Applied Mathematics, 2013, 4, 35-40
Published Online November 2013 (http://www.scirp.org/journal/am)
http://dx.doi.org/10.4236/am.2013.411A3006

Scientific Research

Canonical and Boundary Representations on Rank One Para-Hermitian Spaces

Anatoli A. Artemov
G. R. Derzhavin Tambov State University, Tambov, Russia
Email: tria@tsu.tmb.ru

Received July 24, 2013; revised August 24, 2013; accepted September 2, 2013

ABSTRACT

This work studies the canonical representations (Berezin representations) for para-Hermitian symmetric spaces of rank one. These spaces are exhausted up to the covering by spaces G/H with $G=\mathrm{SL}(n,\mathbb{R})$, $H=\mathrm{GL}(n-1,\mathbb{R})$. For Hermitian symmetric spaces G/K, canonical representations were introduced by Berezin and Vershik-Gelfand-Graev. They are unitary with respect to some invariant non-local inner product (the Berezin form). We consider canonical representations in a wider sense: we give up the condition of unitarity and let these representations act on spaces of distributions. For our spaces G/H, the canonical representations turn out to be tensor products of representations of maximal degenerate series and contragredient representations. We decompose the canonical representations into irreducible constituents and decompose boundary representations.

Keywords: Para-Hermitian Symmetric Spaces; Overgroups; Canonical Representations; Boundary Representations; Poisson and Fourier Transforms

1. Introduction

Canonical representations on Hermitian symmetric spaces G/K were introduced by Vershik-Gelfand-Graev [1] (for the Lobachevsky plane) and Berezin [2]. They are unitary with respect to some invariant non-local inner product (the Berezin form). Molchanov's idea is that it is natural to consider canonical representations in a wider sense: to give up the condition of unitarity and let these representations act on sufficiently extensive spaces, in particular, on distributions. Moreover, the notion of canonical representation (in this wide sense) can be extended to other classes of semisimple symmetric spaces G/H, in particular, to para-Hermitian symmetric spaces, see [3]. Moreover, sometimes it is natural to consider several spaces G/H_i together, possibly with different H_i, embedded as open G-orbits into a compact manifold Ω, where G acts, so that Ω is the closure of these orbits.

Canonical representations can be constructed as follows. Let $\tilde{G}$ be a group containing G (an overgroup), $\tilde{R}$ a series of representations of $\tilde{G}$ induced by characters of some parabolic subgroup $\tilde{P}$ associated with G/H and acting on functions on Ω. The canonical representations R of G are restrictions of $\tilde{R}$ to G.

In this talk we carry out this program for para-Hermitian symmetric spaces of rank one. These spaces are exhausted up to the covering by spaces G/H with $G=\mathrm{SL}(n,\mathbb{R})$, $H=\mathrm{GL}(n-1,\mathbb{R})$. For these spaces G/H, an overgroup is the direct product $G\times G$ and canonical representations turn out to be tensor products of representations of maximal degenerate series and contragredient representations. These tensor products are studied in [4], see also [5]. So we lean essentially on these papers [4,5]. We decompose canonical representations into irreducible constituents and decompose boundary representations. Notice that in our case the inverse of the Berezin transform $Q_{\mu,\nu}$ can be easily written: precisely it is the Berezin transform $Q_{-\mu-n,\nu}$.

Canonical and boundary representations for G/H in the case $n=2$ (then G/H is the hyperboloid of one sheet in $\mathbb{R}^3$) were studied in [6]. For the two-sheeted hyperboloid in $\mathbb{R}^3$, it was done in [7].

In this paper we present only the main results. The detailed theory of canonical and boundary representations, for example, on a sphere with an action of the generalized Lorentz group, can be seen in [8].

Let us introduce some notation and agreements.

By $\mathbb{N}$ we denote $\{0,1,2,\cdots\}$. The sign $\equiv$ denotes the congruence modulo 2.

For a character of the group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ we shall use the following notation

$$t^{\lambda,\varepsilon} = |t|^{\lambda} \operatorname{sgn}^{\varepsilon} t,$$

where $t \in \mathbb{R}^*$, $\lambda \in \mathbb{C}$, $\varepsilon = 0,1$.

For a manifold M, let $\mathcal{D}(M)$ denote the Schwartz space of compactly supported infinitely differentiable $\mathbb{C}$-valued functions on M, with a usual topology, and $\mathcal{D}'(M)$ the space of distributions on M—of anti-linear continuous functionals on $\mathcal{D}(M)$.

2. The Space G/H and the Manifold Ω

We consider the symmetric space G/H where $G = \mathrm{SL}(n,\mathbb{R})$, $H = \mathrm{GL}(n-1,\mathbb{R})$, $n \geq 3$.

