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A Permutation Test for Unit Root in an Autoregressive Model

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ABSTRACT

A permutation test (based on a finite random sample of permutations) for unit root in an autoregressive process is considered. The test can easily be carried out in practice and the proposed permutation test is neither limited to large sample sizes nor normal white noises. Simulations show that the power of the permutation test is reasonable when sample sizes are small or when the white noises have a heavy tailed distribution. The test is shown to be consistent.

Keywords: Permutation Test; Autoregressive; Nonstationary

1. Introduction

Let $Y_1, Y_2, \cdots, Y_{n+1}$ be $n+1$ observations of the real valued autoregressive model

$$Y_t = aY_{t-1} + e_t,$$

for $0 < a \le 1$ and e_t is a sequence of independent identically distributed random variables with mean zero and variance σ^2 and $Y_1 = 0$. Tests of

$$H_0 : a = 1 \quad \text{versus} \quad H_A : 0 < a < 1,$$

are often referred to as tests for unit root. The hypothesis that $a = 1$ is of interest in applications because it corresponds to the hypothesis that it is appropriate to transform the times series by difference. In [1] and [2], the authors derived the limit distribution of the statistic $n(\hat{a} - 1)$ with

$$\hat{a} = \left(\sum_{i=2}^{n+1} Y_{t-1}^2\right)^{-1} \sum_{t=2}^{n+1} Y_t Y_{t-1}$$

under the unit root assumption $a = 1$. However, [1] and [2] are limited to large sample sizes or normal white noises. When sample sizes are small and white noises are from distributions with heavy tails, to test for the presence of unit root, we can use the permutation test proposed in the paper.

Under H_0, Y_t is not stationary and the variance of Y_t is $t\sigma^2$. When H_0 is true, Y_t is sometimes called a random walk and $X_t = Y_{t+1} - Y_t$, $t = 1, 2, \cdots, n$ are independent identically distributed r.v.'s. In Economics, it is important to characterize the velocity, or stock price as a random walk. Another way of phrasing H_0 is that whatever determinants of velocity and their individual stochastic structure may be, their combined effect is such that successive changes in velocity are essentially independent. This would imply that of the past history available at any given date only the current observation is relevant for prediction. Testing for unit root is equivalent to testing for serial independence in sequence $X_1, \cdots, X_n$. For some literature on tests for serial dependence, see [3] and [4] and the references therein. We define

$$T_n = \sum_{i=1}^{n-1} X_i X_{i+1},$$

and for each random permutation of the vector $(X_1, \cdots, X_n)$, say, $(X_{1\ell}, \cdots, X_{n\ell})$, denote

$$T_{n\ell} = \sum_{i=1}^{n-1} X_{i\ell} X_{(i+1)\ell}.$$

If H_0 holds then $(X_1, \cdots, X_n) = (e_2, \cdots, e_{n+1})$. The distribution of a random vector of i.i.d. random variables is invariant under any permutation of its coordinates. Thus T_n and $T_{n\ell}$ have the same distribution if H_0 is true. Based on the invariance of the distribution of the statistic T_n under permutations when H_0 holds, we propose a permutation test for unit root using T_n as our pivot test statistic. This test is easy to perform with a computer and the test makes little assumptions on the probability distribution of the white noise and also it works for small samples. Construction of this test will be presented in Section 2. The consistency of our test is shown in Section 3. A simulation study of our test is provided in Section 4. Time series of velocity of money observed between 1869 and 1960 is investigated in Section 5.

2. Steps Used in Our Permutation Test

We assume that the white noise e_t is a sequence of independent identically distributed random variables with mean zero and variance σ^2. In addition, $E\left(e_t^4\right) < \infty$.

We will often write T_n simply as T for brevity. Note that T is more likely to be negative under H_A (see Lemma 3.2 below). To summarize, the permutation test is carried out as follows.

1) Set a predetermined level α. Permute the n observations $\{X_1, X_2, \cdots, X_n\}$. There are a total of $n!$ permutations. For each permutation, compute the T statistic. Under H_0, the T statistics have the same probability distribution for all of the $n!$ permutations. The T statistic computed from the observations (not permuted) is referred to as T_{obs}.

