

# Stabilization of Functional System with Markovian Switching

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## ABSTRACT

There are many papers related to stability, some on suppression or on stabilization are one type of them. Functional differential systems are common and important in practice. They are special situations of neutral differential systems and generalization of ordinary differential systems. We discussed conditions on suppression on functional system with Markovian switching in our previous work: "Suppression of Functional System with Markovian Switching". Based on it, by slightly modifying and adding some conditions, we get this paper. In this paper, we will study a functional system whose coefficient satisfies the local Lipschitz condition and the one-sided polynomial growth condition under Markovian switching. By introducing two appropriate intensity Brownian noise, we find the potential explosion system stabilized.

**Keywords:** Stochastic Functional System; Brownian Noise; Markovian Switching; Boundedness; Stabilization

## 1. Introduction

There are many papers which discuss stability of systems. It is called a stabilization problem when we impose such conditions on a given unstable system to make it stable. There have been rich literatures on this topic, here we only mention [1-4]. It talks about suppression of noise in [1,2]. It shows similar stabilization phenomena in stochastic systems as those in deterministic systems in [3,4]. They all indicate clearly that different structures of environmental noise may have different effects on the deterministic system. On the other hand, there are also many papers related to stabilization of functional systems, such as [5-8]. [5] investigates a stochastic Lotka-Volterra system with infinite delay, whose initial data come from an admissible Banach space  $C$ , and show that its unique global positive solution has asymptotic boundedness property by using the exponential martingale inequality. [6] studies existence and uniqueness of the global positive solution of stochastic functional Kolmogorov-type system and its asymptotic bound properties and moment average boundedness in time under the traditionally diagonally dominant condition. [7] studies the same problems as [6] under some other conditions. [8] discusses stabilization of a given unstable nonlinear functional system by introducing two Brownian noise.

Many practical systems may experience abrupt changes

in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. The hybrid systems have been used to describe such situations. Along the trajectories of the Markovian jump system, the mode switches from one value to another in a random way are governed by a Markov process with discrete state space. [9,10] studied the stability of a jump system. Feng *et al.* [11] systematically studied stochastic stability properties of jump linear systems and the relationship among various moment and sample path stability properties. Shen and Wang [12] presented new exponential stability results for recurrent neural networks with Markovian switching. Wang *et al.* [13] dealt with the problem of state estimation for a class of delayed neural networks with Markovian jumping parameters without the traditional monotonicity and smoothness assumptions on the activation function.

Taking both the environmental noise and jump into account, the system under consideration becomes a stochastic differential system with Markovian switching (SDSwMS), which has received a lot of attention (see [14-24]) recently. [17] provided some useful conditions on the exponential stability for general nonlinear SDSwMSs, which was improved by himself in Mao *et al.* [19]. Yuan and Lygeros [20] investigated almost sure ex-

ponential stability for a class of switching diffusion processes. [25] discusses the asymptotic stability and exponential stability of SDSwMSs, whose coefficients are assumed to satisfy the local Lipschitz condition and the polynomial growth condition.

Motivated by [25,26] and some other literatures, we will investigate suppression and stabilization by noise of functional differential system with Markov chains, whose coefficient satisfies the local Lipschitz condition and the one-sided polynomial growth condition. For a given unstable functional system with Markovian switching

$$\dot{x}(t) = f(t, x_t, r(t)), \tag{1}$$

where

$$x = (x_1, \dots, x_n)^T,$$

$$f = (f_1, \dots, f_n)^T : R_+ \times C([- \tau, 0]; R^n) \times S \rightarrow R^n,$$

