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Point Correspondences between *N* + 1 Hypersurfaces of Projective Spaces and (*N* + 1)-Webs

V. S. Bolodurin
Algebra and Geometry Department, Pedagogical University, Orenburg, Russia
Email: Bolvikser@mail.ru

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ABSTRACT

For a correspondence in question we establish a sequence of fundamental geometrical objects of the correspondence and find invariant normalizations of the first and second orders of all hupersurfaces under the correspondence. We single out main tensors of the correspondence and establish a connection between the geometry of point correspondences between $n+1$ hypersurfaces of projective spaces and the theory of multidimensional $(n+1)$-webs.

Keywords: Hypersurface; Point Correspondence; Invariant Normalization; Multidimensional $(n+1)$-Webs

1. Introduction

Differentional geometry of point correspondences between projective, affine and euclid spaces of equal dimensions were studied and were studing by scientists till 1920. One can finds the analysis of obtained results to 1964 in the paper [1] by Ryzhkov.

Among all papers devoted to the theory of point correspondences between two three-dimensional spaces we must note papers [2] written by Svec, [3] written by Murracchini, [4] written by Mihailescu and [5] by Vranceanu. They introduce characteristic directions of point correspondences, consider some special classes of correspondences, show connections of point correspondences between spaces with different parts of differentional geometry.

Properties of point correspondences between n-dimensional projective, affine and euclid spaces are studied by Ryzhkov [6], Sokolova [7] and Pavljuchenko [8].

A straight line $\left[\underset{\xi}{M}_0 \underset{\xi}{M}_n\right]$, passing through the point $\underset{\xi}{M}_0$, is called a first order normal of a hypersurface of n-dimensional projective space in the point $\underset{\xi}{M}_0$, if the straight line has no other points with the tangent hyperplane of the hupersurface [9]. We call a $(n-1)$-dimensional plane as the second order normal of the hypersurface in the point $\underset{\xi}{M}_0$, if the tangent hyperplane of the hypersurface in the point $\underset{\xi}{M}_0$ includes this $(n-1)$-dimensional plane and this $(n-1)$-dimensional plane does not pass through the point $\underset{\xi}{M}_0$.

It is known that the main problem of nonmetric differentional geometry of a surface is a construction of invariant normalization of this surface. To construct an invariant first normal in a point of a surface it is necessary to use third-order differential neighbourhood of the point [10]. In our previous papers we showed that to construct an invariant first normal in points of two surfaces under point correspondences it is sufficient to use a second-order differential neighbourhood of corresponding points, but to construct an invariant second normal in points of two surfaces under point correspondences it is necessary to use third-order differential neighbourhood of the point.

In the current paper we will find invariant normalizations of the first and second orders of all hupersurfaces under the correspondence.

There exists a connection between the geometry of point correspondences between three spaces or surfaces and the theory of multidimensional 3-webs (Akivis [11]). We showed it in papers [12,13], devoted point correspondences between three projective spaces and between three hupersurfaces of projective spaces.

The theory of of multidimensional $(n+1)$-webs is constructed in the paper [14] by Goldberg. In the current paper we will consider a connection between the geometry of point correspondences between $n+1$ hypersurfaces of projective spaces and the theory of multidimensional $(n+1)$-webs.

In the way of the investigation we use the exterior differentiation, tensor analysis and G.F.Laptev invariant methods [15].

2. Main Equations of Correspondence, the Sequence of Main Geometrical Objects

Let us consider $n + 1$ smooth hypersurfaces $\underset{\xi}{V}_n \subset \underset{\xi}{P}_{n+1}$ $(\xi,\eta,\theta = 0,1,2,3,\cdots,n)$ of projective spaces and a point correspondence $C:\underset{1}{V}_n \times\cdots\times \underset{n}{V}_n \to \underset{0}{V}_n$ between these hypersurfaces.

Let $\underset{\xi}{M}_0$ be corresponding points of hypersurfaces $\underset{\xi}{V}_n$. A correspondence $C:\underset{1}{V}_n \times\cdots\times \underset{n}{V}_n \to \underset{0}{V}_n$ generates C^3_{n+1} families point subcorrespondences $C:\underset{\xi}{V}_n \times \underset{\eta}{V}_n \to \underset{\theta}{V}_n$ obtained by fixation of $n - 2$ corresponding points and generates C^2_{n+1} point mappings $\underset{\xi\eta}{T}:\underset{\xi}{V}_n \to \underset{\eta}{V}_n$ by fixation of n–1 corresponding points.

Mappings $\underset{\xi\eta}{T}:\underset{\xi}{V}_n \to \underset{\eta}{V}_n$ must be regular in neighbourhoods of points under correspondences of surfaces $\underset{\xi}{V}_n$, $\underset{\eta}{V}_n$ and have the inverse mappings.

We will assume, that surfaces $\underset{\xi}{V}_n$ belong to different projective spaces $\underset{\xi}{P}_n$. The geometry of correspondences under consideration will be studied according to the transformation group, which is a direct product of projective transformation groups of spaces $\underset{\xi}{P}_n$.

With any point $\underset{\xi}{M}_0 \in \underset{\xi}{V}_n$ we associate a projective moving frame consisting of the point $\underset{\xi}{M}_0$ points $\underset{\xi}{M}_i$ $(i,j,k,\cdots = 1,2,\cdots,n)$ of the tangent hyperplane of the hypersurface $\underset{\xi}{V}_n$ in the point $\underset{\xi}{M}_0$ and a point $\underset{\xi}{M}_{n+1}$ outside the tangent hyperplane.

The equations of infinitesimal displacement of our projective frames $\left\{\underset{\xi}{M}_0, \underset{\xi}{M}_i, \underset{\xi}{M}_{n+1}\right\}$ have the form:

$$d\underset{\xi}{M}_u = \underset{\xi}{\omega}{}^v_u \underset{\xi}{M}_v, \qquad (1)$$

where $(u,v,w = 0,1,\cdots,n+1)$ $\underset{\xi}{\omega}{}^v_u$ are 1-forms containing parameters, on which the family of frames in question depends, and their differentials. The forms $\underset{\xi}{\omega}{}^v_u$ satisfy the structural equations of projective space:

$$d\underset{\xi}{\omega}{}^v_u = \underset{\xi}{\omega}{}^w_u \wedge \underset{\xi}{\omega}{}^v_w.$$

We can write equations of hypersurfaces $\underset{\xi}{V}_n$ as follows:

$$\underset{\xi}{\omega}{}^{n+1}_0 = 0. \qquad (2)$$

The Pfaffian forms $\underset{\xi}{\omega}{}^i_0$ define displacements of corresponding points $\underset{\xi}{M}_0$ of hypersurfaces $\underset{\xi}{V}_n$. It follows that the forms $\underset{\xi}{\omega}{}^i_0$ satisfy the following linear relations:

$$\underset{0}{t}{}^i_j \underset{0}{\omega}{}^j_0 + \underset{1}{t}{}^i_j \underset{1}{\omega}{}^j_0 + \cdots + \underset{n}{t}{}^i_j \underset{n}{\omega}{}^j_0 = 0. \qquad (3)$$

Since for $\xi = \text{const}$ forms $\underset{\xi}{\omega}{}^i_0$ are linearly independent, therefore the following conditions are true:

$$\det\left|\underset{\xi}{t}{}^i_j\right| \neq 0.$$

We can transform all frames of projective spaces in points $\underset{\xi}{M}_0$ by setting $\underset{\xi}{M}_i = \underset{\xi}{t}{}^j_i \underset{\xi}{M}'_j$. For new frames $\left\{\underset{\xi}{M}_0, \underset{\xi}{M}'_i, \underset{\xi}{M}_{n+1}\right\}$ we will have $\underset{\xi}{\omega}'^i_0 = \underset{\xi}{t}{}^i_j \underset{\xi}{\omega}{}^j_0$. By Equations (3) relations between forms $\underset{\xi}{\omega}'^i_0$ take the simplest case. Let us suppose that necessary transformations of frames are done and we can write relations between forms $\underset{\xi}{\omega}{}^i_0$ of frames $\left\{\underset{\xi}{M}_0, \underset{\xi}{M}_i, \underset{\xi}{M}_{n+1}\right\}$ as follows

$$\underset{0}{\omega}{}^i_0 + \underset{1}{\omega}{}^i_0 + \cdots + \underset{n}{\omega}{}^i_0 = 0. \qquad (4)$$

Geometrically Equations (4) mean that frames in points $\underset{\xi}{M}_0$ of spaces $\underset{\xi}{P}_{n+1}$ are chosen so that directions in points $\underset{\xi}{M}_0$, $\underset{\eta}{M}_0$ are corresponding by mappings $\underset{\xi\eta}{T}$.