2) Compute the p-value as the proportion of T's less than or equal to T_{obs}, that is,

$$p\text{-value} = \frac{\text{number of } T's \le T_{obs}}{n!}.$$

Conclude that the test is statistically significant if the computed p-value is less than or equal to α.

This test is limited by prohibitive calculation and hard to carry out if n is a large number. Instead of using all $n!$ permutations to compute the p-value, we obtain a random sample R of permutations and then carry out the test as follows:

1) Set a predetermined level α. Compute the T statistic for each of the R sampled permutation.

2) Compute the p-value as the proportion of T's less than or equal to T_{obs}, that is,

$$p\text{-value} = \frac{\text{number of } T's \le T_{obs}}{R}.$$

Conclude that the test is statistically significant if the computed p-value is less than or equal to α.

The approximate p-value is now equal to the fraction of T's that are less than or equal to T_{obs}. The theory of the binomial distribution tells us that the approximate value has about a 95% chance of being within

$$\pm 2\sqrt{p(1-p)/R}$$

of the true p-value. We will denote the permutation test based on a random sample of R permutations by ϕ_R.

Remark. Consider the lag one autocorrelation

$$r_1 = \frac{\sum_{i=1}^{n-1}\left(X_i - \bar{X}\right)\left(X_{i+1} - \bar{X}\right)}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}.$$

Using T is "almost" equivalent to using r_1 to perform our test. We leave it to the reader to verify this.

3. Consistency of ϕ_R

Consistency of hypothesis tests is a desirable property. In this section, we will show that the permutation test based on random sampling R permutations is consistent, that is, the probability of correctly rejecting the null hypothesis H_0 tends to 1 as the sample size goes to infinity when $0 < a < 1$. In other words,

Theorem 3.1 *Suppose H_a is an arbitrary simple hypothesis that the autoregressive parameter a is between* 0 *and* 1, *that is $H_a \in H_A$. Then*

$$P_{H_a}\left[\text{Reject } H_0\right] \to 1$$

as $n \to \infty$.

Lemma 3.1 *Under H_A, for any integer $m \ge 1$,*

$$E\left(X_1 X_{1+m}\right) = \frac{\sigma^2}{1-a^2}\left(2a^m - a^{m-1} - a^{m+1}\right).$$

Proof. We can write Y_t as

$$Y_t = \sum_{u=0}^{\infty} a^u e_{t-u}. \tag{1}$$

Using (1), it is easily seen that under H_A, the AR(1) process Y_t has mean zero with

$$\operatorname{Var}\left(Y_t\right) = \frac{\sigma^2}{1-a^2},$$

and

$$E\left(Y_t Y_{t+m}\right) = \frac{\sigma^2}{1-a^2} a^m.$$

Clearly,

$$E\left(X_1 X_{1+m}\right) = E\left(Y_2 Y_{2+m} - Y_2 Y_{1+m} - Y_1 Y_{2+m} + Y_1 Y_{1+m}\right). \quad \square$$

In particular,

$$E\left(X_1 X_2\right) = -\frac{\sigma^2\left(1-a\right)}{\left(1+a\right)},$$

which is negative under H_A. Note

$$E\left(Y_1^2Y_2^2\right)$$
$$= E\left(\sum_{u=0}^{\infty} a^u e_{1-u} \sum_{v=0}^{\infty} a^v e_{1-v} \sum_{p=0}^{\infty} a^p e_{2-p} \sum_{q=0}^{\infty} a^q e_{2-q}\right)$$
$$= \sum_{u=0}^{\infty} a^{2+4u} E\left(e_{1-u}^4\right) + 2\sum_{u=0}^{\infty}\sum_{v\neq u} a^{1+2u} a^{1+2v}\sigma^4 + \sum_{u=0}^{\infty}\sum_{v\neq 1+u} a^{2u} a^{2v}\sigma^4$$
$$= \frac{a^2}{1-a^4} E\left(e_t^4\right) + 2\left[\frac{a^2}{\left(1-a^2\right)^2} - \frac{a^2}{1-a^4}\right]\sigma^4$$
$$+ \left[\frac{1}{\left(1-a^2\right)^2} - \frac{a^2}{1-a^4}\right]\sigma^4. \tag{2}$$

Lemma 3.2 *Under* H_A, T_n/n *converges in probability to* $E\left(X_1X_2\right) = -\frac{\sigma^2\left(1-a\right)}{1+a}$