The group G acts on the space $\mathrm{Mat}(n,\mathbb{R})$ by

$$x \mapsto g^{-1}xg.$$

Let us write matrices in $\mathrm{Mat}(n,\mathbb{R})$ in block form according to the partition $n = (n-1)+1$ of n. Let us take the matrix

$$x^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The subgroup H is just the stabilizer of this point x^0, this subgroup consists of block diagonal matrices:

$$h = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}, \ \alpha \in \mathrm{GL}(n-1,\mathbb{R}), \ \delta = (\det\alpha)^{-1}.$$

Thus, our space G/H is the G-orbit of x^0, it consists of matrices of rank one and trace one.

Equip $\mathbb{R}^n$ with the standard inner product $\langle x,y\rangle$, let $|x| = \langle x,x\rangle^{1/2}$. Let S be the sphere $|s|=1$. Let $\mathrm{d}s$ be the Euclidean measure on S. The group G acts on S by $s \mapsto sg/|sg|$.

Let $\mathcal{C}$ be a cone in $\mathrm{Mat}(n,\mathbb{R})$ consisting of matrices $x \neq 0$ of rank one. Therefore, the space G/H is the section of $\mathcal{C}$ by the hyperplane $\mathrm{tr}\,x = 1$.

Introduce a norm $\|x\|$ in $\mathrm{Mat}(n,\mathbb{R})$ by

$$\|x\| = \{\mathrm{tr}(xx')\}^{1/2},$$

where the prime denotes matrix transposition.

Let Ω be the section of $\mathcal{C}$ by $\|x\|=1$.

Define a map $S \times S \to \Omega$ by

$$(s,t) \mapsto t's = u.$$

It is a two-fold covering. The measure $\mathrm{d}s$ defines a measure $\mathrm{d}u$ on Ω by

$$\int_{\Omega} f(u)\,\mathrm{d}u = \frac{1}{2}\int_{S\times S} f(t's)\,\mathrm{d}s\mathrm{d}t, \ u = t's.$$

The action of the group G on S gives the following action of G on Ω:

$$u \mapsto \frac{g^{-1}ug}{\|g^{-1}ug\|}.$$

In particular, the subgroup $K = \mathrm{SO}(n)$, a maximal compact subgroup, acts on Ω by translations:

$$u \mapsto k^{-1}uk.$$

Let us consider on Ω the function

$$p = p(u) = \langle s,t\rangle, \ u = t's. \tag{1}$$

The action on Ω has three orbits: namely, two open orbits (of dimension $2n-2$): $\Omega^+ = \{p>0\}$ and $\Omega^- = \{p<0\}$ and one orbit of dimension $2n-3$: $\Gamma = \{p=0\}$. The orbit Γ is a Stiefel manifold, it is the boundary of $\Omega^{\pm}$. Denote $\Omega' = \Omega^+ \cup \Omega^-$. Each of orbits $\Omega^{\pm}$ can be identified with the space G/H. The map is constructed by means of generating lines of the cone $\mathcal{C}$.

3. Maximal Degenerate Series Representations

Recall [4] maximal degenerate series representations $\pi^{\pm}_{\mu,\nu}$, $\mu \in \mathbb{C}$, $\nu = 0,1$, of the group G. Let $\mathcal{D}_{\nu}(S)$ be the subspace of $\mathcal{D}(S)$ consisting of functions φ of parity ν: $\varphi(-s) = (-1)^{\nu}\varphi(s)$. The representations $\pi^{\pm}_{\mu,\nu}$ act on $\mathcal{D}_{\nu}(S)$ by

$$\left(\pi^{-}_{\mu,\nu}(g)\varphi\right)(s) = \varphi\left(\frac{sg}{|sg|}\right)|sg|^{\mu},$$

$$\left(\pi^{+}_{\mu,\nu}(g)\varphi\right)(s) = \varphi\left(\frac{sg'^{-1}}{|sg'^{-1}|}\right)|sg'^{-1}|^{\mu}.$$

4. Representations of G Associated with G/H

Recall [5] a series of representations $T_{\sigma,\varepsilon}$ of the group G associated with the space G/H.

Denote by $\mathcal{D}_{\varepsilon}(\Gamma)$ the space of functions φ in $\mathcal{D}(\Gamma)$ of parity $\varepsilon = 0,1$:

$$\varphi(-\gamma) = (-1)^{\varepsilon}\varphi(\gamma).$$

The representation $T_{\sigma,\varepsilon}$ acts on $\mathcal{D}_{\varepsilon}(\Gamma)$ by

$$T_{\sigma,\varepsilon}(g)\varphi(\gamma) = \varphi\left(\frac{g^{-1}\gamma g}{\|g^{-1}\gamma g\|}\right)\|g^{-1}\gamma g\|^{\sigma}. \tag{2}$$