To find equations of a mapping $\underset{\xi\eta}{T}:\underset{\xi}{V}_n \to \underset{\eta}{V}_n$ we fix points $\underset{\theta}{M}_0$, where $\theta \neq \xi,\eta$. Using Equations (2), (4), we have

$$\underset{\xi}{\omega}{}^{n+1}_0 = 0,\ \underset{\eta}{\omega}{}^{n+1}_0 = 0,\ \underset{\xi}{\omega}{}^i_0 + \underset{\eta}{\omega}{}^i_0 = 0. \qquad (5)$$

Consider projective mappings $\underset{\xi\eta}{K}$, where

$$\underset{\xi\eta}{K}\underset{\xi}{M}_0 = \underset{\eta}{M}_0,\ \underset{\xi\eta}{K}\underset{\xi}{M}_i = -\underset{\eta}{M}_i,\ \underset{\xi\eta}{K}\underset{\xi}{M}_{n+1} = \underset{\eta}{M}_{n+1}.$$

By Equations (1), (5) the following relations satisfy projective mappings:

$$\underset{\xi\eta}{K}\, d\underset{\xi}{M}_0 = d\underset{\eta}{M}_0 + \underset{\xi}{\theta}_1 \underset{\eta}{M}_0,$$

where $\underset{\xi}{\theta}_1$—a quantity of the first order according to $\underset{\xi}{\omega}{}^v_u$. The projective mapping $\underset{\xi\eta}{K}$ has a first order tangency with the mapping $\underset{\xi\eta}{T}_0:\underset{\xi}{V}_n \to \underset{\eta}{V}_n$ in corresponding points $\underset{\xi}{M}_0$, $\underset{\eta}{M}_0$.

Equations (2), (4) are main equations of our problem. With the help of exterior differentiation of these equations and applying Cartan's lemma we obtain

$$\underset{\xi}{\omega}{}_i^{n+1} = \underset{\xi}{\lambda}{}_{ij}\underset{\xi}{\omega}{}_0^j,\ \underset{\alpha}{\Omega}{}_j^i - \underset{0}{\Omega}{}_j^i = \sum_{\beta} \underset{\alpha\beta}{\lambda}{}_{jk}^i \underset{\beta}{\omega}{}_0^k, \tag{6}$$

where $\underset{\xi}{\lambda}{}_{ij} = \underset{\xi}{\lambda}{}_{ji}$, $\underset{\alpha\beta}{\lambda}{}_{jk}^i = \underset{\beta\alpha}{\lambda}{}_{kj}^i$, $\underset{\xi}{\Omega}{}_j^i = \underset{\xi}{\omega}{}_j^i - \delta_j^i \underset{\xi}{\omega}{}_0^0$, $(\alpha,\beta,\gamma = 1,2,3,\cdots,n)$.

Note that quadratic forms $\varphi_{\xi} = \underset{\xi}{\lambda}{}_{ij}\underset{\xi}{\omega}{}_0^i\underset{\xi}{\omega}{}_0^j$ are asymptotic quadratic forms of hypersurfaces $\underset{\xi}{V}{}_n$.

Now in the family of frames we have equations of mapping $\underset{\alpha 0}{T}$ in the way

$$\begin{aligned} &\underset{\alpha}{\omega}{}_i^{n+1} = \underset{\alpha}{\lambda}{}_{ij}\underset{\alpha}{\omega}{}_0^j,\ \underset{0}{\omega}{}_i^{n+1} = \underset{0}{\lambda}{}_{ij}\underset{\alpha}{\omega}{}_0^j,\ \underset{\alpha}{\omega}{}_0^{n+1} = 0,\\ &\underset{0}{\omega}{}_0^{n+1} = 0,\ \underset{\alpha}{\omega}{}_0^i + \underset{0}{\omega}{}_0^i = 0,\ \underset{\alpha}{\Omega}{}_j^i - \underset{0}{\Omega}{}_j^i = \underset{\alpha\alpha}{\lambda}{}_{jk}^i\underset{\alpha}{\omega}{}_0^k, \end{aligned} \tag{7}$$

and similar for $\underset{\alpha\beta}{T}$

$$\begin{aligned} &\underset{\alpha}{\omega}{}_i^{n+1} = \underset{\alpha}{\lambda}{}_{ij}\underset{\alpha}{\omega}{}_0^j,\ \underset{\beta}{\omega}{}_i^{n+1} = \underset{\beta}{\lambda}{}_{ij}\underset{\alpha}{\omega}{}_0^j,\ \underset{\alpha}{\omega}{}_0^{n+1} = 0,\\ &\underset{\beta}{\omega}{}_0^{n+1} = 0,\ \underset{\alpha}{\omega}{}_0^i + \underset{\beta}{\omega}{}_0^i = 0,\ \underset{\alpha}{\Omega}{}_j^i - \underset{\beta}{\Omega}{}_j^i = \underset{\alpha\beta}{\lambda}{}'^i_{jk}\underset{\alpha}{\omega}{}_0^k. \end{aligned} \tag{7'}$$

where $\underset{\alpha\beta}{\lambda}{}'^i_{jk} = \left(\underset{\alpha\alpha}{\lambda}{}_{jk}^i + \underset{\beta\beta}{\lambda}{}_{jk}^i - 2\underset{\alpha\beta}{\lambda}{}_{(jk)}^i\right)$ and $\underset{\alpha\beta}{\lambda}{}'^i_{jk} = \underset{\alpha\beta}{\lambda}{}'^i_{kj}$.

To continue the system of Equations (6) we use exterior differentiation of these equations and Cartan's lemma. We obtain new equations:

$$\begin{aligned} &\nabla_{\xi}\underset{\xi}{\lambda}{}_{jk} = \underset{\xi}{\lambda}{}_{jk}\left(\underset{\xi}{\omega}{}_0^0 - \underset{\xi}{\omega}{}_{n+1}^{n+1}\right) + \underset{\xi}{\lambda}{}_{ijk}\underset{\xi}{\omega}{}_0^k,\\ &\nabla\underset{\alpha\beta}{\lambda}{}_{jk}^i = \underset{\alpha\beta}{\delta}\left(\delta_{(k}^i\underset{\alpha}{\omega}{}_{j)}^0 + \delta_{(k}^i\underset{\beta}{\omega}{}_{j)}^0\right) + 2\delta_{(k}^i\underset{0}{\omega}{}_{j)}^0\\ &\qquad - \underset{\alpha\beta}{\delta}\underset{\alpha}{\lambda}{}_{jk}\underset{\alpha}{\omega}{}_{n+1}^i - \underset{0}{\lambda}{}_{jk}\underset{0}{\omega}{}_{n+1}^i + \sum_{\gamma}\underset{\alpha\beta\gamma}{\lambda}{}_{jkl}^i\underset{\gamma}{\omega}{}_0^l. \end{aligned} \tag{8}$$

To write these equations we used operators ∇ and ∇_{ξ}. Operator ∇ is defined by forms $\underset{0}{\Omega}{}_j^i$ and we have

$$\nabla\lambda_{jk}^i = d\lambda_{jk}^i - \lambda_{lk}^i\underset{0}{\Omega}{}_j^l - \lambda_{jl}^i\underset{0}{\Omega}{}_k^l + \lambda_{jk}^l\underset{0}{\Omega}{}_l^i,$$

and similarly operators ∇_{ξ} are defined by forms $\underset{\xi}{\Omega}{}_j^i$.

Quantities $\underset{\xi}{\lambda}{}_{ijk}$ are symmetric with respect to the indices *i*, *j* and *k*, for quantities $\underset{\alpha\beta\gamma}{\lambda}{}_{jkl}^i$ some additional finite conditions are true.

The system of quantities $\underset{\xi}{\lambda}{}_{jk}, \underset{\alpha\beta}{\lambda}{}_{jk}^i$ define the geometrical object according to G.F.Laptev invariant methods [15]. This object is the fundamental geometrical object of second order of point correspondence $C : \underset{1}{V}{}_n \times\cdots\times \underset{n}{V}{}_n \to \underset{0}{V}{}_n$.

