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L^{∞}-Error Estimate of Schwarz Algorithm for Noncoercive Variational Inequalities

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ABSTRACT

The Schwarz method for a class of elliptic variational inequalities with noncoercive operator was studied in this work. The author proved the error estimate in L^{∞}-norm for two domains with overlapping nonmatching grids using the geometrical convergence of solutions and the uniform convergence of subsolutions.

KEYWORDS

Variational Inequalities; Schwarz Method; Subsolutions; L^{∞}-Error Estimates

1. Introduction

More than one hundred years ago, Schwarz algorithms were proposed for proving the solvability of PDEs on a complicated domain. With parallel calculators, this rediscovery of these methods as algorithms of calculations was based on a modern variational approach. Pierre-Louis Lions was the starting point of an intense research activity to develop this tool of calculation, see, e.g., [1,2] and the references therein [3-9].

In this paper, we give a new approach to the finite element approximation for the problem of variational inequality with noncoercive operator. This problem arises in stochastic control (see [10]). We consider a domain which is the union of two overlapping sub-domains where each sub-domain has its own generated triangulation. To prove the main result of this work, we construct two sequences of subsolutions and we estimate the errors between Schwarz iterates and the subsolutions. The proof stands on a Lipschitz continuous dependency with respect to the source term for variational inequality, while in [5] the proof stands on a Lipschitz continuous dependency with respect to the boundary condition.

The paper is organized as follows. In Sections 2, we introduce the continuous and discrete obstacle problem as well as Schwarz algorithm with two sub-domains and give the geometrical convergence theorem. In Section 3, we establish two sequences of subsolutions and their error estimates and prove a main result concerning the error estimate of solution in the L^{∞}-norm, taking into account the combination of geometrical convergence and uniform convergence [11,12] of finite element approximation.

2. Schwarz Algorithm for Variational Inequalities with Noncoercive Operator

2.1. Notations and Assumptions

Let's consider functions

$$a_{ij}, a_j, a_0 \in C^2\left(\bar{\Omega}\right); 1 \le i, j \le 2 \tag{1}$$

such that

$$\sum_{1\le i,j\le 2} a_{ij}\xi_i\xi_j \ge \alpha \sum_{1\le i\le 2}\xi_i^2;\ \xi\in R^2, \alpha>0 \tag{2}$$

$$a_{ij}(x)=a_{ji}(x);\ a_0(x)\ge\alpha_0>0 \tag{3}$$

where Ω is a connected bounded domain in R^2 with sufficiently regular boundary $\partial\Omega$.

We define a second order differential operator

$$A=-\sum_{1\le i,j\le 2}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right)+\sum_{1\le j\le 2}a_j\frac{\partial}{\partial x_j}+a_0 \tag{4}$$

where the bilinear form associated: $\forall u,v\in H^1(\Omega)$

$$a(u,v)=\int_\Omega\left(\sum_{1\le i,j\le 2}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\sum_{1\le j\le 2}a_j\frac{\partial u}{\partial x_j}v+a_0uv\right)\mathrm{d}x \tag{5}$$

Let f be a function in

$$L^\infty(\Omega) \tag{6}$$

an obstacle

$$\psi\in W^{2,\infty}(\Omega) \tag{7}$$

a regular function g defined on $\partial\Omega$ such that

$$g\in W^{2,p}(\Omega), 2\le p<\infty, g\le\psi \text{ on } \partial\Omega \tag{8}$$

AM and a nonempty convex set

$$K_{(\psi,g)}=\left\{v\in H^1(\Omega)/v-g\in H_0^1(\Omega), v\le\psi \text{ in } \Omega\right\}. \tag{9}$$

We assume there exists $\lambda>0$ large enough and a constant $\beta>0$ such that

$$\beta\|v\|^2_{H^1(\Omega)}\le a(v,v)+\lambda\int_\Omega v^2\mathrm{d}x. \tag{10}$$

Putting

$$b(u,v)=a(u,v)+\lambda\int_\Omega uv\mathrm{d}x\quad \forall u,v\in H^1(\Omega) \tag{11}$$

then the bilinear form $b(.,.)$ is strongly coercive.