Let $\langle\psi,\varphi\rangle_{\Gamma}$ denote the following sesqui-linear form

$$\langle\psi,\varphi\rangle_{\Gamma} = \int_{\Gamma}\psi(\gamma)\overline{\varphi(\gamma)}\mathrm{d}\gamma. \tag{3}$$

Define an operator $A_{\sigma,\varepsilon}$ on $\mathcal{D}_{\varepsilon}(\Gamma)$ by

$$A_{\sigma,\varepsilon}\varphi(\gamma) = \int_{\Gamma}\{\mathrm{tr}(\gamma\beta)\}^{1-n-\sigma,\varepsilon}\varphi(\beta)\mathrm{d}\beta.$$

It intertwines $T_{\sigma,\varepsilon}$ and $T_{1-n-\sigma,\varepsilon}$. The operator $A_{\sigma,\varepsilon}$ is a meromorphic function of σ. Let us normalize this operator (multiplying it by a function of σ) such that the normalized operator $\tilde{A}_{\sigma,\varepsilon}$ is an entire non-vanishing function of σ.

There are three series of unitarizable irreducible representations. The *continuous series* consists of $T_{\sigma,\varepsilon}$ with $\sigma=(1-n)/2+i\rho$, $\rho\in\mathbb{R}$, the inner product is (3). The *complementary series* consists of $T_{\sigma,\varepsilon}$ with $-n/2<\sigma<(2-n)/2$, the inner product is $\langle A_{\sigma,\varepsilon}\psi,\varphi\rangle_\Gamma$ with a factor. The *discrete series* consists of the representations $\tilde{T}_{\sigma(m),\varepsilon}$ where $\sigma(m)=(2-n)/2+m$, $m\in\mathbb{N}$, $\varepsilon=0,1$, which are factor representations of $T_{\sigma(m),\varepsilon}$ on the quotient spaces $\mathcal{D}_\varepsilon(\Gamma)/\mathrm{Ker}\tilde{A}_{\sigma(m),\varepsilon}$. The representations $\tilde{T}_{\sigma(m),\varepsilon}$ with the same m and different ε are equivalent. It is convenient to take $\varepsilon=\varepsilon(m)$ where $\varepsilon(m)=0$ for odd n and $\varepsilon(m)\equiv\sigma(m)+1$ for even n. The inner product is induced by the form $\langle A_{\sigma(m),\varepsilon(m)}\psi,\varphi\rangle_\Gamma$.

5. Canonical Representations

We define *canonical representations* $R_{\mu,\nu}$, $\mu\in\mathbb{C}$, $\nu=0,1$, of the group G as tensor products:

$$R_{\mu,\nu}=\pi^-_{-\mu-n}\otimes\pi^+_{-\mu-n}.$$

They can be realized on Ω: let $\mathcal{D}_\nu(\Omega)$ denote the subspace of $\mathcal{D}(\Omega)$ consisting of functions f of parity ν: $f(-u)=(-1)^\nu f(u)$, then the representation $R_{\mu,\nu}$ acts on $\mathcal{D}_\nu(\Omega)$ by a formula similar to (2):

$$R_{\mu,\nu}(g)f(u)=f\left(\frac{g^{-1}ug}{\|g^{-1}ug\|}\right)\|g^{-1}ug\|^{-\mu-n}.$$

The inner product

$$\langle f,h\rangle_\Omega=\int_\Omega f(u)\overline{h(u)}\mathrm{d}u \tag{4}$$

is invariant with respect to the pair $(R_{\mu,\nu},R_{-\bar{\mu}-n,\nu})$, *i.e.*

$$\langle R_{\mu,\nu}(g)f,h\rangle_\Omega=\langle f,R_{-\bar{\mu}-n,\nu}(g^{-1})h\rangle_\Omega. \tag{5}$$

Consider an operator $Q_{\mu,\nu}$ on $\mathcal{D}_\nu(\Omega)$ defined by

$$Q_{\mu,\nu}f(u)=c(\mu,\nu)\int_\Omega\{\mathrm{tr}(uv)\}^{\mu,\nu}f(v)\mathrm{d}v,$$

It turns out that the composition $Q_{-\mu-n,\nu}Q_{\mu,\nu}$ is equal to the identity operator E up to a factor. We can take $c(\mu,\nu)$ such that

$$Q_{-\mu-n,\nu}Q_{\mu,\nu}=E,$$

namely,

$$\frac{1}{c(\mu,\nu)}=2^n\pi^{n-2}\Gamma(-\mu-n-1)\Gamma(\mu+1)\times\left[\cos\left(\mu+\frac{n}{2}\right)\pi-\cos\left(\nu+\frac{n}{2}\right)\pi\right].$$

With the form (4) the operator $Q_{\mu,\nu}$ interacts as follows:

$$\langle Q_{\mu,\nu}f,h\rangle_\Omega=\langle f,Q_{\bar{\mu},\nu}h\rangle_\Omega. \tag{6}$$

This operator $Q_{\mu,\nu}$ intertwines the representations $R_{\mu,\nu}$ and $R_{-\mu-n,\nu}$, *i.e.*

$$R_{-\mu-n,\nu}(g)Q_{\mu,\nu}=Q_{\mu,\nu}R_{\mu,\nu}(g),\ g\in G.$$

Let us call it the *Berezin transform*.