$x_t \in C([- \tau, 0]; R^n)$  is defined by  $x_t(\theta) = x(t + \theta)$ , by introducing two independent scalar Brownian noise under some conditions, we get a stochastic functional system which admits a unique global positive solution. Furthermore, choosing appropriate intensity noise, we can get an exponential stable stochastic functional system

$$\begin{aligned} dx(t) &= f(t, x_t, r(t))dt + g(t, x_t, r(t))dB_1(t) \\ &\quad + h(t, x_t, r(t))dB_2(t), \end{aligned} \tag{2}$$

on  $t \geq 0$ , where  $B_i(t) (i=1,2)$  is a scalar Brownian motion, and

$$f = (f_1, \dots, f_n)^T : R_+ \times C([- \tau, 0]; R^n) \times S \rightarrow R^n,$$

$$g = (g_1, \dots, g_n)^T : R_+ \times C([- \tau, 0]; R^n) \times S \rightarrow R^n,$$

$$h = (h_1, \dots, h_n)^T : R_+ \times C([- \tau, 0]; R^n) \times S \rightarrow R^n.$$

In the next section we will give some necessary notations and lemmas. In Section 3, we will give the main results of this paper.

## 2. Preliminaries

Throughout this paper, unless otherwise specified, let  $\|\cdot\|$  be the Euclidean norm in  $R^n$ . If  $A$  is a vector or matrix, its transpose is denoted by  $A^T$ . If  $A$  is a matrix, its trace norm is denoted by  $|A| = \sqrt{\text{trace}(A^T A)}$ . Denote the inner product of  $X, Y \in R^n$  by  $\langle X, Y \rangle$  or  $X^T Y$ . Let  $i, j, k, N$  be positive integers. Let  $a \vee b$  denote the maximum of  $a$  and  $b$ , while  $a \wedge b$  the minimum of  $a$  and  $b$ . Let  $R_+ = [0, \infty)$ ,  $R_{++} = (0, \infty)$ . Denote by  $C = C([- \tau, 0]; R^n)$  the family of continuous functions from  $[- \tau, 0]$  to  $R^n$  with the norm  $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$ ,

which forms a Banach space. Let  $C_+ = C([- \tau, 0]; R_+^n)$  and  $C_{++} = C([- \tau, 0]; R_{++}^n)$ . Let  $C^2$  denote the family of functions  $V(t, x_t, i)$  on  $C \times R_+^n \times S$  which are continuously twice differentiable in  $x$  and once in  $t$ .

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions (i.e. it is increasing and right continuous while  $\mathcal{F}_0$  contains all  $\mathbb{P}$ -null sets). Let  $L_{\mathcal{F}_t}^p(\Omega, R^n)$  denote the family of  $R^n$ -valued  $\mathcal{F}_t$ -measurable random variables  $\xi$  with  $E|\xi|^p < \infty$ . Denote the family of  $R^n$ -valued bounded  $\mathcal{F}_t$ -measurable random variables by  $L_{\mathcal{F}_t}^b(\Omega, R^n)$ . If  $x(t)$  is an  $R^n$ -valued process on  $t \in [- \tau, \infty)$ , let

$$x_t = \{x(t + \theta) : -\tau \leq \theta \leq 0, t \geq 0\}.$$

If  $M$  is a continuous local martingale, denote the quadratic variation of  $M$  by  $\langle M, M \rangle_t$ . Let

$$B_i(t) (i=1,2), t \geq 0$$

be independent scalar Brownian motion defined on the probability space. Let  $r(t), t \geq 0$  be a right-continuous Markov chain on the probability space taking values in a definite state space  $S = \{1, 2, \dots, N\}$  with the generator  $\Gamma = (\gamma_{ij})_{N \times N}$  given by

$$P(r(t + \delta) = j | r(t) = i) = \begin{cases} \gamma_{ij} \delta + o(\delta) & \text{if } i \neq j \\ 1 + \gamma_{ii} \delta + o(\delta) & \text{if } i = j \end{cases}$$

where  $\delta > 0$ ,  $\gamma_{ij}$  is the transition rate from  $i$  to  $j$  and  $\gamma_{ij} > 0$  if  $i \neq j$  while  $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$ . We assume

that the Markov chain  $r(\cdot)$  is independent of the Brownian motion  $B(\cdot)$ . For any initial value

$$x(0) = \xi \in C([- \tau, 0]; R^n),$$

denote the solution of the corresponding initial value problem by  $x(t; 0, \xi, r(0))$  or simply  $x(t)$  on  $t \geq 0$ . In order to obtain the main results, we need the following assumptions.