Let $u\in K_{(\psi,g)}$ be the solution of variational inequality (V.I)

$$a(u,v-u)\ge(f,v-u)\quad\forall v\in K_{(\psi,g)} \tag{12}$$

which is equivalent to

$$\begin{cases}\text{Find } u\in K_{(\psi,g)} \text{ solution of}\\ b(u,v-u)\ge(f+\lambda u,v-u)\quad\forall v\in K_{(\psi,g)}\end{cases} \tag{13}$$

$(.,.)$ denotes the usual inner product in $L^2(\Omega)$.

We define $\bar{u}=\sigma(f+\lambda w)\in K_{(\psi,g)}$ the solution of the following V.I

$$b(\bar{u},v-\bar{u})\ge(f+\lambda w,v-\bar{u})\quad\forall v\in K_{(\psi,g)} \tag{14}$$

where $w\in L^\infty(\Omega)$ and σ is a mapping from $L^\infty(\Omega)$ into itself.

Remark 1. *We call quasi-variational inequality (Q.V.I) if the right hand side* $(f+\lambda u)$ *depends of solution* u, *in the contrary case we call variational inequality (V.I).*

2.2. Some Preliminary Results on the V.I Noncoercive

Thanks to [10], the problem (12) has one and only one solution, moreover u satisfies the regularity property

$$u\in W^{2,p}(\Omega), 2\le p<\infty.$$

We give a monotonicity property of the solution with respect to both the source term, the boundary condition and the obstacle. Let $(f,\psi,g);(\tilde{f},\tilde{\psi},\tilde{g})$ be a pair of data and $u=\partial(f,\psi,g);\tilde{u}=\partial(\tilde{f},\tilde{\psi},\tilde{g})$ the corresponding solution of V.I (12).

Lemma 1 [10] *Under the preceding notations and assumptions* (1) *to* (11), *if* $f \geq \tilde{f}, \psi \geq \tilde{\psi}$ *and* $g \geq \tilde{g}$, *then* $\partial(f,\psi,g) \geq \partial(\tilde{f},\tilde{\psi},\tilde{g})$.

Let X be the set of sub-solutions of the Q.V.I, ie all the $\overline{w} \in K_{(\psi,g)}$ such that

$$b(\overline{w},v) \leq (f+\lambda\overline{w},v),\ v \geq 0, v \in H_0^1(\Omega) \tag{15}$$

that is equivalent to

$$a(\overline{w},v) \leq (f,v),\ v \geq 0, v \in H_0^1(\Omega).$$

Lemma 2 [10] *Under the preceding notations and assumptions* (1) *to* (11), *the solution* u *of problem* (12) *is the maximum element of the set* X .

We show the Lipschitz property, which gives the continuous dependance to the data f .

Lemma 3 *Under the preceding notations and assumptions* (1) *to* (11), *we have*

$$\|u-\tilde{u}\|_{L^\infty(\Omega)} \leq \max\left\{c\|f-\tilde{f}\|_{L^\infty(\Omega)}, \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}, \|\psi-\tilde{\psi}\|_{L^\infty(\Omega)}\right\}$$

where c is an independent constant of data.

Proof *Firstly, let*

$$\Phi = \max\left\{\frac{1}{\alpha_0}\|f-\tilde{f}\|_{L^\infty(\Omega)}, \|g-\tilde{g}\|_{L^\infty(\partial\Omega)}, \|\psi-\tilde{\psi}\|_{L^\infty(\Omega)}\right\}$$