Let $\mathcal{D}'_\nu(\Omega)$ be the space of distributions on Ω of parity ν. We extend $R_{\mu,\nu}$ and $Q_{\mu,\nu}$ to $\mathcal{D}'_\nu(\Omega)$ by (5) and (6) respectively and retain their names and the notation.

Let us introduce the following Hermitian form $(f,h)_{\mu,\nu}$ on $\mathcal{D}_\nu(\Omega)$:

$$(f,h)_{\mu,\nu}=\frac{1}{2}\langle Q_{\mu,\nu}f,h\rangle_\Omega$$

Let us call this form the *Berezin form*.

6. Boundary Representations

The canonical representation $R_{\lambda,\nu}$ gives rise to two representations $L_{\lambda,\nu}$ and $M_{\lambda,\nu}$ associated with the boundary Γ of the manifolds $\Omega^\pm$ (*boundary representations*). The first one acts on distributions concentrated at Γ, the second one acts on jets orthogonal to Γ.

We can introduce "polar coordinates" on Ω corresponding to the foliation of Ω into K-orbits. The K-orbits are level surfaces of the function p, see (1). For $-1<p<1$ the K-orbits are diffeomorphic to Γ. In these coordinates the measure $\mathrm{d}u$ on Ω is

$$\mathrm{d}u=(1-p^2)^{(n-3)/2}\mathrm{d}p\mathrm{d}\gamma,$$

where $\mathrm{d}\gamma$ is the measure on Γ.

Let f be a function in $\mathcal{D}_\nu(\Omega)$. Consider it as a function of polar coordinates. Consider its Taylor series $a_0+a_1p+a_2p^2+\cdots$ in powers of p. Here $a_m=a_m(f)$ are functions in $\mathcal{D}(\Gamma)$. Denote by $\Sigma^k_\nu(\Omega)$, $k\in\mathbb{N}$, $\nu=0,1$, the space of distributions in $\mathcal{D}'_\nu(\Omega)$, having the form

$$\sum_{m=0}^k\varphi_m(\gamma)\delta^{(m)}(p),$$

where $\varphi_m\in\mathcal{D}_{\nu-m}(\Gamma)$, δ is the Dirac delta function on the real line, $\delta^{(m)}$ its derivatives. Let $\Sigma_\nu(\Omega)=\bigcup\Sigma^k_\nu(\Omega)$.

Denote by $a^*_m(f)$ Taylor coefficients of the function

$\left(1-p^2\right)^{(n-3)/2} f(u)$. The distribution $\varphi(\gamma)\delta^{(m)}(p)$ acts on a function $f\in\mathcal{D}_\nu(\Omega)$ as follows:

$$\left\langle\varphi\delta^{(m)}(p),f\right\rangle_\Omega=(-1)^m m!\left\langle\varphi,a_m^*(f)\right\rangle_\Gamma. \quad (7)$$

Denote by $L_{\mu,\nu}$ the restriction of $R_{\mu,\nu}$ to $\Sigma_\nu(\Omega)$. This representation is written as a upper triangular matrix with the diagonal $T_{1-n-\mu+m,\nu-m}$, $m\in\mathbb{N}$.

Distributions in $\Sigma_\nu^k(\Omega)$ can be extended in a natural way to a space wider than $\mathcal{D}_\nu(\Omega)$. Namely, let $\mathcal{T}_\nu^k(\Omega)$ be the space of functions f of class C^∞ on $\Omega^\pm$ and Γ of parity ν and having the Taylor decomposition of order k:

$$f(u)=a_0+a_1p+\cdots+a_kp^k+o\left(p^k\right),$$

where $a_m\in\mathcal{D}(\Gamma)$. Then (7) keeps for $f\in\mathcal{T}_\nu^k(\Omega)$ with $m\le k$.

Let $a(f)$ denote the column of Taylor coefficients $a_m(f)$. The representation $M_{\mu,\nu}$ acts on these columns:

$$M_{\mu,\nu}(g)a(f)=a\left(R_{\mu,\nu}(g)f\right).$$

It is written as a lower triangular matrix with the diagonal $T_{-n-\mu-m,\nu-m}$, $m\in\mathbb{N}$.

The boundary representations $L_{\mu,\nu}$ and $M_{\mu,\nu}$ are in a duality.

7. Poisson and Fourier Transforms

Let us write operators $P_{\mu,\nu;\sigma,\varepsilon}$ and $F_{\mu,\nu;\sigma,\varepsilon}$ intertwining representations $R_{\mu,\nu}$ and $T_{\sigma,\varepsilon}$. We call them *Poisson and Fourier transforms associated with canonical representations*.