(H<sub>1</sub>) There are some nonnegative constants  $\alpha, \lambda_i, \mu_i$  such that

$$\begin{aligned} \varphi^T(0) f(t, \varphi, i) &\leq |\varphi(0)|^2 \left[ \lambda_i |\varphi(0)|^\alpha \right. \\ &\quad \left. + \mu_i \int_{-\tau}^0 |\varphi(\theta)|^\alpha d\eta(\theta) + o(|\varphi(0)|^\alpha) \right] \end{aligned} \tag{3}$$

for all  $(t, \varphi, i) \in R_+ \times C \times S$ , where  $\eta$  is a probability measure on  $[- \tau, 0]$  and  $o(y)$  means some functions satisfying  $\lim_{y \rightarrow \infty} \frac{o(y)}{y} = 0$ .

(H<sub>2</sub>) For every integer  $j \geq 1$ , there is a  $K_j > 0$  such that

$$\begin{aligned}
 &|f(t, \varphi, i) - f(t, \psi, i)| \vee |g(t, \varphi, i) - g(t, \psi, i)| \\
 &\vee |h(t, \varphi, i) - h(t, \psi, i)| \leq K_j \|\varphi - \psi\| \tag{4}
 \end{aligned}$$

for all  $(t, \varphi, i), (t, \psi, i) \in R_+ \times C \times S$  with  $\|\varphi\| \vee \|\psi\| \leq j$ .

(H<sub>3</sub>) There are some nonnegative constants  $\beta, \bar{w}_i, \hat{w}_i, \bar{q}_i, \hat{q}_i, \bar{\gamma}_i, \hat{\gamma}_i$  and probability measure  $\eta$  such that for any  $(t, \varphi, i) \in R_+ \times C \times S$  satisfying  $x \neq 0$ ,

$$|\varphi^T(0)g(t, \varphi, i)| \geq \bar{w}_i |\varphi(0)|^{\beta+1} - \hat{\gamma}_i \int_{-\tau}^0 |\varphi(\theta)|^{\beta+1} d\eta(\theta) + o(|\varphi(0)|^{\beta+1}) \tag{5}$$

$$|g(t, \varphi, i)| \leq \hat{w}_i |\varphi(0)|^\beta + \bar{\gamma}_i \int_{-\tau}^0 |\varphi(\theta)|^\beta d\eta(\theta) + o(|\varphi(0)|^\beta) \tag{6}$$

$$|\varphi^T(0)h(t, \varphi, i)| \geq \bar{q}_i |\varphi(0)|^2 - \hat{\sigma}_i \int_{-\tau}^0 |\varphi(\theta)|^2 d\eta(\theta) + o(|\varphi(0)|^2) \tag{7}$$

$$|h(t, \varphi, i)| \leq \hat{q}_i |\varphi(0)| + \bar{\sigma}_i \int_{-\tau}^0 |\varphi(\theta)| d\eta(\theta) + o(|\varphi(0)|). \tag{8}$$

*Definition 1:* The irreducibility of the Markov chain means that the Markov chain has a unique stationary (probability) distribution  $\pi = (\pi_1, \pi_2, \dots, \pi_N) \in R^{1 \times N}$ , which can be determined by solving the following linear equation

$$\pi \Gamma = 0 \tag{9}$$

subject to

$$\sum_{i=1}^N \pi_i = 1 \text{ and } \pi_i > 0, \forall i \in S. \tag{10}$$

*Lemma 1:* [27] Let (H<sub>2</sub>) hold, for any initial value  $x_0 = \xi \in C$ , system (2) has a unique maximal local strong solution on  $[-\tau, \tau_e)$ , where  $\tau_e$  is the explosion time.

### 3. Main Results

Similar to the proof of Theorem 1 in [28], we slightly

modify the condition on the coefficient of (1) and obtain the following theorem.