we have

$$a(\Phi, v-u) = \Phi.(a_0, v-u)$$

then

$$a(u+\Phi, v-u) \geq (f+a_0\Phi, v-u)$$

and

$$a(u+\Phi, (v+\Phi)-(u+\Phi)) \geq (f+a_0\Phi, (v+\Phi)-(u+\Phi))$$

if we put

$$u+\Phi = \hat{u} \text{ and } v+\Phi = \hat{v}$$

then

$$a(\hat{u}, \hat{v}-\hat{u}) \geq (f+a_0\Phi, \hat{v}-\hat{u})$$

therefore

$$\partial(f+a_0\Phi, \psi+\Phi, g+\Phi) = \hat{u} = \partial(f,\psi,g)+\Phi.$$

Secondly, it is clear that

$$\tilde{f} \leq f + \|f-\tilde{f}\|_{L^\infty(\Omega)} \leq f + \frac{a_0}{\alpha_0}\|f-\tilde{f}\|_{L^\infty(\Omega)} \leq f + a_0\Phi$$

and

$$\tilde{\psi} \leq \psi + \Phi$$

$$\tilde{g} \leq g + \Phi$$

so, due to lemma 1, we get

$$\partial(\tilde{f},\tilde{\psi},\tilde{g}) \leq \partial(f+a_0\Phi, \psi+\Phi, g+\Phi) = \partial(f,\psi,g)+\Phi$$

which gives

$$\partial(\tilde{f},\tilde{\psi},\tilde{g}) - \partial(f,\psi,g) \leq \Phi$$

by changing the roles of (f,ψ,g) and $(\tilde{f},\tilde{\psi},\tilde{g})$, we obtain

$$\partial(f,\psi,g)-\partial\left(\tilde{f},\tilde{\psi},\tilde{g}\right)\le\Phi$$

which completes the proof.

Remark 2 *If* $\psi=\tilde{\psi}$ *and* $g=\tilde{g}$ *, then we have*

$$\|u-\tilde{u}\|_{L^\infty(\Omega)}\le c\left\|f-\tilde{f}\right\|_{L^\infty(\Omega)}.$$

Let Ω be decomposed into triangles and let τ^h denote the set of those elements; $h>0$ is the mesh-size. We assume the triangulation τ^h is regular and quasi-uniform. Let V_h denote the standard piecewise linear finite element space and by $\varphi_i, i=1,2,\cdots,m(h)$, the basis functions of the space V_h. Let r_h be the usual restriction operator in Ω. The discrete counterpart of (13) consists of finding $u_h\in K^h_{(\psi,g)}$ solution of

$$b(u_h,v_h-u_h)\ge(f+\lambda u_h,v_h-u_h)\ \ \forall v_h\in K^h_{(\psi,g)} \tag{16}$$

where

$$K^h_{(\psi,g)}=\left\{v_h\in V_h\,/\,v_h=\pi_h g \text{ on } \partial\Omega, v_h\le r_h\psi \text{ in } \tau^h\right\} \tag{17}$$

π_h is an interpolation operator on $\partial\Omega$.

We shall assume that the matrix B defined by

$$B_{ij}=b\left(\varphi_i,\varphi_j\right)=a\left(\varphi_i,\varphi_j\right)+\lambda\int_\Omega\varphi_i\varphi_j\mathrm{d}x \tag{18}$$

is M -matrix [13] (*i.e*. angles of triangles of τ^h are $\le\pi/2$).

2.3. The Continuous Schwarz Algorithm

Consider the model obstacle problem: find $u\in K_{(\psi,0)}$ such that

$$b(u,v-u)\ge(f+\lambda u,v-u)\ \ \forall v\in K_{(\psi,0)} \tag{19}$$

where $K_{(\psi,0)}$ defined in (9) with $g=0$.

We decompose Ω into two overlapping polygonal subdomains Ω_1 and Ω_2 such that

$$\Omega=\Omega_1\cup\Omega_2,\ \Omega_1\cap\Omega_2\ne\varnothing$$

and u satisfies the local regularity property

$$u|_{\Omega_i}\in W^{2,p}(\Omega_i),\ 2\le p<\infty$$

we denote $\partial\Omega_i$ the boundary of Ω_i, and $\Gamma_i=\partial\Omega_i\cap\Omega_j$. The intersection of $\overline{\Gamma}_i$ and $\overline{\Gamma}_j$; $i\ne j$ is assumed to be empty. We will always assume to simplify that Γ_1,Γ_2 are smooth.