If we continue Equations (8), we obtain the system of differentional equations of a sequence of fundamental geometrical objects of point correspondence under consideration

$$\underset{\xi}{\lambda}{}_{jk}, \underset{\alpha\beta}{\lambda}{}_{jk}^i, \underset{\alpha\beta\gamma}{\lambda}{}_{jkl}^i, \cdots$$

3. Characteristic Directions of Point Correspondences

Let us consider a mapping $\underset{\xi\eta}{T} : \underset{\xi}{V}{}_n \to \underset{\eta}{V}{}_n$. If frames are fixed in corresponding points of hypersurfaces $\underset{\xi}{V}{}_n$, $\underset{\eta}{V}{}_n$, then the object $\underset{\theta}{\lambda}{}_{jk}^i$ define the quadratic transformation of tangent directions of hypersurfaces

$$\underset{\xi}{\omega}{}_0^i \to \Omega_{\xi}^i = \underset{\theta}{\lambda}{}_{jk}^i\underset{\xi}{\omega}{}_0^j\underset{\xi}{\omega}{}_0^k.$$

In geometry of point correspondences [1] directions are said to be characteristic if they are invariant according to these quadratic transformations. They must satisfy a system of equations

$$\underset{\theta}{\lambda}{}_{jk}^i\underset{\xi}{\omega}{}_0^j\underset{\xi}{\omega}{}_0^k = \theta\,\underset{\xi}{\omega}{}_0^i. \tag{9}$$

A geodesic curve of hypersurface $\underset{\xi}{V}{}_n$, connected with the family of first order normals, is called a curve, whose 2-dimensional osculant plane passes through corresponding first order normals of hypersurface in every point (see for exsample [9]). If Pfaffian forms $\underset{\xi}{\omega}{}^i$ define a tangent direction to a curve ℓ in a point $\underset{\xi}{M}{}_0$, then relations

$$\nabla_{\xi}\underset{\xi}{\omega}{}^i = \theta\underset{\xi}{\omega}{}^i$$

are the condition of the geometrical second order tangency of the curve ℓ and a geodesic curve having the same tangent direction in this point $\underset{\xi}{M}{}_0$.

Characteristic directions have the following property. If a curve $\ell \in \underset{\xi}{V}{}_n$ and a geodesic curve have second order tangency along a characteristic direction in the point $\underset{\xi}{M}{}_o \in \underset{\xi}{V}{}_n$, then the image $\underset{\xi\eta}{T}(\ell) \in \underset{\eta}{V}{}_n$ of the curve under $\underset{\xi\eta}{T} : \underset{\xi}{V}{}_n \to \underset{\eta}{V}{}_n$ has the similar property in the point $\underset{\eta}{M}{}_o \in \underset{\eta}{V}{}_n$ by the corresponding characteristic direction. It follows from Equations (7,) (7'), (9) and relations

$$\nabla_{\xi}\underset{\xi}{\omega}{}^i + \nabla_{\eta}\underset{\eta}{\omega}{}^i = \underset{\theta}{\lambda}{}_{jk}^i\underset{\xi}{\omega}{}_0^j\underset{\xi}{\omega}{}_0^k.$$

From geometric meaning of characteristic directions it is clear, that they depend on the choice of first order normals of a hypersurface and do not depend on the choice of second order normals.

We can rewrite Equations (9) in this way

$$\underset{\xi}{\omega}{}_0^{[l}\underset{\eta}{\lambda}{}^{i]}_{jk}\underset{\xi}{\omega}{}_0^{j}\underset{\xi}{\omega}{}_0^{k}=0.$$

We obtained equations of cubic cones. Characteristic directions are common generatrices of these cones.

Let us assume, that any direction $\underset{\xi}{\omega}{}_0^i$ in a point $\underset{\xi}{M}{}_0$ by some choice of a first order normal on hypersurfaces $\underset{\xi}{V}{}_n$ is characteristic for a mapping $\underset{\xi\eta}{T}:\underset{\xi}{V}{}_n\to\underset{\eta}{V}{}_n$. Then the last equations must be sutisfied for any magnitudes $\underset{\xi}{\omega}{}_0^i$. Therefore, the following conditions are true for simillar correspondences

$$\underset{\xi}{\delta}{}^{[l}_{(m}\underset{\eta}{\lambda}{}^{i]}_{jk)}=0.$$

After calculations we get the relations:

$$\underset{\alpha\alpha}{\lambda}{}^i_{jk}=\frac{1}{n+1}\left(\delta^i_j\underset{\alpha\alpha}{\lambda}{}^l_{lk}+\delta^i_k\underset{\alpha\alpha}{\lambda}{}^l_{lj}\right),$$
$$\underset{\alpha\beta}{\lambda}{}^i_{(jk)}=\frac{1}{n+1}\left(\delta^i_j\underset{\alpha\beta}{\lambda}{}^l_{(lk)}+\delta^i_k\underset{\alpha\beta}{\lambda}{}^l_{(lj)}\right),\qquad(10)$$

where $\alpha\neq\beta$.

Theorem 1. *If any direction* $\underset{\xi}{\omega}{}_0^i$ *in a point* $\underset{\xi}{M}{}_0$ *by any choice of first order normals on hypersurfaces* $\underset{\xi}{V}{}_n$ *is characteristic for a mapping* $\underset{\xi\eta}{T}:\underset{\xi}{V}{}_n\to\underset{\eta}{V}{}_n$, *then for* $n>1$ *hypersurfaces* $\underset{\xi}{V}{}_n$ *degenerate into hyperplanes and the correspondence becomes Godeux*'*s homography.*

Really, let conditions of the theorem be true in corresponding points $\underset{\xi}{M}{}_0$ of all hypersurfaces $\underset{\xi}{V}{}_n$ according to some first order normals $\left[\underset{\xi}{M}{}_0\underset{\xi}{M}{}_{n+1}\right]$, then relations (10) are satisfied. We transform first order normals on hypersurfaces $\underset{\xi}{V}{}_n$ as follows

$\underset{\xi}{M}{}'_{n+1}=\underset{\xi}{t}{}^i\underset{\xi}{M}{}_i+\underset{\xi}{M}{}_{n+1}$, where $\underset{\xi}{t}{}^i$ are arbitrary quantities.

We denote the values quantities $\underset{\alpha\alpha}{\lambda}{}^i_{jk},\underset{\alpha\beta}{\lambda}{}'^i_{(jk)}$ for new frames $\left\{\underset{\xi}{M}{}_0,\underset{\xi}{M}{}_i,\underset{\xi}{M}{}'_{n+1}\right\}$ of hypersurfaces of the correspondence as $\underset{\alpha\alpha}{\lambda}{}'^i_{jk},\underset{\alpha\beta}{\lambda}{}''^i_{(jk)}$.

Calculations show that

$$\underset{\alpha\alpha}{\lambda}{}'^i_{jk}=\underset{\alpha\alpha}{\lambda}{}^i_{jk}-\underset{\alpha}{\lambda}{}_{jk}\underset{\alpha}{t}{}^i-\underset{0}{\lambda}{}_{jk}\underset{0}{t}{}^i,\quad\underset{\alpha\beta}{\lambda}{}''^i_{(jk)}=\underset{\alpha\beta}{\lambda}{}'^i_{(jk)}-\underset{0}{\lambda}{}_{jk}\underset{0}{t}{}^i.$$

Since any direction $\underset{\xi}{\omega}{}_0^i$ is characteristic according to first order normals on hypersurfaces $\underset{\xi}{V}{}_n$, then quantities $\underset{\alpha\alpha}{\lambda}{}'^i_{jk},\underset{\alpha\beta}{\lambda}{}''^i_{(jk)}$ must also satisfy relations (10).