Proof. Under H_A, time series Y_t is stationary and so is time series X_tX_{t+1}. Thus

$$E\left(\frac{T_n}{n-1}\right) = E\left(X_1X_2\right),$$

it suffices to show that

$$\operatorname{Var}\left(\frac{T_n}{n-1}\right) \to 0.$$

$$\operatorname{Var}\left(T_n\right)$$
$$= \left(n-1\right)\operatorname{Var}\left(X_1X_2\right) + 2\sum_{1\le i<j\le n-1}\operatorname{Cov}\left(X_iX_{i+1}, X_jX_{j+1}\right)$$
$$= \left(n-1\right)\operatorname{Var}\left(X_1X_2\right) + 2\sum_{j=2}^{n-1}\left(n-j\right)\operatorname{Cov}\left(X_1X_2, X_jX_{j+1}\right)$$
$$= \left(n-1\right)\operatorname{Var}\left(X_1X_2\right) + 2\left(n-2\right)\operatorname{Cov}\left(X_1X_2, X_2X_3\right)$$
$$+ 2\left(n-3\right)\operatorname{Cov}\left(X_1X_2, X_3X_4\right)$$
$$+ 2\sum_{j=4}^{n-1}\left(n-j\right)\left[E\left(X_1X_2X_jX_{j+1}\right) - \left(E\left(X_1X_2\right)\right)^2\right]. \tag{3}$$

By (3), it is sufficient to show that

$$\frac{\sum_{j=4}^{n-1}\left(n-j\right)\left[E\left(X_1X_2X_jX_{j+1}\right) - \left(E\left(X_1X_2\right)\right)^2\right]}{\left(n-1\right)^2} \to 0$$

as $n\to\infty$.

For $j\ge 4$, consider $E\left(X_1X_2X_jX_{j+1}\right)$. Following the proof of (2), we have

$$E\left(X_1X_2X_jX_{j+1}\right)$$
$$= \frac{-a^{2j-3} + 3a^{2j-2} - 3a^{2j-1} + 2a^{2j}}{1-a^4} E\left(e_t^4\right)$$
$$+ \frac{3a^{2j+2} - 3a^{2j+1} - a^{2j+3}}{1-a^4} E\left(e_t^4\right)$$
$$+ \frac{1-4a+6a^2-4a^3+a^4}{\left(1-a^2\right)^2}\sigma^4$$
$$+ \frac{3a^{2j-3} - 9a^{2j-2} + 9a^{2j-1} - 6a^{2j}}{1-a^4}\sigma^4$$
$$+ \frac{9a^{2j+1} - 9a^{2j+2} + 3a^{2j+3}}{1-a^4}\sigma^4$$
$$+ \frac{2a^{2j-4} - 8a^{2j-3} + 12a^{2j-2} - 8a^{2j-1} + 2a^{2j}}{\left(1-a^2\right)^2}\sigma^4.$$

It is not hard to see that

$$\frac{1}{\left(n-1\right)^2}\sum_{j=4}^{n-1}\left(n-j\right)\left(-a^{2j-3} + 3a^{2j-2} - 3a^{2j-1} + 2a^{2j}\right)$$
$$+ \frac{1}{\left(n-1\right)^2}\sum_{j=4}^{n-1}\left(n-j\right)\left(-3a^{2j+1} + 3a^{2j+2} - a^{2j+3}\right) \to 0$$

and

$$\sum_{j=4}^{n-1}\left(n-j\right)\frac{1-4a+6a^2-4a^3+a^4}{\left(n-1\right)^2\left(1-a^2\right)^2}$$
$$+ \sum_{j=4}^{n-1}\left(n-j\right)\frac{2a^{2j-4} - 8a^{2j-3} + 12a^{2j-2} - 8a^{2j-1} + 2a^{2j}}{\left(n-1\right)^2\left(1-a^2\right)^2}$$
$$- \sum_{j=4}^{n-1}\left(n-j\right)\frac{\left(1-a\right)^2}{\left(n-1\right)^2\left(1+a\right)^2} \to 0$$

as $n\to\infty$. This completes the proof. □

On the left of **Figures 1** and **2**, it shows the long run behavior of T_n/n based on 10,000 simulations from AR(1) model with white noise from normal (0,1) and uniform (−1,1) respectively.