For $w\in C^0\left(\overline{\Gamma}_i\right)$, we define

$$V_i^{(w)}=\left\{v\in H^1(\Omega_i)\,/\,v=0 \text{ on } \partial\Omega\cap\partial\Omega_i, v=w \text{ on } \Gamma_i\right\};\ i=1,2.$$

We associate with problem (19) the following system: find $(u_1,u_2)\in V_1^{(u_2)}\times V_2^{(u_1)}$ solution of

$$\begin{cases} b_1(u_1,v-u_1)\ge(f_1+\lambda u_1,v-u_1)\ \forall v\in V_1^{(u_2)}, \\ u_1\le\psi, v\le\psi \text{ in } \Omega_1 \\ b_2(u_2,v-u_2)\ge(f_2+\lambda u_2,v-u_2)\ \forall v\in V_2^{(u_1)}, \\ u_2\le\psi, v\le\psi \text{ in } \Omega_2 \end{cases} \tag{20}$$

where

$$b_i(u,v)=\int_{\Omega_i}\left(\sum_{1\le l,j\le 2}a_{lj}\frac{\partial u}{\partial x_l}\frac{\partial v}{\partial x_j}+\sum_{1\le j\le 2}a_j\frac{\partial u}{\partial x_j}v+a_0uv+\lambda uv\right)\mathrm{d}x;\ i=1,2$$

$$f_i=f|_{\Omega_i},\ u_i=u|_{\Omega_i}.$$

Starting from $u^0=\psi$, we define the continuous Schwarz sequences $\left(u_1^{n+1}\right)$ on Ω_1 such that $u_1^{n+1}\in V_1^{\left(u_2^n\right)}$ solves

$$\begin{cases} b_1\left(u_1^{n+1}, v-u_1^{n+1}\right) \geq \left(f_1+\lambda u_1^n, v-u_1^{n+1}\right) \ \forall v \in V_1^{\left(u_2^n\right)} \\ u_1^{n+1} \leq \psi, v \leq \psi \text{ in } \Omega_1 \end{cases} \tag{21}$$

and $\left(u_2^{n+1}\right)$ on Ω_2 such that $u_2^{n+1} \in V_2^{\left(u_1^{n+1}\right)}$ solves

$$\begin{cases} b_2\left(u_2^{n+1}, v-u_2^{n+1}\right) \geq \left(f_2+\lambda u_2^n, v-u_2^{n+1}\right) \ \forall v \in V_2^{\left(u_1^{n+1}\right)} \\ u_2^{n+1} \leq \psi, v \leq \psi \text{ in } \Omega_2 \end{cases} \tag{22}$$

where

$$u_1^0 = u^0 \text{ in } \Omega_1,\ u_2^0 = u^0 \text{ in } \Omega_2,$$

$$u_1^{n+1} = 0 \text{ in } \bar{\Omega} \setminus \bar{\Omega}_1 \text{ and } u_2^{n+1} = 0 \text{ in } \bar{\Omega} \setminus \bar{\Omega}_2.$$

The following geometrical convergence is due to ([2], pages 51-63)

Theorem 1 *The sequences* $\left(u_1^{n+1}\right)$ *and* $\left(u_2^{n+1}\right), n \geq 0$ *of the Schwarz algorithm converge geometrically to the solution of the problem* (20)*. More precisely, there exist two constants* $k_1, k_2 \in \left]0,1\right[$ *such that for all* $n > 0$

$$\left\|u_1 - u_1^{n+1}\right\|_{L^\infty(\Omega_1)} \leq k_1^n k_2^n \left\|u^0 - u\right\|_{L^\infty(\Gamma_1)}$$

$$\left\|u_2 - u_2^{n+1}\right\|_{L^\infty(\Omega_2)} \leq k_1^{n+1} k_2^n \left\|u^0 - u\right\|_{L^\infty(\Gamma_2)}$$