*Theorem 1:* Let (H<sub>1</sub>) - (H<sub>3</sub>) hold, for any initial value  $x_0 = \xi \in C$ , if

$$2\bar{\gamma}_i^2 - 2\bar{w}_i\hat{\gamma}_i + 2\hat{w}_i^2 < \bar{w}_i^2 \text{ and } \beta > \frac{\alpha}{2} + 1,$$

then there exists a unique global solution  $x(t)$  of system (2) on all  $t \in [0, \infty)$  a.s.

Similarly to that in [28], we define the stopping time

$$\tau_k = \inf \{t \in [0, \tau_e) : |x(t)| \geq k\}, \tag{11}$$

and a  $C^2$ -function  $V(x) = (1 + |x|^2)^{\frac{p}{2}}$ , for any  $p \in (0, 1)$ .

Using the Itô formula and the Young inequality, for any  $p \in (0, 1)$ , by (H<sub>1</sub>) and (H<sub>3</sub>), we get

$$\begin{aligned}
 \mathcal{L}V(t, x, i) \leq & H(x) + \frac{p\mu_i\alpha}{\alpha+p} \left[ \int_{-\tau}^0 |x(t+\theta)|^{\alpha+p} d\eta(\theta) - |x|^{\alpha+p} \right] \\
 & + \frac{p(p-2)(\beta+1)\bar{w}_i\hat{\gamma}_i}{2\beta+p-2} \left[ \int_{-\tau}^0 |x(t+\theta)|^{2\beta+p-2} d\eta(\theta) - |x|^{2\beta+p-2} \right] \\
 & + 2(p-2)\bar{q}_i\hat{\sigma}_i \left[ \int_{-\tau}^0 |x(t+\theta)|^p d\eta(\theta) - |x|^p \right] \\
 & + \frac{2p\beta\bar{\gamma}_i^2}{2\beta+p-2} \left[ \int_{-\tau}^0 |x(t+\theta)|^{2\beta+p-2} d\eta(\theta) - |x|^{2\beta+p-2} \right] + \bar{\sigma}_i^2 \left[ \int_{-\tau}^0 |x(t+\theta)|^p d\eta(\theta) - |x|^p \right],
 \end{aligned} \tag{12}$$

where

$$H(x) = p(\lambda_i + \mu_i)|x|^{\alpha+p} + \frac{p(p-2)}{2} \left( \bar{w}_i^2 + 2\bar{w}_i\hat{\gamma}_i + \frac{2\hat{w}_i^2 + 2\bar{\gamma}_i^2}{p-2} \right) |x|^{2\beta+p-2} + o(|x|^{\alpha+p}) + o(|x|^{2\beta+p-2}). \tag{13}$$

These results will be used in the following.

#### 3.1. Boundedness

*Theorem 2:* Let (H<sub>1</sub>) - (H<sub>3</sub>) hold, for any initial value

$x_0 = \xi \in C$  and  $p \in (0, 1)$ , if  $2\bar{\gamma}_i^2 - 2\bar{w}_i\hat{\gamma}_i + 2\hat{w}_i^2 < \bar{w}_i^2$  and  $\beta > \frac{\alpha}{2} + 1$ , then there exists a constant  $M_p$  such that the global solution  $x(t)$  of system (2) has the

property that

$$\limsup_{t \rightarrow \infty} E|x(t)|^p \leq M_p, \tag{14}$$

where  $M_p$  is dependent on  $p$  and independent of the

initial value  $\xi$ , that is,  $x(t)$  is bounded in moment.