Let us consider the object $\underset{\alpha\beta}{\lambda}{}''^i_{(jk)}$. We have

$$\frac{1}{n+1}\left(\delta^i_j\underset{\alpha\beta}{\lambda}{}''^l_{(lk)}+\delta^i_k\underset{\alpha\beta}{\lambda}{}''^l_{(lj)}\right)$$
$$=\frac{1}{n+1}\left(\delta^i_j\underset{\alpha\beta}{\lambda}{}'^l_{(lk)}+\delta^i_k\underset{\alpha\beta}{\lambda}{}'^l_{(lj)}\right)-\underset{0}{\lambda}{}_{jk}\underset{0}{t}{}^i.$$

After substituting the values $\underset{\alpha\beta}{\lambda}{}''^l_{(lk)}=\underset{\alpha\beta}{\lambda}{}'^l_{(lk)}-\underset{0}{\lambda}{}_{lk}\underset{0}{t}{}^l$, and considering similar terms we obtain

$$\left(\frac{1}{n+1}\left(\delta^i_j\underset{0}{\lambda}{}_{lk}+\delta^i_k\underset{0}{\lambda}{}_{lj}\right)-\delta^i_l\underset{0}{\lambda}{}_{jk}\right)\underset{0}{t}{}^l=0.$$

These relations must be true for any values $\underset{0}{t}{}^l$, then

$$\frac{1}{n+1}\left(\delta^i_j\underset{0}{\lambda}{}_{lk}+\delta^i_k\underset{0}{\lambda}{}_{lj}\right)-\delta^i_l\underset{0}{\lambda}{}_{jk}=0.$$

Contructing these relations with respect to the indices i and l, we arrive at the equation $\underset{0}{\lambda}{}_{jk}=0$ for $n>1$. In a similar way we get $\underset{\alpha}{\lambda}{}_{jk}=0$.

It is known that hypersurfaces degenerate into hyperplanes if the asymptotic tensors $\underset{\xi}{\lambda}{}_{ij}=0$.

In this case a point correspondence $C:\underset{1}{V}{}_n\times\cdots\times\underset{n}{V}{}_n\to\underset{0}{V}{}_n$ between hypersurfaces transforms into a point correspondence $C:\underset{1}{P}{}_n\times\cdots\times\underset{n}{P}{}_n\to\underset{0}{P}{}_n$ between hyperplanes. Since quantities $\underset{\alpha\alpha}{\lambda}{}^i_{jk},\underset{\alpha\beta}{\lambda}{}^i_{(jk)}$ satisfy relations (10), then mappings $\underset{\xi\eta}{T}$ degenerate in projective mappings. Correspondences between projective spaces having similar properties are called Godeux's homography.

4. Invariant Normalizations of Hypersurfaces under Point Correspondences

Moving frames of hypersurfaces $\underset{\xi}{V}{}_n$ under the correspondence depend on parameters of two types. There exsist principal parameters determined displacements of corresponding points $\underset{\xi}{M}{}_0$ of hypersurfaces $\underset{\xi}{V}{}_n$. Since points $\underset{\xi}{M}{}_0$ are connected by the correspondence the number of independent principal parameters is equal to n^2. By the Equations (4) 1-forms $\underset{\alpha}{\omega}{}_0^i$ are independent linear combinations of differentials of principal parameters.

The Pfaffian forms $\underset{\alpha}{\omega}{}_u^v$ depend linearly on differentials of principal parameters and differentials of other parameters. The other parameters define trasformations of moving frames for fixing points $\underset{\xi}{M}{}_0$. We denote values of forms $\underset{\xi}{\omega}{}^{v}_{u}$ as $\underset{\xi}{\pi}{}^{v}_{u}=\underset{\xi}{\omega}{}^{v}_{u}(\delta)$ for fixing principal parameters.

We denote as ∇_{δ}, $\nabla_{\xi\delta}$ values of operators ∇, ∇_{ξ} and denote as π^{i}_{j} values of the Pfaffian forms $\underset{0}{\Omega}{}^{i}_{j}$ for fixing principal parameters.

By Equation (6) we have:

$$\underset{0}{\Omega}{}^{i}_{j}(\delta)=\pi^{i}_{j}=\underset{\xi}{\pi}{}^{i}_{j}-\delta^{i}_{j}\underset{\xi}{\pi}{}^{0}_{0},$$

it follows $\nabla_{\xi\delta}=\nabla_{\delta}$.

With the help of the operator ∇_{δ} we can write Equation (8) for the case $\underset{\alpha}{\omega}{}^{i}_{0}=0$, as follows:

$$\begin{aligned}&\nabla_{\delta}\underset{\xi}{\lambda}_{jk}=\underset{\xi}{\lambda}_{jk}\left(\underset{\xi}{\pi}{}^{0}_{0}-\underset{\xi}{\pi}{}^{n+1}_{n+1}\right),\\&\nabla_{\delta}\underset{\alpha\alpha}{\lambda}{}^{i}_{jk}=2\delta^{i}_{(k}\underset{\alpha}{\pi}{}^{0}_{j)}-\underset{\alpha}{\lambda}_{jk}\underset{\alpha}{\pi}{}^{i}_{n+1}+2\delta^{i}_{(k}\underset{0}{\pi}{}^{0}_{j)}-\underset{0}{\lambda}_{jk}\underset{0}{\pi}{}^{i}_{n+1},\\&\nabla_{\delta}\underset{\alpha\beta}{\lambda}{}^{i}_{(jk)}=2\delta^{i}_{(k}\underset{0}{\pi}{}^{0}_{j)}-\underset{0}{\lambda}_{jk}\underset{0}{\pi}{}^{i}_{n+1},\end{aligned}\tag{11}$$

where $\alpha\neq\beta$.

It follows from relations (11) that quantities $\underset{\xi}{\lambda}_{ij}$ are relative tensors.

It is known that the main problem of nonmetric differentional geometry of a surface is a construction of invariant normalization of this surface. According to theory [10] for a hypersurface it is necessary to construct on the basis of the sequence of fundamental geometrical objects of the correspondence under consideration some quantities. These quantities must satisfy the following equations:

For the invariant first order normal (straight line)

$$\nabla_{\delta}\underset{\xi}{x}{}^{i}=-\underset{\xi}{x}{}^{i}\left(\underset{\xi}{\pi}{}^{0}_{0}-\underset{\xi}{\pi}{}^{n+1}_{n+1}\right)-\underset{\xi}{\pi}{}^{i}_{n+1},\tag{12}$$

For the point on the invariant first order normal

$$\delta\underset{\xi}{x}=-\underset{\xi}{x}\left(\underset{\xi}{\pi}{}^{0}_{0}-\underset{\xi}{\pi}{}^{n+1}_{n+1}\right)-\underset{\xi}{\pi}{}^{0}_{n+1},\tag{13}$$

For the second order normal ($(n-1)$-dimensional plane inside the tangent hyperplane)

$$\nabla_{\delta}\underset{\xi}{x}_{i}=-\underset{\xi}{\pi}{}^{0}_{i}.\tag{14}$$

Below we will assume, that asymptotic quadratic forms of hypersurfaces $\underset{\xi}{V}_{n}$ are nondegenerate. By virtue of this, $\det\left|\underset{\xi}{\lambda}_{ij}\right|\neq 0$. It follows there exsist tensors $\underset{\xi}{\lambda}{}^{ij}$, symmetric with respect to the indices i, j. These tensors sutisfy conditions $\underset{\xi}{\lambda}{}^{ik}\underset{\xi}{\lambda}_{kj}=\delta^{i}_{j}$. By Equation (11) we have differential equations:

$$\nabla_{\delta}\underset{\xi}{\lambda}{}^{ij}=-\underset{\xi}{\lambda}{}^{ij}\left(\underset{\xi}{\pi}{}^{0}_{0}-\underset{\xi}{\pi}{}^{n+1}_{n+1}\right).$$

By Equation (11) we obtain:

$$\nabla_{\delta}\left(\frac{2}{C_n^2}\underset{0}{\lambda}{}^{ki}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{l}_{(lk)}\right)=-\frac{2}{C_n^2}\underset{0}{\lambda}{}^{ki}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{l}_{(lk)}\left(\underset{0}{\pi}{}^{0}_{0}-\underset{0}{\pi}{}^{n+1}_{n+1}\right)+2(n+1)\underset{0}{\lambda}{}^{ij}\underset{0}{\pi}{}^{0}_{j}-2\underset{0}{\pi}{}^{i}_{n+1},$$

$$\nabla_{\delta}\left(\frac{n+1}{C_n^2}\underset{0}{\lambda}{}^{jk}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{i}_{(jk)}\right)=-\frac{n+1}{C_n^2}\sum_{\alpha\neq\beta}\underset{0}{\lambda}{}^{jk}\underset{\alpha\beta}{\lambda}{}^{i}_{(jk)}\left(\underset{0}{\pi}{}^{0}_{0}-\underset{0}{\pi}{}^{n+1}_{n+1}\right)+2(n+1)\underset{0}{\lambda}{}^{ij}\underset{0}{\pi}{}^{0}_{j}-\left(n^2+n\right)\underset{0}{\pi}{}^{i}_{n+1},$$

where $\alpha\neq\beta$. Note that for $n>1$ quantities

$$\underset{0}{p}{}^{i}=-\frac{1}{n^2+n-2}\left(\frac{2}{C_n^2}\underset{0}{\lambda}{}^{ki}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{l}_{(lk)}-\frac{n+1}{C_n^2}\underset{0}{\lambda}{}^{jk}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{i}_{(jk)}\right)\tag{15}$$

satisfy equations

$$\nabla_{\delta}\underset{0}{p}{}^{i}=-\underset{0}{p}{}^{i}\left(\underset{0}{\pi}{}^{0}_{0}-\underset{0}{\pi}{}^{n+i}_{n+1}\right)-\underset{0}{\pi}{}^{i}_{n+1}.$$