2.4. The Discretization

For $i = 1, 2$; let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i, h_i being the mesh size. We assume that the two triangulations are mutually independent on $\Omega_1 \cap \Omega_2$, where a triangle belonging to one triangulation does not necessarily belong to the other. Let $V_{h_i} = V_{h_i}\left(\Omega_i\right)$ be the space of continuous piecewise linear functions on τ^{h_i} which vanish on $\partial\Omega \cap \partial\Omega_i$. For $w \in C^0\left(\bar{\Gamma}_i\right)$ we define

$$V_{h_i}^{(w)} = \left\{v_{h_i} \in V_{h_i} / v_{h_i} = 0 \text{ on } \partial\Omega \cap \partial\Omega_i, v_{h_i} = \pi_{h_i}(w) \text{ on } \Gamma_i\right\};\ i = 1, 2$$

where π_{h_i} denotes a suitable interpolation operator on Γ_i. We give the discrete counterparts of Schwarz algorithm defined in (21) and (22) as follows.

Starting from $u_{ih_i}^0 = r_{h_i}\psi$, we define the discrete Schwarz sequence $\left(u_{1h_1}^{n+1}\right)$ on Ω_1 such that $u_{1h_1}^{n+1} \in V_{h_1}^{\left(u_{2h_2}^n\right)}$ solves

$$\begin{cases} b_1\left(u_{1h_1}^{n+1}, v_{h_1} - u_{1h_1}^{n+1}\right) \geq \left(f_1 + \lambda u_{1h_1}^n, v_{h_1} - u_{1h_1}^{n+1}\right) \\ \forall v_{h_1} \in V_{h_1}^{\left(u_{2h_2}^n\right)}, u_{1h_1}^{n+1} \leq r_{h_1}\psi, v_{h_1} \leq r_{h_1}\psi \text{ in } \tau^{h_1} \end{cases} \tag{23}$$

and on Ω_2 the sequence $u_{2h_2}^{n+1} \in V_{h_2}^{\left(u_{1h_1}^{n+1}\right)}$ solves

$$\begin{cases} b_2\left(u_{2h_2}^{n+1}, v_{h_2} - u_{2h_2}^{n+1}\right) \geq \left(f_2 + \lambda u_{2h_2}^n, v_{h_2} - u_{2h_2}^{n+1}\right) \\ \forall v_{h_2} \in V_{h_2}^{\left(u_{1h_1}^{n+1}\right)}, u_{2h_2}^{n+1} \leq r_{h_2}\psi, v_{h_2} \leq r_{h_2}\psi \text{ in } \tau^{h_2} \end{cases} \tag{24}$$

We will also always assume that the respective matrices resulting from problems (23) and (24) are M-matrices.

3. Error Analysis

This section is devoted to the proof of the main result of this work. For that, we begin by introducing two auxiliary sequences.

3.1. Auxiliary Schwarz Sequences

To simplify the notation, we take

$$h_1 = h_2 = h,\ r_{h_1} = r_{h_2} = r_h,\ \pi_{h_1} = \pi_{h_2} = \pi_h,$$
$$\|\cdot\|_1 = \|\cdot\|_{L^\infty(\Omega_1)},\quad \|\cdot\|_2 = \|\cdot\|_{L^\infty(\Omega_2)}$$
$$\tau^{h_1} = \tau_1^h,\quad \tau^{h_2} = \tau_2^h$$