*Proof:* For any  $\varepsilon > 0$ , applying the Itô formula to  $e^{\varepsilon t}V(x)$  yields

$$EV(t, x(t), i) = e^{-\varepsilon t}EV(o, x(0), r(0)) + e^{-\varepsilon t}E \int_0^t e^{\varepsilon s} [\mathcal{L}V(s, x_s, r(s)) + \varepsilon V(s, x(s), r(s))] ds.$$

By (12) and (13), we have

$$\begin{aligned} \mathcal{L}V(t, x_t, i) + \varepsilon V(x) &\leq H(x) + \frac{p\mu_i\alpha}{\alpha+p} \left[ \int_{-\tau}^0 |x(t+\theta)|^{\alpha+p} d\eta(\theta) - |x|^{\alpha+p} \right] \\ &\quad + \frac{p(p-2)(\beta+1)\bar{w}_i\hat{\gamma}_i}{2\beta+p-2} \left[ \int_{-\tau}^0 |x(t+\theta)|^{2\beta+p-2} d\eta(\theta) - |x|^{2\beta+p-2} \right] \\ &\quad + 2(p-2)\bar{q}_i\hat{\sigma}_i \left[ \int_{-\tau}^0 |x(t+\theta)|^p d\eta(\theta) - |x|^p \right] \\ &\quad + \frac{2p\beta\bar{\gamma}_i^2}{2\beta+p-2} \left[ \int_{-\tau}^0 |x(t+\theta)|^{2\beta+p-2} d\eta(\theta) - |x|^{2\beta+p-2} \right] \\ &\quad + \bar{\sigma}_i^2 \left[ \int_{-\tau}^0 |x(t+\theta)|^p d\eta(\theta) - |x|^p \right] + \varepsilon(1+|x|^2)^{\frac{p}{2}}. \end{aligned}$$

By the boundedness of polynomial functions, there exists a constant  $\tilde{M}$  such that

$$\mathcal{L}V(t, x_t, i) + \varepsilon V(x) \leq \tilde{M},$$

which implies

$$EV(x(t)) \leq e^{-\varepsilon t}E\left(1+|\xi|^2\right)^{\frac{p}{2}} + \frac{\tilde{M}}{\varepsilon}(1-e^{-\varepsilon t}).$$

Then

$$\limsup_{t \rightarrow \infty} E|x(t)|^p \leq \limsup_{t \rightarrow \infty} E\left(1+|x(t)|^2\right)^{\frac{p}{2}} \leq \frac{\tilde{M}}{\varepsilon} := M_p.$$

That is, the global solution  $x(t)$  of system (2) is

bounded in  $p$ -th moment for any  $p \in (0, 1)$ .

*Theorem 3:* Let (H<sub>1</sub>) - (H<sub>3</sub>) hold, if

$$2\bar{\gamma}_i^2 - 2\bar{w}_i\hat{\gamma}_i + 2\hat{w}_i^2 < \bar{w}_i^2 \quad \text{and} \quad \beta > \frac{\alpha}{2} + 1,$$

then for any initial value  $x_0 = \xi \in C$  and  $p \in (0, 1)$ , the solution  $x(t)$  of system (2) has the property that

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T E|x(t)|^p dt \leq \sum_{i=1}^N \pi_i K_i, \tag{15}$$

where

$$K_i = \sup_{x \in R_+^n} pF_1(x, i) < \infty.$$

$$\begin{aligned} F_1(x, i) &= \left( \lambda_i + \frac{p\mu_i}{\alpha+p} \right) |x|^{\alpha+p} + \hat{w}_i^2(p-1)|x|^{2\beta+p-2} + \hat{q}_i^2(p-1)|x|^p + \frac{\alpha\mu_i}{\alpha+p} \int_{-\tau}^0 |x(t+\theta)|^{\alpha+p} d\eta(\theta) \\ &\quad + \bar{\gamma}_i^2(p-1)|x|^{p-2} \int_{-\tau}^0 |x(t+\theta)|^{2\beta} d\eta(\theta) + \bar{\sigma}_i^2|x|^{p-2} \int_{-\tau}^0 |x(t+\theta)|^2 d\eta(\theta) + o(|x|^{2\beta+p-2}). \end{aligned} \tag{16}$$