The Poisson transform $P_{\mu,\nu;\sigma,\varepsilon}$ is a map $\mathcal{D}(\Gamma)\to C^\infty(\Omega')$ given by

$$P_{\mu,\nu;\sigma,\varepsilon}\varphi(u)=p^{-\mu-\sigma-n,\nu-\varepsilon}\int_\Gamma\{\mathrm{tr}(u\gamma)\}^{\sigma,\varepsilon}\varphi(\gamma)\mathrm{d}\gamma.$$

It intertwines $T_{1-\sigma-n,\varepsilon}$ with $R_{\mu,\nu}$. Here we consider $R_{\mu,\nu}$ as the restriction to $C^\infty(\Omega')$ of the representation $R_{\mu,\nu}$ acting on distributions in $\mathcal{D}'(\Omega)$.

For a K-finite function $\varphi\in\mathcal{D}_\nu(\Gamma)$ and $\sigma\notin(1-n)/2+\mathbb{Z}$ the Poisson transform has the following decomposition in powers of p:

$$\left(P_{\mu,\nu;\sigma,\varepsilon}\varphi\right)(u)=p^{-\mu-\sigma-n,\nu-\varepsilon}\sum_{k=0}^\infty\left(C_{\sigma,\varepsilon,k}\varphi\right)(\gamma)\cdot p^k$$
$$+p^{-\mu+\sigma-1,\nu-\varepsilon}\sum_{k=0}^\infty\left(D_{\sigma,\varepsilon,k}\varphi\right)(\gamma)\cdot p^k,$$

where u has polar coordinates p,γ. Here $C_{\sigma,\varepsilon,k}$ and $D_{\sigma,\varepsilon,k}$ are certain operators acting on $\mathcal{D}_\nu(\Gamma)$. The factors $p^{-\mu-\sigma-n,\nu-\varepsilon}$ and $p^{-\mu+\sigma-1,\nu-\varepsilon}$ give poles of the Poisson transform in σ depending on μ:

$$\sigma=\mu-k,\quad \sigma=1-n-\mu+l, \quad (8)$$

where $k,l\in\mathbb{N}$ and $k\equiv\nu-\varepsilon$, $l\equiv\nu-\varepsilon$. If a pole belongs only to one of series (8), then the pole is simple, and if a pole belongs to both series (8), then $\mu\in(1-n)/2+\mathbb{N}$ and the pole is of the second or first order.

Let the pole $\sigma=\mu-k$, $k\equiv\nu-\varepsilon$, be simple. The residue $\hat{P}_{\mu,\nu,\mu-k}$ of $P_{\mu,\nu;\sigma,\varepsilon}$ at this pole is an operator $\mathcal{D}(\Gamma)\to\Sigma_\nu^k(\Omega)$. Denote the image of this operator by $V_{\mu,\nu,k}$.

The Fourier transform $F_{\mu,\nu;\sigma,\varepsilon}$ is a map $\mathcal{D}_\nu(\Omega)\to\mathcal{D}_\varepsilon(\Gamma)$ given by

$$F_{\mu,\nu;\sigma,\varepsilon}f(\gamma)=\int_\Omega\{\mathrm{tr}(u\gamma)\}^{\sigma,\varepsilon}p^{\mu-\sigma,\nu-\varepsilon}f(u)\mathrm{d}u.$$

It intertwines $R_{\mu,\nu}$ with $T_{\sigma,\varepsilon}$.

The Fourier and Poisson transforms are conjugate to each other:

$$\left\langle F_{\mu,\nu;\sigma,\varepsilon}f,\varphi\right\rangle_\Gamma=\left\langle f,P_{-\bar{\mu}-n,\nu;\bar{\sigma},\varepsilon}\varphi\right\rangle_\Omega.$$

Poles in σ of the Fourier transform are situated at points

$$\sigma=-n-\mu-k,\quad \sigma=\mu+1+l, \quad (9)$$

where $k,l\in\mathbb{N}$ and $k\equiv\nu-\varepsilon$, $l\equiv\nu-\varepsilon$. If a pole belongs only to one of the series (9), then the pole is simple, and if a pole belongs to both series (9), then $\mu\in(-1-n)/2-\mathbb{N}$ and the pole is of the second or first order.