Let $\bar{u}_{ih}^{n+1} = \sigma_h\left(f_i + \lambda u_i^n\right) \in V_{h_i}\left(\Omega_i\right)$ be the solution of discrete V.I

$$\begin{cases} b_i\left(\bar{u}_{ih}^{n+1}, v_h - \bar{u}_{ih}^{n+1}\right) \geq \left(f_i + \lambda u_i^n, v_h - \bar{u}_{ih}^{n+1}\right) \\ \forall v_h \in V_h\left(\Omega_i\right), \bar{u}_{ih}^{n+1} \leq r_h\psi, v_h \leq r_h\psi \text{ in } \tau_i^h \\ \bar{u}_{ih}^{n+1} = \pi_h u_{ih}^{n+1}, v_h = \pi_h u_{ih}^{n+1} \text{ on } \Gamma_i \end{cases} \tag{25}$$

where $u_i^{n+1} = \sigma\left(f_i + \lambda u_i^n\right);\quad i = 1,2$ is the solution of continuous V.I (21) (resp. (22)) and let $\bar{u}_i^{(h),n+1} = \sigma\left(f_i + \lambda u_{ih}^n\right) \in V_i$ be the solution of continuous V.I

$$\begin{cases} b_i\left(\bar{u}_i^{(h),n+1}, v - \bar{u}_i^{(h),n+1}\right) \geq \left(f_i + \lambda u_{ih}^n, v - \bar{u}_i^{(h),n+1}\right) \\ \forall v \in H^1\left(\Omega_i\right), \bar{u}_i^{(h),n+1} \leq \psi, v \leq \psi \text{ in } \Omega_i \\ \bar{u}_i^{(h),n+1} = u_i^{n+1},\ v = u_i^{n+1} \text{ on } \Gamma_i \end{cases} \tag{26}$$

where $u_{ih}^{n+1} = \sigma_h\left(f_i + \lambda u_{ih}^n\right);\quad i = 1,2$ is the solution of discrete V.I. (23) (resp. (24)).

It is clear that $\bar{u}_{ih}^n$ is the finite element approximation of u_i^n. Then, as $\left\|f_i + \lambda u_i^n\right\|_i \leq c$ (independent of n), therefore, we apply the error estimate for variational inequality (see [11,12]), we get

$$\left\|u_i^n - \bar{u}_{ih}^n\right\|_i \leq ch^2\left|\log h\right|^3 \tag{27}$$

similarly, we have

$$\left\|u_{ih}^n - \bar{u}_i^{(h),n}\right\|_i \leq ch^2\left|\log h\right|^3. \tag{28}$$

3.2. Sequences of Sub-Solutions

The following theorems will play a important role in proving the main result of this paper.

3.2.1. Part One—Discrete Sub-Solution

We construct a discrete function α_{ih}^n near u_i^n such that: $\alpha_{ih}^n \leq u_{ih}^n$.

Theorem 2 *Let $\bar{u}_{ih}^{n+1}$ be the solution of* (25). *Then there exists a function α_{ih}^n and a constant c independent of h and n, such that*

$$\alpha_{ih}^n \leq u_{ih}^n$$
$$\left\|\alpha_{ih}^n - u_i^n\right\|_i \leq ch^2\left|\log h\right|^3$$

Proof *Let us give the proof for $i = 1$. The one for $i = 2$ is similar. Indeed, $\bar{u}_{1h}^{n+1}$ being the solution of V.I* (25) *for $i = 1$, it is easy to show that $\bar{u}_{1h}^{n+1}$ is also a subsolution, i.e*

$$\begin{cases} b_1\left(\bar{u}_{1h}^{n+1}, \varphi_l\right) \leq \left(f_1 + \lambda u_1^n, \varphi_l\right)\ \forall \varphi_l \geq 0; l = 1,\cdots,m(h) \\ \bar{u}_{1h}^{n+1} \leq r_h\psi \text{ in } \tau_1^h, \bar{u}_{1h}^{n+1} = \pi_h u_{2h}^n \text{ on } \Gamma_1 \end{cases}$$

then

$$\begin{cases} b_1\left(\bar{u}_{1h}^{n+1}, \varphi_l\right) \leq \left(f_1 + \lambda\left\|u_1^n - \bar{u}_{1h}^n\right\|_1 + \lambda\bar{u}_{1h}^n, \varphi_l\right)\ \forall \varphi_l \geq 0 \\ l = 1,\cdots,m(h),\ \bar{u}_{1h}^{n+1} \leq r_h\psi \text{ in } \tau_1^h, \bar{u}_{1h}^{n+1} = \pi_h u_{2h}^n \text{ on } \Gamma_1 \end{cases}$$