*Proof:* By Theorem 1, there a.s. exists a unique global solution  $x(t)$  to system (2) on  $t \in [0, \infty)$  a.s. Let  $\bar{V}(x) = |x|^p$ , by the Itô formula, we have

$$d\bar{V}(t, x(t), i) = \mathcal{L}\bar{V}(t, x_t, i)dt + p|x|^{p-2}x^T [g(t, x_t, i)dB_1(t) + h(t, x_t, i)dB_2(t)],$$

where

$$\begin{aligned} \mathcal{L}\bar{V}(t, x_t, i) &= p|x|^{p-2}x^T f(t, x_t, i) + \frac{p(p-2)}{2}|x|^{p-4} \left( |x^T g(t, x_t, i)|^2 + |x^T h(t, x_t, i)|^2 \right) \\ &\quad + \frac{p}{2}|x|^{p-2} \left( |g(t, x_t, i)|^2 + |h(t, x_t, i)|^2 \right). \end{aligned}$$

By (H<sub>1</sub>) and (H<sub>3</sub>), we have

$$\begin{aligned} \mathcal{L}\bar{V}(t, x_t, i) \leq & p|x|^p \left[ \lambda_i |x|^\alpha + \mu_i \int_{-\tau}^0 |x(t+\theta)| d\eta(\theta) + o(|x|^\alpha) \right] \\ & + p(p-1)|x|^{p-2} \left\{ \left[ \hat{w}_i^2 |x|^{2\beta} + \bar{\gamma}_i^2 \int_{-\tau}^0 |x(t+\theta)|^{2\beta} d\eta(\theta) + o(|x|^{2\beta}) \right] \right. \\ & \left. + \left[ \hat{q}_i^2 |x|^2 + \bar{\sigma}_i^2 \int_{-\tau}^0 |x(t+\theta)|^2 d\eta(\theta) + o(|x|^2) \right] \right\}. \end{aligned}$$

Let  $\tau_k$  be the same stopping time as defined in the proof of Theorem 1. By (13) and (14), we have

$$0 \leq |\xi(0)|^p + E \int_0^{\tau_k \wedge T} K_{r(t)} dt - E \int_0^{\tau_k \wedge T} |x(t)|^p dt.$$

Then as  $k \rightarrow \infty$ , we have

$$E \int_0^T |x(t)|^p dt \leq |\xi(0)|^p + E \int_0^T K_{r(t)} dt.$$

That is,

$$\frac{1}{T} E \int_0^T |x(t)|^p dt \leq \frac{|\xi(0)|^p}{T} + E \left( \frac{1}{T} \int_0^T K_{r(t)} dt \right). \quad (17)$$

By the ergodic and irreducibility property of the Markov chain, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T K_{r(t)} dt = \sum_{i=1}^N \pi_i K_i.$$

Hence,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T E |x(t)|^p dt \leq \sum_{i=1}^N \pi_i K_i,$$

$$\varphi_i = \max_{x \geq 0} \left\{ (\lambda_i + \mu_i) x^\alpha - \frac{1}{2} (2\bar{\gamma}_i^2 + 2\bar{w}_i \hat{\gamma}_i + 2\hat{w}_i^2 - \bar{w}_i^2) x^{2\beta+2} + (\hat{q}_i^2 + 2\bar{\sigma}_i^2 + 2\bar{q}_i \hat{\sigma}_i - \bar{q}_i^2) \right\}, \quad (20)$$

then for any initial value  $x_0 = \xi \in C$ , satisfying  $x(0) \neq 0$ , the global solution of system (2) has the property that

$$\limsup_{t \rightarrow \infty} \frac{\ln |x(t)|}{t} \leq -(\bar{q}_i^2 - \varphi_i), \text{ a.s.} \quad (21)$$

$$\begin{aligned} \ln |x(t)| = & \ln |\xi(0)| + \int_0^t |x(s)|^{-2} x^T(s) f(s, x_s, r(s)) ds + M_1(t) \\ & + M_2(t) + \frac{1}{2} \int_0^t |x(s)|^{-2} \left[ |g(s, x_s, r(s))|^2 + |h(s, x_s, r(s))|^2 \right] ds \\ & - \int_0^t |x(s)|^{-4} \left[ |x^T(s) g(s, x_s, r(s))|^2 + |x^T(s) h(s, x_s, r(s))|^2 \right] ds, \end{aligned}$$