Let the pole $\sigma=-n-\mu-k$, $k\equiv\nu-\varepsilon$, be simple. The residue $\hat{F}_{\mu,\nu,-n-\mu-k}$ of $F_{\mu,\nu;\sigma,\varepsilon}$ at this pole is a "boundary" operator $B_{\mu,\nu,k}:\mathcal{D}_\nu(\Omega)\to\mathcal{D}(\Gamma)$, $k\equiv\nu$. The operator $B_{\mu,\nu,k}$ is defined in terms of Taylor coefficients $a_m(f)$: it is a linear combination of functions $D_{-\mu-n-k,\nu,k-m}\left(a_m^*(f)\right)$. Therefore, we may consider the following operator $B_{\mu,\nu}$ acting on columns $a=(a_0,a_1,a_2,\cdots)$ of functions $a_k\in\mathcal{D}_\nu(\Gamma)$: this operator to any column a assigns the column $B_{\mu,\nu}a=\left(B_{\mu,\nu,0}a,B_{\mu,\nu,1}a,B_{\mu,\nu,2}a,\cdots\right)$ of functions in the same space $\mathcal{D}_\nu(\Gamma)$—by the same formulas without f. This operator $B_{\mu,\nu}$ is given by a lower triangular matrix.

8. Decomposition of Boundary Representations

The meromorphic structure of the Poisson and Fourier transforms is a basis for decompositions of boundary representations $L_{\mu,\nu}$ and $M_{\mu,\nu}$.

Let the pole $\sigma=\mu-k$ of the Poisson transform is simple, in particular, it happens when $\mu\notin(1-n)/2+\mathbb{N}$.

Then the boundary representation $L_{\mu,\nu}$ is diagonalizable which means that $\Sigma_\nu(\Omega)$ decomposes into the direct sum of $V_{\mu,\nu,k}, k \in \mathbb{N}$, and the restriction of $L_{\mu,\nu}$ to $V_{\mu,\nu,k}$ is equivalent to $T_{1-n-\mu+k,\varepsilon+k}$ (by means of $\hat{P}_{\mu,\nu,\mu-k}$).

If a pole is of the second order, then the decomposition of $L_{\mu,\nu}$ contains a finite number of Jordan blocks, this number depends on μ.

Let the pole $\sigma = -n - \mu - k$ of the Fourier transform is simple, in particular, when $\mu \notin (-1-n)/2 - \mathbb{N}$. Then the matrix $M_{\mu,\nu}$ is diagonalizable which means that $B^{-1}_{\mu,\nu} M_{\mu,\nu} B_{\mu,\nu}$ is a diagonal matrix. Its diagonal is $T_{-n-\mu-k,\nu-k}$, $k \in \mathbb{N}$.

If a pole is of the second order, then the decomposition of $M_{\mu,\nu}$ contains a finite number of Jordan blocks, this number depends on μ.

9. Decomposition of Canonical Representations

Let us write decomposition of canonical representations. We restrict ourselves to a generic case: μ lies in strips

$$I_k = \left\{ \mu : \frac{-n-1}{2} + k < \operatorname{Re}\mu < \frac{-n+1}{2} + k \right\},\ k \in \mathbb{Z}.$$

Case (A): $\mu \in I_0$.

Theorem 1 *Let* $\mu \in I_0$. *Then the canonical representation* $R_{\mu,\nu}$ *decomposes—as the quasiregular representation* [5]*—into irreducible unitary representations of continuous and discrete series with multiplicity one. Namely, let us assign to a function* $f \in \mathcal{D}_\nu(\Omega)$ *the family of its Fourier components* $F_{\mu,\nu;\sigma,\varepsilon} f$, $\sigma = (1-n)/2 + i\rho$, $\rho \in \mathbb{R}$, $\varepsilon = 0,1$, *and* $F_{\mu,\nu;1-n-\sigma(m),\varepsilon(m)}$, $m \in \mathbb{N}$. *This correspondence is* G*-equivariant. There is an inversion formula*:

$$f = \int_{-\infty}^{\infty} \omega(\sigma,\varepsilon) \sum_\varepsilon P_{\mu,\nu;1-n-\sigma,\varepsilon} F_{\mu,\nu;\sigma,\varepsilon} f \Big|_{\sigma=(1-n)/2+i\rho} \mathrm{d}\rho + \sum_{m=0}^{\infty} \omega_m P_{\mu,\nu;\sigma(m),\varepsilon(m)} F_{\lambda,\nu;1-n-\sigma(m),\varepsilon(m)} f \tag{10}$$

and a "Plancherel formula" for the Berezin form:

$$\begin{aligned}(f,h)_{\mu,\nu} &= \int_{-\infty}^{\infty} \omega(\sigma,\varepsilon) \sum_\varepsilon \Lambda(\mu,\nu;\sigma,\varepsilon) \\ &\times \left\langle F_{\lambda,\nu;\sigma,\varepsilon} f, F_{\bar{\mu},\nu;1-n-\bar{\sigma},\varepsilon} h \right\rangle_\Gamma \Big|_{\sigma=(1-n)/2+i\rho} \mathrm{d}\rho \\ &+ \sum_{m=0}^{\infty} \omega_m \Lambda(\mu,\nu;\sigma(m),\varepsilon(m)) \\ &\times \left\langle F_{\mu,\nu;1-n-\sigma(m),\varepsilon(m)} f, F_{\bar{\mu},\nu;\sigma(m),\varepsilon(m)} h \right\rangle_\Gamma .\end{aligned} \tag{11}$$