so, due to lemma 2 (discrete case), it follows that

$$\bar{u}_{1h}^{n+1} \leq \tilde{u}_{1h}^{n+1} = \partial_h\left(\tilde{f}_1, r_h\psi, \pi_h u_{2h}^n\right) \tag{29}$$

where

$$\tilde{f}_1 = f_1 + \lambda \left\| u_1^n - \overline{u}_{1h}^n \right\|_1$$

setting $u_{1h}^{n+1} = \partial_h \left(f_1, r_h\psi, \pi_h u_{2h}^n \right)$ and using both remak2 (discrete case) and estimate (27), we get

$$\left\| u_{1h}^{n+1} - \tilde{u}_{1h}^{n+1} \right\|_1 \le c \left\| \tilde{f}_1 - f_1 \right\|_1 \le c\lambda \left\| u_1^n - \overline{u}_{1h}^n \right\|_1 \le ch^2 \left| \log h \right|^3 \tag{30}$$

which combined with (29) yields

$$\overline{u}_{1h}^{n+1} \le u_{1h}^{n+1} + ch^2 \left| \log h \right|^3$$

Thus, we choose

$$\alpha_{1h}^{n+1} = \overline{u}_{1h}^{n+1} - ch^2 \left| \log h \right|^3$$

then

$$\alpha_{1h}^{n+1} \le u_{1h}^{n+1}$$

and

$$\begin{aligned} \left\| \alpha_{1h}^{n+1} - u_1^{n+1} \right\|_1 &\le \left\| \overline{u}_{1h}^{n+1} - ch^2 \left| \log h \right|^3 - u_1^{n+1} \right\|_1 \le \left\| \overline{u}_{1h}^{n+1} - u_1^{n+1} \right\|_1 + ch^2 \left| \log h \right|^3 \\ &\le ch^2 \left| \log h \right|^3 + ch^2 \left| \log h \right|^3 \le ch^2 \left| \log h \right|^3 \end{aligned}$$

3.2.2. Part Two—Continuous Sub-Solution

We construct a continuous function $\beta_i^{(h),n}$ near u_{ih}^n such that: $\beta_i^{(h),n} \le u_i^n$.

Theorem 3 *Let* $\overline{u}_i^{(h),n+1}$ *be the solution of* (26). *Then there exists a function* $\beta_{ih}^{(h),n}$ *and a constant* c *independent of* h *and* n, *such that*

$$\beta_i^{(h),n} \le u_i^n$$
$$\left\| \beta_i^{(h),n} - u_{ih}^n \right\|_i \le ch^2 \left| \log h \right|^3$$

Proof *Let us give the proof for* $i = 1$. *The one for* $i = 2$ *is similar. indeed,* $\overline{u}_1^{(h),n+1}$ *being the solution of V.I* (26) *for* $i = 1$, *it is also a subsolution, i.e.*

$$\begin{cases} b_1\left(\overline{u}_1^{(h),n+1}, w\right) \le \left(f_1 + \lambda u_{1h}^n, w \right) \ \forall w \in H_0^1\left(\Omega_1\right), w \ge 0 \\ \overline{u}_1^{(h),n+1} \le \psi \text{ in } \Omega_1, \overline{u}_1^{(h),n+1} = u_2^n \text{ on } \Gamma_1 \end{cases}$$

then

$$\begin{cases} b_1\left(\overline{u}_1^{(h),n+1}, w\right) \le \left(f_1 + \lambda \left\| u_{1h}^n - \overline{u}_1^{(h),n} \right\|_1 + \lambda \overline{u}_1^{(h),n}, w \right) \\ \forall w \in H_0^1\left(\Omega_1\right), w \ge 0, \overline{u}_1^{(h),n+1} \le \psi \text{ in } \Omega_1 \\ \overline{u}_1^{(h),n+1} = u_2^n \text{ on } \Gamma_1 \end{cases}$$

so, making use of lemma 2, we obtain

$$\overline{u}_1^{(h),n+1} \le \tilde{u}_1^{n+1} = \partial\left(\tilde{f}_1, \psi, u_2^n \right) \tag{31}$$

where

$$\tilde{f}_1 = f_1 + \lambda \left\| u_{1h}^n - \overline{u}_1^{(h),n} \right\|_1$$