where  $I$  is an identity matrix and

$$\begin{aligned} M_1(t) &= \int_0^t |x(s)|^{-2} x^T(s) g(s, x_s, r(s)) dB_1(s), \\ M_2(t) &= \int_0^t |x(s)|^{-2} x^T(s) h(s, x_s, r(s)) dB_2(s). \end{aligned}$$

Clearly  $M_1(t)$  and  $M_2(t)$  are continuous local

as required.

### 3.2. Stabilization of Noise

The following lemma can be obtained by slightly modifying the proof of Mao [2].

*Lemma 2:* Let (H<sub>1</sub>) - (H<sub>3</sub>) hold, for any initial value  $x_0 = \xi \in C$  with  $x(0) \neq 0$ , the global solution of system (2) has the property that

$$P\{x(t) \neq 0 \text{ on } t \geq 0\} = 1. \quad (18)$$

where  $\tau_e$  is the explosion time.

*Theorem 4:* Let (H<sub>1</sub>) - (H<sub>3</sub>) hold, assume that

$$2\bar{\gamma}_i^2 - 2\bar{w}_i \hat{\gamma}_i + 2\hat{w}_i^2 < \bar{w}_i^2 \quad \text{and} \quad \beta > \frac{\alpha}{2} + 1.$$

If

$$\bar{q}_i^2 - \varphi_i > 0, \quad (19)$$

where

That is, the solution to system (2) is a.s. exponentially stable.

*Proof:* By Lemma 2 and Theorem 1,  $x(t) \neq 0, \forall t \in \mathbb{R}_+$  a.s. Thus, applying the Itô formula to  $\ln |x(t)|$  yields

martingales with the quadratic variation

$$\begin{aligned} \langle M_1, M_1 \rangle_t &= \int_0^t |x(s)|^{-4} |x^T(s) g(s, x_s, r(s))|^2 ds, \\ \langle M_2, M_2 \rangle_t &= \int_0^t |x(s)|^{-4} |x^T(s) h(s, x_s, r(s))|^2 ds. \end{aligned}$$

By (H<sub>3</sub>),

$$\limsup_{t \rightarrow \infty} \frac{\langle M_2, M_2 \rangle_t}{t} \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t |x(s)|^{-2} \left[ 2\hat{q}_i^2 |x|^2 + 2\bar{\sigma}_i^2 \int_{-\tau}^0 |x(s+\theta)|^2 d\eta(\theta) + o(|x|^2) \right] ds \leq 2\hat{q}_i^2 + 2\bar{\sigma}_i^2 < \infty.$$

Applying the strong law of large number,

$$\lim_{t \rightarrow \infty} \frac{M_2(t)}{t} = 0, \text{ a.s.}$$

For any  $\zeta > 1$  and each integer  $m > 0$ , by the exponential martingale inequality,

$$M_1(t) \leq \frac{1}{2} \int_0^t |x(s)|^{-4} |x^\top(s)g(s, x_s, r(s))|^2 ds + \zeta \ln(t+1).$$