Therefore, by Equation (12) the quantities $\underset{0}{p}{}^{i}$ define the invariant first order normal geometrical object of the hypersurface $\underset{0}{V}_{n}$. From Equation (11) we have

$$\nabla_{\delta}\left(\underset{\alpha\alpha}{\lambda}{}^{i}_{jk}-\frac{1}{C_n^2}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{i}_{(jk)}\right)=2\delta^{i}_{(k}\underset{\alpha}{\pi}{}^{0}_{j)}-\underset{\alpha}{\lambda}_{jk}\underset{\alpha}{\pi}{}^{i}_{n+1}.$$

It follows that quantities

$$\underset{\alpha}{p}{}^{i}=-\frac{1}{n^2+n-2}\left(2\underset{\alpha}{\lambda}{}^{ki}\left(\underset{\alpha\alpha}{\lambda}{}^{l}_{lk}-\frac{1}{C_n^2}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{l}_{(lk)}\right)-(n+1)\underset{\alpha}{\lambda}{}^{jk}\left(\underset{\alpha\alpha}{\lambda}{}^{i}_{jk}-\frac{1}{C_n^2}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{i}_{(jk)}\right)\right)\tag{15'}$$

satisfy Equation (12) and define the invariant first order normal geometrical objects of the hypersurfaces $\underset{\alpha}{V}_{n}$.

To construct the invariant second order normal geometrical object of the hypersurface $\underset{\xi}{V}_{n}$ we consider quantities

$$\begin{aligned}&\underset{0}{p}_{k}=\frac{1}{n+1}\left(\underset{0}{\lambda}_{lk}\underset{0}{p}{}^{l}-\frac{1}{C_n^2}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{l}_{(lk)}\right),\\&\underset{\alpha}{p}_{k}=\frac{1}{n+1}\left(\underset{\alpha}{\lambda}_{lk}\underset{\alpha}{p}{}^{l}-\underset{\alpha\alpha}{\lambda}{}^{l}_{lk}+\frac{1}{C_n^2}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{l}_{(lk)}\right),\end{aligned}\tag{16}$$

Calculations show that quantities $\underset{\xi}{p}_{k}$ satisfy Equation (14).

Thus, it is proved.

Theorem 2. *If asymptotic quadratic forms of* $n+1$ *hypersurfaces* $\underset{\xi}{V}_{n}$ *are nondegenerate and* $n>1$, *then a point correspondence* $C:\underset{1}{V}_{n}\times\cdots\times\underset{n}{V}_{n}\to\underset{0}{V}_{n}$ *between these hypersurfaces determine invariant first and second* *orders normals for all hypersurfaces in a second-order differential neighbourhood of corresponding points.*

Note that to find necessary objects we used quantities $\sum\limits_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^{i}_{(jk)}$. A quantity $\underset{\alpha\beta}{\lambda}{}^{i}_{(jk)}$ may be used instead of the previous one. In general cases there exist C_n^2 different quantities $\underset{\alpha\beta}{\lambda}{}^{i}_{(jk)}$. Therefore, different invariant normalizations of hypersurfaces exist. In the paper we used a symmetrical case.

Below we will suppose that $n>2$. The case $n=2$ is considered in paper [12].

5. The Main Tensors of the Point Correspondence between *n* + 1 Hypersurfaces

Let us use the quantities $\underset{\xi}{p}{}^{i},\underset{\xi}{p}{}_{k}$ for construction of invariant frames of the correspondence. We introduce an invariant family of frames $\left\{\underset{\xi}{N}{}_{0},\underset{\xi}{N}{}_{i},\underset{\xi}{N}{}_{n+1}\right\}$, defined by points

$$\underset{\xi}{N}{}_{0}=\underset{\xi}{M}{}_{0},\ \ \underset{\xi}{N}{}_{i}=\underset{\xi}{M}{}_{i}+\underset{\xi}{p}{}_{i}\underset{\xi}{M}{}_{0},\ \ \underset{\xi}{N}{}_{n+1}=\underset{\xi}{p}{}^{i}\underset{\xi}{M}{}_{i}+\underset{\xi}{M}{}_{n+1}.$$

We denote Pfaffian forms of infinitesimal displacement of these frames as $\underset{\xi}{\sigma}{}^{u}_{v}$. Then relations between 1-forms $\underset{\xi}{\sigma}{}^{u}_{v}$ and $\underset{\xi}{\omega}{}^{u}_{v}$ can be written as follows

$$\begin{aligned}
&\underset{\xi}{\sigma}{}^{0}_{0}=\underset{\xi}{\omega}{}^{0}_{0}-\underset{\xi}{p}{}_{i}\underset{\xi}{\omega}{}^{i}_{0},\ \ \underset{\xi}{\sigma}{}^{i}_{0}=\underset{\xi}{\omega}{}^{i}_{0},\ \ \underset{\xi}{\sigma}{}^{n+1}_{i}=\underset{\xi}{\omega}{}^{n+1}_{i},\\
&\underset{\xi}{\sigma}{}^{j}_{i}=\underset{\xi}{\omega}{}^{j}_{i}+\underset{\xi}{p}{}_{i}\underset{\xi}{\omega}{}^{j}_{0}-\underset{\xi}{\omega}{}^{n+1}_{i}\underset{\xi}{p}{}^{j},\\
&\underset{\xi}{\sigma}{}^{i}_{n+1}=\nabla_{\xi}\underset{\xi}{p}{}^{i}+\underset{\xi}{\omega}{}^{i}_{n+1}+\underset{\xi}{p}{}^{i}\left(\underset{\xi}{\omega}{}^{0}_{0}-\underset{\xi}{\omega}{}^{n+1}_{n+1}\right)-\underset{\xi}{p}{}^{i}\underset{\xi}{p}{}^{t}\underset{\xi}{\omega}{}^{n+1}_{t},\\
&\underset{\xi}{\sigma}{}^{0}_{i}=\nabla_{\xi}\underset{\xi}{p}{}_{i}+\underset{\xi}{\omega}{}^{0}_{i}-\underset{\xi}{p}{}_{i}\underset{\xi}{p}{}_{j}\underset{\xi}{\omega}{}^{j}_{0}+\underset{\xi}{p}{}_{j}\underset{\xi}{p}{}^{j}\underset{\xi}{\omega}{}^{n+1}_{i}.
\end{aligned}\tag{17}$$

By Equations (12), (14) quantities $\nabla_{\xi}\underset{\xi}{p}{}^{i}+\underset{\xi}{\omega}{}^{i}_{n+1}+\underset{\xi}{p}{}^{i}\left(\underset{\xi}{\omega}{}^{0}_{0}-\underset{\xi}{\omega}{}^{n+1}_{n+1}\right)$, $\nabla_{\xi}\underset{\xi}{p}{}_{i}+\underset{\xi}{\omega}{}^{0}_{i}$ depend on differentials of principal parameters, therefore we can write forms $\underset{\xi}{\sigma}{}^{0}_{i}$ and $\underset{\xi}{\sigma}{}^{i}_{n+1}$ as follows

$$\underset{\xi}{\sigma}{}^{0}_{i}=\sum_{\alpha}\underset{\xi\alpha}{a}{}_{ij}\underset{\alpha}{\sigma}{}^{j}_{0},\ \ \underset{\xi}{\sigma}{}^{i}_{n+1}=\sum_{\alpha}\underset{\xi\alpha}{a}{}^{i}_{j}\underset{\alpha}{\sigma}{}^{j}_{0}.\tag{18}$$