Lemma 3.3 *Under* H_A, $E\left(X_{1\ell}X_{2\ell}\right) \to 0$ *as* $n\to\infty$.

Proof. For a random permutation, $\ell \le R$, of $X_1, X_2, \cdots X_n$.

$$P\left(X_{1\ell}X_{2\ell} = X_iX_j\right) = \frac{1}{n\left(n-1\right)}$$

for any $1\le i\ne j\le n$.

$$E\left(X_{1\ell}X_{2\ell}\right) = E\left(E\left[X_{1\ell}X_{2\ell}\middle| X_1,\cdots,X_n\right]\right)$$
$$= \sum_{1\le i<j\le n}\frac{2}{n\left(n-1\right)}E\left(X_iX_j\right). \tag{4}$$

For $i < j$, by stationarity and Lemma 3.1,

$$E(X_i X_j) = E(X_1 X_{j-i+1}) = \frac{\sigma^2}{1-a^2}\left(2a^{j-i} - a^{j-i-1} - a^{j-i+1}\right). \quad (5)$$

From (4) and (5),

$$E(X_{1\ell} X_{2\ell}) = \frac{2\sigma^2}{1-a^2}\frac{na-(n-1)-a^n}{n(n-1)}. \quad (6)$$

The proof follows from (6). □

Lemma 3.4 *Under* H_A, $E(X_{1\ell}X_{2\ell}X_{3\ell}X_{4\ell}) \to 0$ *as* $n \to \infty$.

Proof.

$$E(X_{1\ell}X_{2\ell}X_{3\ell}X_{4\ell}) = \sum_{1\le i<j<k<l\le n} \frac{4!E(X_iX_jX_kX_l)}{n(n-1)(n-2)(n-3)}. \quad (7)$$

Note

$$\begin{aligned} &E(X_iX_jX_kX_l) \\ &= E(Y_{i+1}Y_{j+1}Y_{k+1}Y_{l+1}) - E(Y_{i+1}Y_{j+1}Y_{k+1}Y_l) \\ &-E(Y_{i+1}Y_{j+1}Y_kY_{l+1}) + E(Y_{i+1}Y_{j+1}Y_kY_l) \\ &-E(Y_{i+1}Y_jY_{k+1}Y_{l+1}) + E(Y_{i+1}Y_jY_{k+1}Y_l) \\ &+E(Y_{i+1}Y_jY_kY_{l+1}) - E(Y_{i+1}Y_jY_kY_l) \\ &-E(Y_iY_{j+1}Y_{k+1}Y_{l+1}) + E(Y_iY_{j+1}Y_{k+1}Y_l) \\ &+E(Y_iY_{j+1}Y_kY_{l+1}) - E(Y_iY_{j+1}Y_kY_l) \\ &+E(Y_iY_jY_{k+1}Y_{l+1}) - E(Y_iY_jY_{k+1}Y_l) \\ &-E(Y_iY_jY_kY_{l+1}) + E(Y_iY_jY_kY_l). \end{aligned}$$

After lengthy calculation of $E(X_iX_jX_kX_l)$, we know that to show (7) converges to zero when n goes to infinity, it is sufficient to show

$$\sum_{1\le i<j<k<l\le n} \frac{2a^{j+k+l} - 3a^{j+l+k-1} + 3a^{j+l+k-2} - a^{j+l+k-3}}{n^4a^{3i}} + \sum_{1\le i<j<k<l\le n} \frac{-a^{j+l+k+3} + 3a^{j+l+k+2} - 3a^{j+l+k+1}}{n^4a^{3i}} \to 0;$$

$$\sum_{1\le i<j<k<l\le n} \frac{6a^{j+l-k-i} - 4a^{j+l-k-i-1} - 4a^{j+l-k-i+1}}{n^4} + \sum_{1\le i<j<k<l\le n} \frac{a^{j+l-k-i-2} + a^{j+l-k-i+2}}{n^4} \to 0;$$

$$\sum_{1\le i<j<k<l\le n} \frac{12a^{k+l-j-i} - 8a^{k+l-j-i-1} - 8a^{k+l-j-i+1}}{n^4} + \sum_{1\le i<j<k<l\le n} \frac{2a^{k+l-j-i-2} + 2a^{k+l-j-i+2}}{n^4} \to 0,$$

which are easy to see. □

Lemma 3.5 *Under* H_A, $T_{n\ell}/n$ *converges to* 0 *in probability for all* $1 \le \ell \le R$.