Here $\omega(\sigma,\varepsilon)$ and ω_m stand for the Plancherel measure for G/H, see [5], the factor Λ is given by following formula:

$$\begin{aligned}&\Lambda(\mu,\nu;\sigma,\varepsilon) \\ &= \frac{\Gamma(-\mu+\sigma)\Gamma(-\mu-n+1+\sigma)}{\Gamma(-\mu)\Gamma(-\mu-n+1)} \cdot \frac{1-(-1)^\nu \cos\mu\pi}{\sin\mu\pi} \\ &\times \frac{\sin(\mu+n/2)\pi + (-1)^{\varepsilon+\nu}\sin(\sigma+n/2)\pi}{\cos(n/2)\pi - (-1)^\nu \cos(\mu+n/2)\pi}.\end{aligned}$$

Case (B): $\mu \in I_{k+1}, k \in \mathbb{N}$.

Here we continue decomposition (10) analytically in μ from I_0 to I_{k+1}, $k \in \mathbb{N}$. Some poles in σ of the integrand intersect the integrating line—the line $\operatorname{Re}\sigma = (1-n)/2$. They are poles $\sigma = \mu - m$ and $\sigma = 1 - n - \mu + m$ of the Poisson transform with $0 \le m \le k$. They give additional summands to the right hand side. So after the continuation we obtain:

$$f = \int_{-\infty}^{\infty} + \sum_{m=0}^{\infty} + \sum_{m=0}^{k} \pi_{\mu,\nu,m}(f), \tag{12}$$

where the integral and the series mean the same as in (10) and

$$\pi_{\mu,\nu,m} = L(\mu,\nu,m) \cdot \hat{P}_{\mu,\nu,\mu-m} \circ F_{\mu,\nu;1-n-\mu+m,\nu-m},$$

$L(\mu,\nu,m)$ are some numbers.

Similarly, the continuation of (11) gives

$$\begin{aligned}(f,h)_{\mu,\nu} &= \int_{-\infty}^{\infty} + \sum_{m=0}^{\infty} + \sum_{m=0}^{k} M(\mu,\nu,m) \\ &\times \left\langle F_{\mu,\nu;1-n-\mu+m,\nu-m} f, F_{\bar{\mu},\nu;\bar{\mu}-m,\nu-m} h \right\rangle_\Gamma ,\end{aligned} \tag{13}$$

where the integral and the series mean the same as in (11) and $M(\mu,\nu,m)$ are some numbers.

The operators $\pi_{\mu,\nu,m}$, $m \le k$, can be extended from $\mathcal{D}_\nu(\Omega)$ to the space $\Sigma^k_\nu(\Omega)$ and therefore to the sum

$$\mathcal{D}^k_\nu(\Omega) = \mathcal{D}_\nu(\Omega) + \Sigma^k_\nu(\Omega).$$

Then these operators $\pi_{\mu,\nu,m}$ turn out to be projection operators onto $V_{\mu,\nu,m}$. Moreover, there are some "orthogonality relations" for them. Decomposition (13) can also be extended to the space $\mathcal{D}^k_\nu(\Omega)$. This decomposition is a "Pythagorean theorem" for decomposition (12).

Theorem 2 *Let* $\mu \in I_{k+1}$, $k \in \mathbb{N}$. *Then the space* $\mathcal{D}_\nu(\Omega)$ *has to be completed to* $\mathcal{D}^k_\nu(\Omega)$. *On this space the representation* $R_{\mu,\nu}$ *splits into the sum of two terms*: *the first one decomposes as* $R_{\mu,\nu}$ *does in Case* (*A*), *the second one decomposes into the sum of* $k+1$ *irreducible representations* $T_{1-n-\mu+m,\nu-m}$, $m = 0,1,\cdots,k$. *Namely, let us assign to any* $f \in \mathcal{D}^k_\nu(\Omega)$ *the family*

$$\left\{ F_{\mu,\nu;\sigma,\varepsilon} f, F_{\mu,\nu;1-n-\sigma(m),\varepsilon(m)} f, \pi_{\mu,\nu,m}(f) \right\}$$

where $\sigma=(1-n)/2+i\rho$, $n\in\mathbb{N}$, $m=0,1,\cdots,k$. *This correspondence is* G*-equivariant. There is an inverse formula*, *see* (12), *and a* "*Plancherel formula*", *see* (13).

Case (C): $\mu\in I_{-k-1}, k\in\mathbb{N}$.