By new frames Equations (4), (6) of the correspondence C can be written in the form:

$$\begin{aligned}
&\underset{\xi}{\sigma}{}^{n+1}_{i}=\underset{\xi}{\lambda}{}_{ij}\underset{\xi}{\sigma}{}^{j}_{0},\ \ \underset{0}{\sigma}{}^{i}_{0}+\underset{1}{\sigma}{}^{i}_{0}+\cdots+\underset{n}{\sigma}{}^{i}_{0}=0,\\
&\underset{\alpha}{\Sigma}{}^{i}_{j}-\underset{0}{\Sigma}{}^{i}_{j}=\sum_{\beta}\underset{\alpha\beta}{a}{}^{i}_{jk}\underset{\beta}{\sigma}{}^{k}_{0},
\end{aligned}\tag{19}$$

where $\underset{\xi}{\Sigma}{}^{i}_{j}=\underset{\xi}{\sigma}{}^{i}_{j}-\delta^{i}_{j}\underset{\xi}{\sigma}{}^{0}_{0}$ and

$$\begin{aligned}
&\underset{\alpha\alpha}{a}{}^{i}_{jk}=\underset{\alpha\alpha}{\lambda}{}^{i}_{jk}+2\delta^{i}_{(j}\underset{\alpha}{p}{}_{k)}-\underset{\alpha}{\lambda}{}_{jk}\underset{\alpha}{p}{}^{i}+2\delta^{i}_{(j}\underset{0}{p}{}_{k)}-\underset{0}{\lambda}{}_{jk}\underset{0}{p}{}^{i},\\
&\underset{\alpha\beta}{a}{}^{i}_{jk}=\underset{\alpha\beta}{\lambda}{}^{i}_{jk}+2\delta^{i}_{(j}\underset{0}{p}{}_{k)}-\underset{0}{\lambda}{}_{jk}\underset{0}{p}{}^{i},\ \ \alpha\neq\beta.
\end{aligned}\tag{20}$$

Calculations show, that quantities $\underset{\alpha\alpha}{a}{}^{i}_{jk},\underset{\alpha\beta}{a}{}^{i}_{jk}$ satisfy equations

$$\nabla_{\delta}\underset{\alpha\alpha}{a}{}^{i}_{jk}=0,\nabla_{\delta}\underset{\alpha\beta}{a}{}^{i}_{jk}=0.$$

Therefore, quantities $\underset{\alpha\alpha}{a}{}^{i}_{jk},\underset{\alpha\beta}{a}{}^{i}_{jk}$ are absolute tensors of a second-order differential neighbourhood of the correspondence. They satisfy some additional conditions:

$$\underset{\alpha\alpha}{a}{}^{i}_{jk}=\underset{\alpha\alpha}{a}{}^{i}_{kj},\ \underset{\alpha\beta}{a}{}^{i}_{jk}=\underset{\beta\alpha}{a}{}^{i}_{kj},\ \sum_{\alpha\neq\beta}\underset{\alpha\beta}{a}{}^{l}_{(lk)}=0,\ \ \underset{\alpha\alpha}{a}{}^{l}_{lk}=0,$$

$$\underset{\alpha}{\lambda}{}^{jk}\left(\underset{\alpha\alpha}{a}{}^{i}_{jk}-\frac{1}{C_n^2}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{a}{}^{i}_{(jk)}\right)=0,\ \underset{0}{\lambda}{}^{jk}\sum_{\alpha\neq\beta}\underset{\alpha\beta}{a}{}^{i}_{(jk)}=0.$$

By relations (7), (7'), (19) in the family of new frames we have equations of mapping $\underset{\alpha0}{T}$ in the way

$$\begin{aligned}
&\underset{\alpha}{\sigma}{}^{n+1}_{i}=\underset{\alpha}{\lambda}{}_{ij}\underset{\alpha}{\sigma}{}^{j}_{0},\ \underset{0}{\sigma}{}^{n+1}_{i}=\underset{0}{\lambda}{}_{ij}\underset{0}{\sigma}{}^{j}_{0},\ \underset{\alpha}{\sigma}{}^{n+1}_{0}=0,\\
&\underset{0}{\sigma}{}^{n+1}_{0}=0,\ \underset{\alpha}{\sigma}{}^{i}_{0}+\underset{0}{\sigma}{}^{i}_{0}=0,\ \ \underset{\alpha}{\Sigma}{}^{i}_{j}-\underset{0}{\Sigma}{}^{i}_{j}=\underset{\alpha\alpha}{a}{}^{i}_{jk}\underset{\alpha}{\sigma}{}^{k}_{0},
\end{aligned}$$

and similar for $\underset{\alpha\beta}{T}$

$$\begin{aligned}
&\underset{\alpha}{\sigma}{}^{n+1}_{i}=\underset{\alpha}{\lambda}{}_{ij}\underset{\alpha}{\sigma}{}^{j}_{0},\ \underset{\beta}{\sigma}{}^{n+1}_{i}=\underset{\beta}{\lambda}{}_{ij}\underset{\beta}{\sigma}{}^{j}_{0},\ \underset{\alpha}{\sigma}{}^{n+1}_{0}=0,\\
&\underset{\beta}{\sigma}{}^{n+1}_{0}=0,\ \underset{\alpha}{\sigma}{}^{i}_{0}+\underset{\beta}{\sigma}{}^{i}_{0}=0,\ \ \underset{\alpha}{\Sigma}{}^{i}_{j}-\underset{\beta}{\Sigma}{}^{i}_{j}=\underset{\alpha\beta}{a}{}'^{i}_{jk}\underset{\alpha}{\sigma}{}^{k}_{0}.
\end{aligned}$$

where $\underset{\alpha\beta}{a}{}'^{i}_{jk}=\left(\underset{\alpha\alpha}{a}{}^{i}_{jk}+\underset{\beta\beta}{a}{}^{i}_{jk}-2\underset{\alpha\beta}{a}{}^{i}_{(jk)}\right)$ and $\underset{\alpha\beta}{a}{}'^{i}_{jk}=\underset{\beta\alpha}{a}{}'^{i}_{kj}$.

We will call tensors $\underset{\alpha\alpha}{a}{}^{i}_{jk},\underset{\alpha\beta}{a}{}'^{i}_{jk}$ as main tensors of the correspondence. Tensors $\underset{\alpha\alpha}{a}{}^{i}_{jk},\underset{\alpha\beta}{a}{}'^{i}_{jk}$ define quadratic transformations $\underset{\xi}{\sigma}{}^{i}_{0}\to\underset{\xi}{\sigma}{}^{i}=\underset{\theta}{a}{}^{i}_{jk}\underset{\xi}{\sigma}{}^{j}_{0}\underset{\xi}{\sigma}{}^{k}_{0}$, generated invariant charictiristic directions in corresponding points of hypersurfaces.

Let us consider correspondences C if there are relations

$$\underset{\alpha\alpha}{a}{}^{i}_{jk}=\underset{\alpha\beta}{a}{}'^{i}_{jk}=0.$$

A point correspondence $C:\underset{1}{V}{}_{n}\times\cdots\times\underset{n}{V}{}_{n}\to\underset{0}{V}{}_{n}$ is called geodesic, if any tangent directions of hypersurfaces $\underset{\xi}{V}{}_{n}$ in corresponding points $\underset{\xi}{M}{}_{0}$ became charictiristic for mappings $\underset{\xi\eta}{T}:\underset{\xi}{V}{}_{n}\to\underset{\eta}{V}{}_{n}$ by some choice of the first order normals in these points.

It is true.