From (H<sub>1</sub>) and (H<sub>3</sub>),

$$\begin{aligned} \ln|x(t)| &\leq \ln|\xi| + M_2(t) + \zeta \ln(t+1) + \int_0^t \left[ \lambda_i |x|^\alpha + \mu_i \int_{-\tau}^0 |x(s+\theta)|^\alpha d\eta(\theta) + o(|x|^\alpha) \right] ds \\ &\quad - \frac{1}{2} \int_0^t |x(s)|^{-4} \left[ \bar{w}_i^2 |x|^{2\beta+2} - 2\bar{w}_i \hat{\gamma}_i |x|^{\beta+1} \int_{-\tau}^0 |x(s+\theta)|^{\beta+1} d\eta(\theta) + o(|x|^{2\beta+2}) \right] ds \\ &\quad - \int_0^t |x(s)|^{-4} \left[ \bar{q}_i^2 |x|^4 - 2\bar{q}_i \hat{\sigma}_i |x|^2 \int_{-\tau}^0 |x(s+\theta)|^2 d\eta(\theta) + o(|x|^4) \right] ds \\ &\quad + \frac{1}{2} \int_0^t |x(s)|^{-2} \left[ 2\hat{w}_i^2 |x|^{2\beta} + 2\bar{\gamma}_i^2 \int_{-\tau}^0 |x(s+\theta)|^{2\beta} d\eta(\theta) \right. \\ &\quad \left. + o(|x|^{2\beta}) + 2\hat{q}_i^2 |x|^2 + 2\bar{\sigma}_i^2 \int_{-\tau}^0 |x(s+\theta)|^2 d\eta(\theta) + o(|x|^2) \right] ds \\ &\leq \ln|\xi| + M_2(t) + \zeta \ln(t+1) + \int_0^t \bar{H}(s) ds + \mu_i \int_0^t \left[ \int_{-\tau}^0 |x(s+\theta)|^\alpha d\eta(\theta) - |x|^\alpha \right] ds \\ &\quad + \frac{1}{2} (2\bar{\gamma}_i^2 + \bar{w}_i \hat{\gamma}_i) \int_0^t \left[ \int_{-\tau}^0 |x(s+\theta)|^{2\beta+2} d\eta(\theta) - |x|^{2\beta+2} \right] ds \\ &\quad + (\bar{\sigma}_i^2 + \bar{q}_i \hat{\sigma}_i) \int_0^t \left[ \int_{-\tau}^0 |x(s+\theta)|^\alpha d\eta(\theta) - |x|^\alpha \right] ds, \end{aligned}$$

where

$$\bar{H}(x) = (\lambda_i + \mu_i) |x|^\alpha - \frac{1}{2} (2\bar{\gamma}_i^2 - 2\bar{w}_i \hat{\gamma}_i + 2\hat{w}_i^2 - \bar{w}_i^2) |x|^{2\beta+2} + (\hat{q}_i^2 + 2\bar{\sigma}_i^2 + 2\bar{q}_i \hat{\sigma}_i - \bar{q}_i^2) |x|^\alpha + o(|x|^{2\beta+2}).$$

By the definition of  $\varphi_i$  in (20),

$$\ln|x(t)| \leq \ln|\xi| - (\bar{q}_i^2 - \varphi_i) t + M_2(t, r) + \zeta \ln(t+1).$$

Applying the strong law of large number to the Brownian motion,

$$\lim_{t \rightarrow \infty} \frac{M_2(t)}{t} = 0, \text{ a.s.}$$

which implies

$$\limsup_{t \rightarrow \infty} \frac{\ln|x(t)|}{t} \leq -(\bar{q}_i^2 - \varphi_i), \text{ a.s.}$$

### 4. Conclusion

In this paper, we study a stochastic functional system with Markovian switching. Motivated by [25,26] and other literatures, we introduce two appropriate intensity Brownian noise to perturb the system so as to suppress its potential explosion and stabilize it. Based on [28], we just slightly modify some conditions on its coefficients

$$P \left\{ \sup_{1 \leq t \leq m} \left[ M_1(t) - \frac{1}{2} \langle M_1, M_1 \rangle_t \right] \geq \zeta \ln m \right\} \leq \frac{1}{m^\zeta}.$$

Since  $\sum_{m=1}^\infty m^{-\zeta} < \infty$ , by the Borel-Cantelli lemma, there exists an  $\bar{\Omega}_0 \subseteq \Omega$  with  $P(\bar{\Omega}_0) = 1$  such that for any  $\omega \in \bar{\Omega}_0$ , when  $m-1 \leq t \leq m$ ,

and add some contents, then we get some new conclusions about boundedness and stabilization of the system.

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