$$\begin{bmatrix}n-1\\1\end{bmatrix}_q-\begin{bmatrix}d-1\\1\end{bmatrix}_q=\frac{q^{d-1}\left(q^{n-d}-1\right)}{q-1}\ge 2.$$

Therefore, there exist two different 2-dimensional subspaces $P\subseteq P',P''\in L_R(n,d)$ such that $P=P'\bigcap P''$. So $P'\vee P''=\{0\}$, $P'\wedge P''=P'+P''$. Hence $r_R(P'\vee P'')+r_R(P'\wedge P'')=2n-3>2n-4=r_R(P')+r_R(P'')$, which implies that $L_R(n,d)$ is not a geometric lattice when $n\ge 3$. □

Lemma 3.2. For any $P,Q\in L_R(n,d)$, suppose that $\dim P=t+s$, $\dim Q=t$ and $\dim(W_0\bigcap P)=m$. Then the Möbius function of $L_R(n,d)$ is

$$\mu(P,Q)=\begin{cases}(-1)^s q^{\binom{s}{2}} & \text{if } P\le Q\ne\{0\} \text{ or } P=Q=\{0\};\\ \sum_{l=1}^{s}(-1)^{s-l+1}\left(\begin{bmatrix}s\\l\end{bmatrix}_q-\begin{bmatrix}m\\l\end{bmatrix}_q\right)q^{\binom{s-l}{2}} & \text{if } P<Q=\{0\};\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The Möbius function of $L_R(n,d)$ is

$$\mu(P,Q)=\begin{cases}(-1)^s q^{\binom{s}{2}} & \text{if } P\le Q\ne\{0\} \text{ or } P=Q=\{0\};\\ \sum_{P\le U<\{0\}}-\mu(P,U) & \text{if } P<Q=\{0\};\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 2.1 implies that

$$\sum_{P\le U<\{0\}} -\mu(P,U) = \sum_{l=1}^{s}(-1)^{s-l+1}\left(\begin{bmatrix} s \\ l \end{bmatrix}_q - \begin{bmatrix} m \\ l \end{bmatrix}_q\right) q^{\binom{s-l}{2}}.$$

Theorem 3.3. The characteristic polynomial of $L_R(n,d)$ is

$$\chi(L_R(n,d),x) = x^n - 1 + \sum_{j=1}^{n}(-1)^{n-j}\left(\begin{bmatrix} n \\ j \end{bmatrix}_q - \begin{bmatrix} d \\ j \end{bmatrix}_q\right) q^{\binom{n-j}{2}}\left(x^j - 1\right).$$

Proof. By Proposition 2.1, we have

$$\begin{aligned}
\chi(L_R(n,d),x) &= \sum_{P\in L_R(n,d)} \mu(F_q^n,P)\, x^{r_R(\{0\})-r_R(P)} \\
&= x^n + \sum_{F_q^n \ne P\in L_R(n,d)} \mu(F_q^n,P)\, x^{\dim(P)} \\
&= x^n + \sum_{j=1}^{n}(-1)^{n-j}\left(\begin{bmatrix} n \\ j \end{bmatrix}_q - \begin{bmatrix} d \\ j \end{bmatrix}_q\right) q^{\binom{n-j}{2}} x^j + \sum_{l=1}^{n}(-1)^{n-l+1}\left(\begin{bmatrix} n \\ l \end{bmatrix}_q - \begin{bmatrix} d \\ l \end{bmatrix}_q\right) q^{\binom{n-l}{2}} \\
&= x^n - 1 + \sum_{j=1}^{n-1}(-1)^{n-j}\left(\begin{bmatrix} n \\ j \end{bmatrix}_q - \begin{bmatrix} d \\ j \end{bmatrix}_q\right) q^{\binom{n-j}{2}}\left(x^j - 1\right).
\end{aligned}$$

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