Theorem 3. *For* $n>1$ *a point correspondence* $C: \underset{1}{V}_n \times\cdots\times \underset{n}{V}_n \to \underset{0}{V}_n$ *will be geodesic if ahd only if main tensors* $\underset{\alpha\alpha}{a}{}^i_{jk} = \underset{\alpha\beta}{a}{}'^i_{jk} = 0.$

Really, let there exist $(n+1)$ families of the first order normals of hypersurfaces under correspondence by them a point correspondence $C: \underset{1}{V}_n \times\cdots\times \underset{n}{V}_n \to \underset{0}{V}_n$ is geodesic. Then relations (10) must be true. In this case as follows from Equations (15), (15') the first order normal objects of hypersurfaces $\underset{\xi}{p}{}^i = 0.$

By setting $\underset{\xi}{p}{}^i = 0$ in relations (16), we get values of second order normal objects of hypersurfaces under correspondence in this way:

$$\underset{0}{p}_k = -\frac{1}{n+1}\sum_{\alpha\neq\beta} \underset{\alpha\beta}{\lambda}{}^l_{(lk)}, \quad \underset{\alpha}{p}_k = \frac{1}{n+1}\left(-\underset{\alpha\alpha}{\lambda}{}^l_{lk} + \sum_{\alpha\neq\beta}\underset{\alpha\beta}{\lambda}{}^l_{(lk)}\right),$$

If we substitute values $\underset{\xi}{p}{}^i$, $\underset{\xi}{p}_k$ in Equation (20) and use relations (10), then we obtain

$$\underset{\alpha\alpha}{a}{}^i_{jk} = \underset{\alpha\beta}{a}{}'^i_{jk} = 0.$$

Conversely, if we use invariant first and second order normals in all hypersurfaces under correspondence and tensors

$$\underset{\alpha\alpha}{a}{}^i_{jk} = \underset{\alpha\beta}{a}{}'^i_{jk} = 0, \tag{21}$$

then relations (10) are true.

Any tangent direction $\underset{\xi}{\sigma}{}^i_0$ becomes charactiristic by invariant first order normals in corresponding points of hypersurfaces. It follows the point correspondence $C: \underset{1}{V}_n \times\cdots\times \underset{n}{V}_n \to \underset{0}{V}_n$ is geodesic.

6. The Whole Projective-Invariant Normalization of Hypersurfaces under the Point Correspondence

To finish normalizations of hypersurfaces under consideration it is necessary to construct objects satisfying Equations (13). We prolong Equations (18). With the help of exterior differentiations and applying Cartan's lemma we obtain new equations:

$$\nabla_\delta \underset{0\alpha}{a}{}^i_j = -\underset{0\alpha}{a}{}^i_j\left(\underset{0}{\pi}{}^{n+1}_{n+1} - \underset{0}{\pi}{}^0_0\right) - \delta^i_j \underset{0}{\pi}{}^0_{n+1},$$

$$\nabla_\delta \underset{\alpha\alpha}{a}{}^i_j = +\underset{\alpha\alpha}{a}{}^i_j\left(\underset{\alpha}{\pi}{}^{n+1}_{n+1} - \underset{\alpha}{\pi}{}^0_0\right) + \delta^i_j \underset{\alpha}{\pi}{}^0_{n+1}.$$

We construct quantities $\underset{0}{p} = \frac{1}{n^2}\sum_\alpha \underset{0\alpha}{a}{}^i_i,\ \underset{\alpha}{p} = -\frac{1}{n}\underset{\alpha\alpha}{a}{}^i_i.$ These quantities satisfy Equations (13) and define invariant points on the first order normals of hypersurfaces $\underset{\xi}{V}_n$.

Let us find a geometrical meaning of chosen invariant points. We consider hypersurfaces $\underset{\alpha}{V}_n$. We fix the hypersurface $\underset{\alpha}{V}_n$, then $\underset{\beta}{\sigma}{}^i_0 = 0, \alpha\neq\beta$. The set of invariant first order normals of the hypersurface $\underset{\alpha}{V}_n$ generates n-parametrical fimily of straight lines. This set is called as a congruence of straight lines.

Let point $\underset{\alpha}{L} = \underset{1}{y}\underset{\alpha}{N}_0 + \underset{\alpha}{N}_{n+1}$ be a focus of the congruence of the straight lines $\left[\underset{\alpha}{N}_0 \underset{\alpha}{N}_{n+1}\right]$, then infinitesimal displacement of focus $\underset{\alpha}{L}$ must belong to the straight line $\left[\underset{\alpha}{N}_0 \underset{\alpha}{N}_{n+1}\right]$. Since

$$d\underset{\alpha}{L} = (.)\underset{\alpha}{N}_0 + (.)\underset{\alpha}{N}_{n+1} + \left(\underset{\alpha}{y}\underset{\alpha}{\sigma}{}^i_0 + \underset{\alpha}{\sigma}{}^i_{n+1}\right)\underset{\alpha}{N}_i,$$

then focuses $\underset{\alpha}{L}$ are obtained by conditions

$$\underset{\alpha}{y}\underset{\alpha}{\sigma}{}^i_0 + \underset{\alpha}{\sigma}{}^i_{n+1} = 0$$

or

$$\left(\underset{\alpha}{y}\delta^i_j + \underset{\alpha}{a}{}^i_j\right)\underset{\alpha}{\sigma}{}^j_0 = 0.$$

To get values $\underset{\alpha}{y}$, defined focuses on the straight line $\left[\underset{\alpha}{N}_0 \underset{\alpha}{N}_{n+1}\right]$, we consider the equation

$$\left|\underset{\alpha}{y}\delta^i_j + \underset{\alpha}{a}{}^i_j\right| = 0.$$

For roots of this equation we have

$$\sum_{i=1}^n \underset{\alpha}{y}_i = -\underset{\alpha}{a}{}^i_i.$$

We can define the harmonic pole [16] on each straight line $\left[\underset{\alpha}{N}_0 \underset{\alpha}{N}_{n+1}\right]$ of the congruence according to the point $\underset{\alpha}{N}_0$ and n focuses by the relation

$$\frac{1}{n}\sum_{i=1}^n \underset{\alpha}{y}_i \underset{\alpha}{N}_0 + \underset{\alpha}{N}_{n+1} = -\frac{1}{n}\underset{\alpha}{a}{}^i_i \underset{\alpha}{N}_0 + \underset{\alpha}{N}_{n+1} = \underset{\alpha}{N}'_{n+1}.$$

Let points $\underset{\xi}{N}_{n+1}$ of frames coinside with invariant points $\underset{\xi}{N}'_{n+1} = \underset{\xi}{p}\underset{\xi}{N}_0 + \underset{\xi}{N}_{n+1}$, where quantities $\underset{\xi}{p}$ are defined by values $\underset{0}{p} = \frac{1}{n^2}\sum_\alpha \underset{0\alpha}{a}{}^i_i,\ \underset{\alpha}{p} = -\frac{1}{n}\underset{\alpha\alpha}{a}{}^i_i.$ Other points of frames we leave without changing. After these transformations quantities $\underset{\xi\alpha}{a}_{ij}$ become absolute tensors and quantities $\underset{\xi\alpha}{a}{}^{i}_{j}$ become relative tensors of the correspondence. Some relations are true

$$\sum_{\alpha} \underset{0\alpha}{a}{}^{i}_{i} = 0, \quad \underset{\alpha\alpha}{a}{}^{i}_{i} = 0.$$

Forms $\underset{\xi}{\sigma}{}^{0}_{n+1}$ will depend only on differentials of principal parameters, that's why they can be written as follows $\underset{\xi}{\sigma}{}^{0}_{n+1} = \sum_{\alpha} \underset{\xi\alpha}{a}{}_{i} \underset{\alpha}{\sigma}{}^{i}_{0}$.

It is proved.

Theorem 4. *For* $n > 1$ *a point correspondence* $C : \underset{1}{V}_n \times \cdots \times \underset{n}{V}_n \to \underset{0}{V}_n$ *define the whole projective-invariant normalization of hypersurfaces in the third differential neighbourhood of corresponding points.*

7. Point Correspondences between (n + 1) Hypersurfaces of Projective Spaces and Multidimensional (n + 1)-Webs

A point correspondence C between $n+1$ hyperspaces $\underset{\xi}{V}_n$ of projective spaces $\underset{\xi}{P}_{n+1}$ is a local differential n-quasigroup from the algebraic point of view. There exists an $(n+1)$-web connected with this n-quasigroup. To find this web it is sufficient to consider a new manifold constructed as $\underset{0}{V}_n \times \underset{1}{V}_n \times \cdots \times \underset{n}{V}_n$. A correspondence C will be determined as an n^2-dimesional smooth submanifold. There exist $n+1$ foliations of codimension n on this submanifold. Each foliation is determined by the hypersurface $\underset{\xi}{V}_n$. These foliations define $(n+1)$-web $W(n + 1, n)$ on the n^2-dimensional submanifold.