The proof follows from (3), Lemma 3.3 and 3.4. □

On the right of **Figures 1** and **2**, it shows the long run behavior of $T_{n\ell}/n$ based on 10,000 simulations from AR(1) model with white noise from normal (0,1) and uniform (−1,1) respectively.

Proof of Theorem 3.1. Clearly, under H_A,

$$P\left[T_{n\ell} \ge T_n, \text{ for all } 1 \le \ell \le R\right] = P\left[T_{n\ell}/n \ge T_n/n, \text{ for all } 1 \le \ell \le R\right],$$

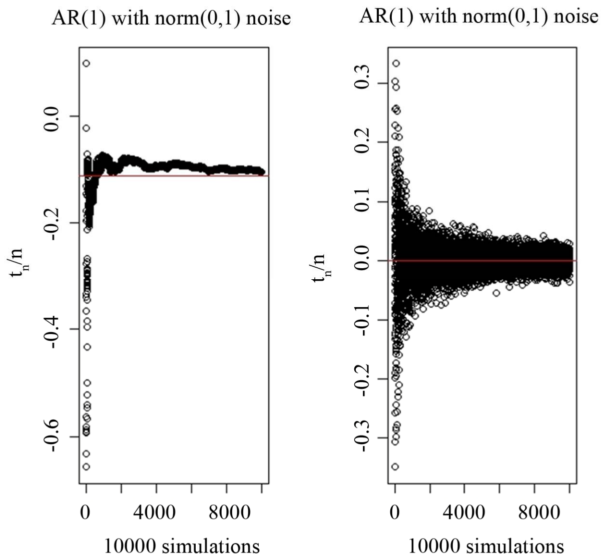

Figure 1. Illustrations of Lemma 3.2 and Lemma 3.5 with normal (0,1) noise and $a = 0.8$.

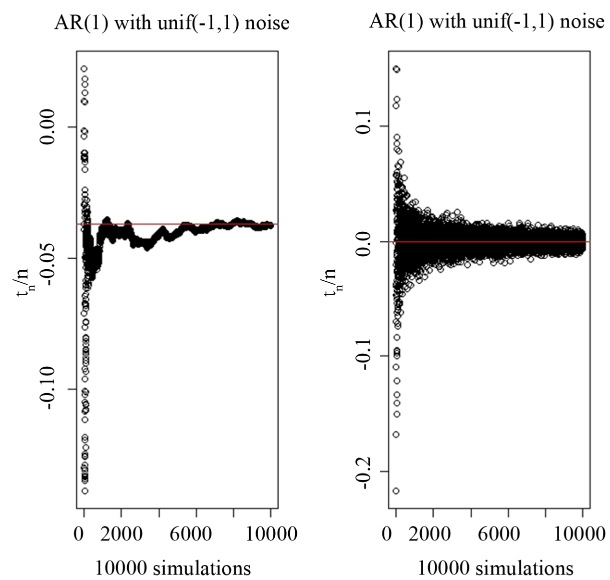

Figure 2. Illustrations of Lemma 3.2 and Lemma 3.5 with uniform (−1,1) noise and $a = 0.8$.

which tends to 1 by Lemma 3.2 and Lemma 3.5. In particular, the probability of rejecting H_0 under H_A tends to 1. □

4. Simulation Study

Consider the model $Y_t = aY_{t-1} + e_t$, $t = 2, \cdots, n+1$, $Y_1 = 0$ where the e_t has contaminated normal distribution. Note the contaminated normal observations were generated in the following way: 70% of the time an observation is generated from a standard normal distribution while 30% of the time it is generated from a normal distribution with mean 0 and standard deviation 25. One thousand samples of size $n = 5$, 25, 50, 100, 250, 500 were generated for $a = 0.2$, 0.4, 0.6, 0.8, 1. Permutation tests based on all permutations when samples are small or randomly selected 1000 permutations for unit root were applied to each sample at the significance level 0.05. The power of the test is tabulated in **Table 1** based on 1000 simulated tests for $n = 5$, 25, 50, 100, 250, 500 and $a = 0.2$, 0.4, 0.6, 0.8. From the table, it is easy to see that the power gets closer to 1 when the sample size increases and this demonstrates the consistency of the proposed permutation test.