Now we continue decomposition (10) analytically in σ from I_0 to I_{-k-1}. Here poles $\sigma=-\mu-n-m$ and $\sigma=\mu+1+m$, $\varepsilon\equiv\nu-m$, $m\le k$, of the integrand (they are poles of the Fourier transform) give additional terms. We obtain

$$f=\int_{-\infty}^{\infty}+\sum_{m=0}^{\infty}+\sum_{m=0}^{k}\Pi_{\mu,\nu,m}(f), \tag{14}$$

where the integral and the series mean the same as in (10) and

$$\Pi_{\mu,\nu,m}=N_{\mu,\nu,m}\cdot P_{\mu,\nu;\mu+1+m,\nu-m}\circ B_{\mu,m},$$

$N_{\mu,\nu,m}$ some numbers. The operators $\Pi_{\mu,\nu,m}$ can be extended to the space $\mathcal{T}_\nu^k(\Omega)$, $m\le k$. Denote by $\mathcal{P}_{\mu,\nu,m}$ the image of $\Pi_{\mu,\nu,m}$. It turns out that the operators $\Pi_{\mu,\nu,m}$ are projection operators onto $\mathcal{P}_{\mu,\nu,m}$ and for them there are some "orthogonality relations".

Now we continue decomposition (11) from I_0 to I_{-k-1}. Poles of the integrand which intersect the integrating line $\mathrm{Re}\,\sigma=(1-n)/2$ and give additional terms (they are poles of both Fourier transforms) turn out fortunately to be of the first order, since at these points the function $\Lambda_{\mu,\nu;\sigma,\varepsilon}$ as a function of σ has zero of the first order. After the continuation we obtain:

$$(f,h)_{\mu,\nu}=\int_{-\infty}^{\infty}+\sum_{m=0}^{\infty}+\sum_{m=0}^{k}K(\mu,\nu,m)\times\left\langle A_{-\lambda-n-m}B_{\mu,m}(f),B_{\bar{\mu},m}(h)\right\rangle_\Gamma, \tag{15}$$

where the integral and the series mean the same as in (11), $K(\mu,\nu,m)$ some numbers. It is a "Pythagorean theorem" for decomposition (14).

Theorem 3 *Let* $\mu\in I_{-k-1}$, $k\in\mathbb{N}$. *Then the representation* $R_{\mu,\nu}$ *considered on the space* $\mathcal{T}_\nu^k(\Omega)$ *splits into the sum of two terms. The first one acts on the subspace of functions* f *such that their Taylor coefficients* $a_m(f)$ *are equal to* 0 *for* $m\le k$ *and decomposes as* $R_{\mu,\nu}$ *does in Case* (*A*), *the second one decomposes into the direct sum of* $k+1$ *irreducible representations* $T_{-\mu-n-m,\nu-m}$, $m\le k$ *acting on the sum of the spaces* $\mathcal{P}_{\mu,\nu,m}$. *There is an inversion formula*, see (14), *and a* "*Plancherel formula*" *for the Berezin form*, *see* (15).

REFERENCES

[1] A. M. Vershik, I. M. Gelfand and M. I. Graev, "Representations of the Group $SL(2,R)$ Where R Is a Ring of Functions," *Uspekhi Matematicheskikh Nauk*, Vol. 28, No. 5, 1973, pp. 83-128.

[2] F. A. Berezin, "Quantization in Complex Symmetric Spaces," *Izvestia Akademii nauk SSSR*, Vol. 39, No. 2, 1975, pp. 363-402.

[3] V. F. Molchanov, "Quantization on Para-Hermitian Symmetric Spaces," *American Mathematical Society Translations*, Vol. 175, Series 2, 1996, pp. 81-95.

[4] G. van Dijk and V. F. Molchanov, "Tensor Products of Maximal Degenerate Series Representations of the Group $SL(n,\mathbb{R})$," *Journal de Mathématiques Pures et Appliquées*, Vol. 78, No. 1, 1999, pp. 99-119. http://dx.doi.org/10.1016/S0021-7824(99)80011-7

[5] G. van Dijk and V. F. Molchanov, "The Berezin form for Rank One Para-Hermitian Symmetric Spaces," *Journal de Mathématiques Pures et Appliquées*, Vol. 77, No. 8, 1998, pp. 747-799. http://dx.doi.org/10.1016/S0021-7824(98)80008-1

[6] V. F. Molchanov, "Canonical and Boundary Representations on a Hyperboloid of One Sheet," *Acta Applicandae Mathematicae*, Vol. 81, No. 1-3, 2004, pp. 191-214.

[7] V. F. Molchanov, "Canonical Representations on the Two-Sheeted Hyperboloid," *Indagationes Mathematicae*, Vol. 16, No. 3-4, 2005, pp. 609-630.

[8] A. A. Artemov, "Canonical and Boundary Representations on a Sphere with an Action of the Generalized Lorentz Group," Monography, The Publishing House of TSU, Tambov, 2010, 235 p.