We introduce additional forms

$$\omega^i_j = \underset{0}{\Omega}{}^{i}_{j} - \frac{1}{C_n^2} \sum_{\alpha\beta} \underset{\alpha\beta}{\lambda}{}^{i}_{(jk)} \underset{0}{\omega}{}^{k}_{0}, \tag{22}$$

and quantities

$$\underset{\alpha\beta}{b}{}^{i}_{jk} = \underset{\alpha\beta}{\lambda}{}^{i}_{jk} - \frac{1}{C_n^2} \sum_{\gamma\neq\delta} \underset{\gamma\delta}{\lambda}{}^{i}_{(jk)},$$

where $\alpha \neq \beta$.

By relations (11) we have

$$\nabla_\delta \underset{\alpha\beta}{b}{}^{i}_{jk} = 0.$$

Therefore, quantities $\underset{\alpha\beta}{b}{}^{i}_{jk}$ determine a tensor of a second-order differential neighbourhood of the correspondence. It can be written as

$$\underset{\alpha\beta}{b}{}^{i}_{jk} = \underset{\alpha\beta}{a}{}^{i}_{jk} - \frac{1}{C_n^2} \sum_{\gamma\neq\delta} \underset{\gamma\delta}{a}{}^{i}_{(jk)}.$$

Using relations (17) we obtain $\underset{0}{\Omega}{}^{i}_{j} - \frac{1}{C_n^2} \sum_{\alpha\beta} \underset{\alpha\beta}{\lambda}{}^{i}_{(jk)} \underset{0}{\omega}{}^{k}_{0} = \underset{0}{\Sigma}{}^{i}_{j} - \frac{1}{C_n^2} \sum_{\alpha\beta} \underset{\alpha\beta}{a}{}^{i}_{(jk)} \underset{0}{\sigma}{}^{k}_{0}$. Therefore, forms ω^i_j do not depend on a choice of frames in corresponding points of hypersurfaces.

To write equations of $(n+1)$-web adjoined to correspondence C we use Equations (4), (22) and structural equations of projective spaces. We obtain

$$\sum_{\xi} \underset{\xi}{\sigma}{}^{i}_{0} = 0, \; d\,\underset{\alpha}{\sigma}{}^{i}_{0} = \underset{\alpha}{\sigma}{}^{j}_{0} \wedge \omega^i_j + \sum_{\alpha\beta} \underset{\alpha\beta}{b}{}^{i}_{jk} \underset{\alpha}{\sigma}{}^{j}_{0} \wedge \underset{\beta}{\sigma}{}^{k}_{0},$$

$$d\,\omega^i_j - \omega^k_j \wedge \omega^i_k = \sum_{\alpha} \underset{\alpha}{b}{}^{i}_{jkl} \underset{\alpha}{\sigma}{}^{k}_{0} \wedge \underset{\alpha}{\sigma}{}^{l}_{0} + \sum_{\alpha\beta} \underset{\alpha\beta}{b}{}^{i}_{jkl} \underset{\alpha}{\sigma}{}^{k}_{0} \wedge \underset{\beta}{\sigma}{}^{l}_{0}.$$

The equations show that forms ω^i_j are the forms of an affine connection assosiated to the web W and tensors $\underset{\alpha\beta}{b}{}^{i}_{jk}$ are the torsion tensor of W [14].

It is known that parallelizable webs [11] are the simplest class of (n + 1)-webs. A correspondence between (n + 1) hypesurfaces of projective spaces is said to be parallelizable if the (n + 1)-web of this correspondence is parallelizable. The necessary and sufficient conditions for correspondence to be parallelizable are relations

$$\underset{\alpha\beta}{b}{}^{i}_{jk} = 0.$$

Calculations show that if hypersurfaces are given then parallelizable correspondences between (n + 1) hypesurfaces of projective spaces exist and depend on $(n+1)n$ functions in n variables.

In paper [11] specific classes of webs are introduced called a class of (2n + 2)-adric webs. For these classes the following relations are true

$$\underset{\alpha\beta}{b}{}^{i}_{(jk)} = 0.$$

Comparing these relations with conditions (21), we note that they are true for geodesic correspondences, that's why the (n + 1)-web adjoined to the geodesic correspondence between (n + 1) hypersurfaces of projective spaces is always (2n + 2)-adric web of type 2.

A point correspodence $C : \underset{1}{V}_n \times \cdots \times \underset{n}{V}_n \to \underset{0}{V}_n$ generates C^3_{n+1} families point subcorrespondences $\underset{\xi\eta\zeta}{C} : \underset{\xi}{V}_n \times \underset{\eta}{V}_n \to \underset{\theta}{V}_n$ obtained by fixation of n − 2 corresponding points. We can adjoin the web $W(3, n)$ to each subcorrespondence $\underset{\xi\eta\zeta}{C}$. Let us find equations of correspondences $\underset{\xi\eta\zeta}{C}$ and equations of three-webs joined to them. Equations of correspondences $\underset{0\eta\zeta}{C}$ can be written in the following way

$$\underset{o}{\sigma}{}^i_0 + \underset{\alpha}{\sigma}{}^i_0 + \underset{\beta}{\sigma}{}^i_0 = 0.$$

Substituting these values into equations of (n + 1)-web we have after transformations

$$d\underset{\alpha}{\sigma}{}^i_0 = \underset{\alpha}{\sigma}{}^j_0 \wedge \left(\underset{0}{\Sigma}{}^i_j + \underset{\alpha\beta}{a}{}^i_{jk} \underset{\beta}{\sigma}{}^k_0 + \underset{\alpha\beta}{a}{}^i_{kj} \underset{\alpha}{\sigma}{}^k_0 \right) + \underset{\alpha\beta}{a}{}^i_{[jk]} \underset{\alpha}{\sigma}{}^j_0 \wedge \underset{\alpha}{\sigma}{}^k_0,$$

$$d\underset{\beta}{\sigma}{}^i_0 = \underset{\beta}{\sigma}{}^j_0 \wedge \left(\underset{0}{\Sigma}{}^i_j + \underset{\alpha\beta}{a}{}^i_{jk} \underset{\beta}{\sigma}{}^k_0 + \underset{\alpha\beta}{a}{}^i_{kj} \underset{\alpha}{\sigma}{}^k_0 \right) - \underset{\alpha\beta}{a}{}^i_{[jk]} \underset{\beta}{\sigma}{}^j_0 \wedge \underset{\beta}{\sigma}{}^k_0.$$

The forms

$$\underset{0}{\Sigma}{}^i_j + \underset{\alpha\beta}{a}{}^i_{jk} \underset{\beta}{\sigma}{}^k_0 + \underset{\alpha\beta}{a}{}^i_{kj} \underset{\alpha}{\sigma}{}^k_0$$

are connection forms of this three-web and the tensor $\underset{\alpha\beta}{a}{}^i_{[jk]} = \underset{\alpha\beta}{b}{}^i_{[jk]}$ is the torsion tensor. If we take a correspondence $\underset{\alpha\beta\gamma}{C}$, then the torsion tensor of three-web adjoined to $\underset{\alpha\beta\gamma}{C}$ can be written as follows

$$\underset{\alpha\beta}{a}{}^i_{[jk]} + \underset{\beta\gamma}{a}{}^i_{[jk]} + \underset{\gamma\alpha}{a}{}^i_{[jk]}.$$

There exist the so-called paratactical three-webs [11]. In accordance with this, point correspondences between (n + 1) hypersurfaces of projective spaces are called paratactical, if all their subcorrespondences $\underset{\alpha\beta\gamma}{C}$ are paratactical ones (torsion tensors are equal zero). The following relations

$$\underset{\alpha\beta}{a}{}^i_{[jk]} = 0$$

are conditions of the existence of paratactical correspondences.

8. Conclusions

We write main equations of a point correspondence between $n+1$ hypersurfaces of projective spaces and construct the sequence of main geometrical objects of the correspondence. we define characteristic directions of a correspondence and prove that there exist invariant characteristic directions.

We construct whole projective-invariant normalizations of all hupersurfaces and prove that invariant first and second orders normals for all hypersurfaces (n > 2) under point correspondences are determined in a second-order differential neighbourhood of corresponding points. We single out main tensors of the correspondence and define some partial cases of correspondences.

We establish a connection between the geometry of point correspondences between $n+1$ hypersurfaces of projective spaces and the theory of multidimensional (n + 1)-webs. In particular we prove that the (n + 1)-web adjoined to the geodesic correspondence between (n + 1) hypersurfaces of projective spaces is always (2n + 2)-adric web of type 2.

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