5. An Example

Reference [5] studied the stochastic structure of velocity in order to determine whether there is a statistical basis for extrapolative prediction. Noting that the velocity of money is defined as the ratio of national income to the stock of money. In the paper they conclude that the logs of the velocity series constructed in [6] are well characterized as a simple random walk. As preliminary analysis, we look at the time series plot of Y_t, the centered logs of velocity. The pattern in the time series plot is typical of a nonstationary series of the sort which displays no affinity for a mean value. We also note that the autocorrelations of centered logs of velocity series are very large and decline slowly with increasing lag (**Figure 3**). Now let us look at the time series plot of X_t, the first differences of centered logs of velocity, and autocorrelations of X_t (**Figure 4**). Judging from the time series plot and autocorrelations of time series X_t, there seems no significant dependence in the time series X_t. Moving on to the formal analysis, we can see that it is reasonable to fit model $Y_t = aY_{t-1} + e_t$ to the centered logs. With an application of permutation test on the centered logs, we obtain the test statistic $T_{obs} = 0.04$, and the *p*-value = 0.7674. The null hypothesis is not rejected at any reasonable level.

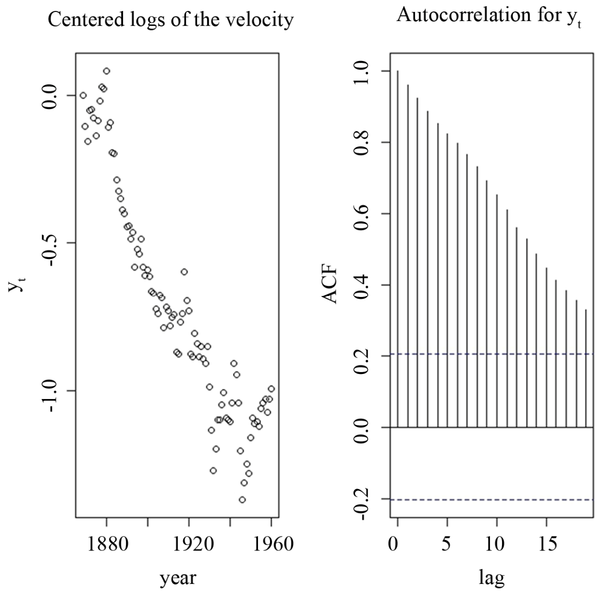

Figure 3. Time series Y_t and its autocorrelations.

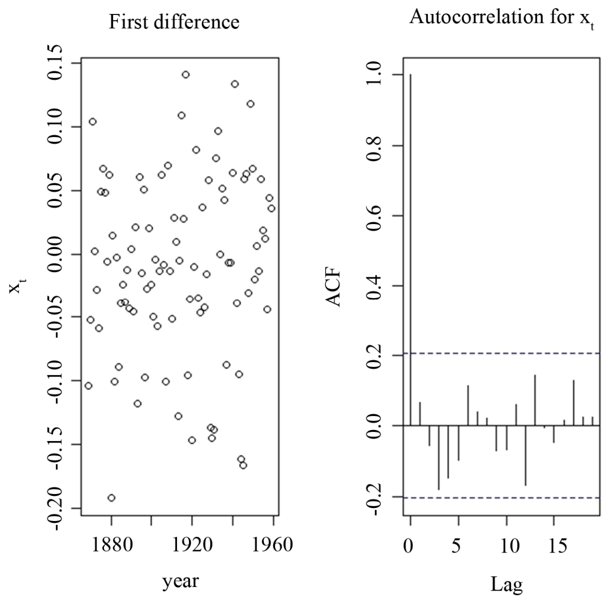

Figure 4. Time series X_t and its autocorrelations.

Table 1. Power of our permutation test.

n	5	25	50	100	250	500
a = 0.2	0.081	0.639	0.905	0.997	1	1
a = 0.4	0.051	0.398	0.71	00.933	1	1
a = 0.6	0. 047	0.19	0.342	0.652	0.954	0.998
a = 0.8	0.045	0.09	0.147	0.196	0.449	0.706
a = 1	0.054	0.046	0.047	0.044	0.048	0.055

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