Setting $u_1^{n+1} = \partial\left(f_1, \psi, u_2^n \right)$ and using both Remark 2 and estimate (28), we get

$$\left\| \tilde{u}_1^{n+1} - u_1^{n+1} \right\|_1 \le c \left\| \tilde{f}_1 - f_1 \right\|_1 \le ch^2 \left| \log h \right|^3 \tag{32}$$

so, combining (31) with estimate (32) yields

$$\overline{u}_1^{(h),n+1} \le u_1^{n+1} + ch^2 \left| \log h \right|^3$$

Finally, choosing

$$\beta_1^{(h),n+1} = \overline{u}_1^{(h),n+1} - ch^2 \left|\log h\right|^3$$

we get immediately the results.

3.3. L^∞-Error Estimate

Theorem 4 $\left(Main\,result\right)$ *Let* $\left(u_i^{n+1}\right)$ (*resp.* $\left(u_{ih}^{n+1}\right)$) *be the solution of* (21), (22) (*resp.* (23), (24)). *Then there exists a constant* c *independent of* h *and* n, *such that*

$$\left\|u_i - u_{ih}^{n+1}\right\|_i \le ch^2 \left|\log h\right|^3$$

$$\left\|u_i - u_{ih}^{n+1}\right\|_{W^{1,\infty}(\Omega_i)} \le ch\left|\log h\right|^3 .$$

Proof *Thanks to theorem* 2 *and theorem* 3, *we have*

$$u_i^{n+1} - u_{ih}^{n+1} \le ch^2 \left|\log h\right|^3$$

$$u_{ih}^{n+1} - u_i^{n+1} \le ch^2 \left|\log h\right|^3$$

therefore

$$\left\|u_i^{n+1} - u_{ih}^{n+1}\right\|_i \le ch^2 \left|\log h\right|^3 ;\ i = 1,2 \tag{33}$$

moreover

$$\left\|u_i - u_{ih}^{n+1}\right\|_i \le \left\|u_i - u_i^{n+1}\right\|_i + \left\|u_i^{n+1} - u_{ih}^{n+1}\right\|_i$$

let $k = \max\left(k_1, k_2\right)$, then making use of Theorem 1 and estimate (33), we get

$$\left\|u_i - u_{ih}^{n+1}\right\|_i \le k^{2n} \left\|u^0 - u\right\|_{L^\infty(\Gamma_i)} + ch^2 \left|\log h\right|^3 \le k^{2n} \left\|\psi - u\right\|_{L^\infty(\Gamma_i)} + ch^2 \left|\log h\right|^3$$
$$\le ck^{2n} + ch^2 \left|\log h\right|^3$$

we choose n such that

$$k^{2n} \le h^2$$

then

$$\left\|u_i - u_{ih}^{n+1}\right\|_i \le ch^2 + ch^2 \left|\log h\right|^3 \le ch^2 \left|\log h\right|^3$$

and by inverse inequality, we get

$$\left\|u_i - u_{ih}^{n+1}\right\|_{W^{1,\infty}(\Omega_i)} \le ch\left|\log h\right|^3$$

4. Conclusion

We have established a convergence order of Schwarz algorithm for two overlapping subdomains with non-matching grids. This approach developed in this paper relies on the geometrical convergence and the error estimate between the continuous and discrete Schwarz iterates. The constant *c* in error estimate is independent of Schwarz iterate *n*.

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