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Doubly and Triply Periodic Waves Solutions for the KdV Equation*

Ying Huang, Dengguo Xu
Department of Mathematics, Chuxiong Normal University, Chuxiong, China
Email: huang11261001@163.cn

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ABSTRACT

Based on the arbitrary constant solution, a series of explicit doubly periodic solutions and triply periodic solutions for the Korteweg-de Vries (KdV) equation are first constructed with the aid of the Darboux transformation method.

Keywords: KdV Equation; Doubly Periodic Solution; Triply Periodic Solution; Darboux Transformation

1. Introduction

The famous KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

is a shallow water wave equation early derived by Korteweg de and Vries, its first application was discovered in the study of collision-free hydro-magnetic waves in 1960. Subsequently, it has arisen in a number of physical contexts, such as stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics and so on. Following the further studies of these physical problems, its exact solutions have attracted much attention and have been extensively studied [1-7]. However, in contrast to solitary wave solutions, the analytic periodic solutions represent only a small subclass of its known solutions, and multi-periodic solutions are scarce. It is always useful to seek more and various multi-periodic solutions for recovering interactions among some simple periodic waves in a nonlinear medium.

We know that the Darboux transformation method is the main method to construct exact multi-soliton solutions, and this method is scarcely used for solving multi-periodic solutions [8-10]. In the paper, not only explicit doubly periodic solutions are available, but also a group of explicit triply periodic solutions is obtained by means of the Darboux transformation method.

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2. Doubly Periodic Solutions

According to [11], the linear system

$$\begin{cases} \Phi_x = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix} \Phi, \\ \Phi_t = \begin{pmatrix} u_x & -(4\lambda + 2u) \\ A & -u_x \end{pmatrix} \Phi, \end{cases} \tag{2}$$

is the Lax pair for Equation (1), with the Darboux matrix

$$D(x,t,\lambda) = \begin{pmatrix} -\sigma_i & 1 \\ \lambda - \lambda_i + \sigma_i^2 & -\sigma_i \end{pmatrix}, \tag{3}$$

where $A = -(4\lambda + 2u)(\lambda - u) + u_{xx}$, $\lambda, \lambda_i\ (i = 0,1,2)$ are the spectral parameters. The monograph [11] further points out, if u_i is a known solution to Equation (1), then

$$u_{i+1} = 2\lambda_i - u_i - 2\sigma_i^2 \tag{4}$$

becomes new solution generated from u_i, with

$$\sigma_i = \frac{a_{21}^{(i)}(x,t,\lambda_i)\mu_i + a_{22}^{(i)}(x,t,\lambda_i)\gamma_i}{a_{11}^{(i)}(x,t,\lambda_i)\mu_i + a_{12}^{(i)}(x,t,\lambda_i)\gamma_i}, \tag{5}$$

where, μ_i and γ_i are arbitrary constants, but $\mu_i^2 + \gamma_i^2 \neq 0$, and $\Phi_i(x,t,\lambda) = \left(a_{jk}^{(i)}(x,t,\lambda)\right)_{2\times 2}$ is the fundamental solution matrix to the lax pair on u_i.

Only solving the fundamental solution matrix of the lax pair corresponding to constant solution u_0, it is possible to construct multi-periodic solutions to the KdV Equation (1). Substituting u_0 into the system (2) yields

$$\begin{cases} \Phi_x = \begin{pmatrix} 0 & 1 \\ \lambda - u_0 & 0 \end{pmatrix} \Phi, \\ \Phi_t = -(4\lambda + 2u_0) \begin{pmatrix} 0 & 1 \\ \lambda - u_0 & 0 \end{pmatrix} \Phi. \end{cases} \quad (6)$$

If setting $\xi = x - (4\lambda + 2u_0)t$, then we can assert that both the system (6) and the following linear system

$$\Phi_\xi = \begin{pmatrix} 0 & 1 \\ \lambda - u_0 & 0 \end{pmatrix} \Phi$$

have exactly the same solutions. Under the condition for $u_0 > \lambda$, by the eigenvalue method, we obtain the complex-valued fundamental solution matrix to the above system

$$\begin{pmatrix} e^{ia\xi} & e^{-ia\xi} \\ ia e^{ia\xi} & -ia e^{-ia\xi} \end{pmatrix}, \quad (7)$$

where $a = a(\lambda) = \sqrt{u_0 - \lambda}$. Because the real and imaginary parts of a complex-valued solution are also solutions, we thus take

$$\Phi_0(x,t,\lambda) = \begin{pmatrix} \cos\theta & \sin\theta \\ -a\sin\theta & a\cos\theta \end{pmatrix} \quad (8)$$

as the fundamental solution matrix to the the system (6), where $\theta = \theta(\lambda) = a\xi$.

For simplicity, we setting $a_i = a(\lambda_i), \theta_i = \theta(\lambda_i), \Gamma_{ij} = a_i^2 - a_2^j = \lambda_j - \lambda_i$, $(i, j = 0,1,2)$.

From (5), we have

$$\sigma_0 = a_0 \frac{-\mu_0 \sin\theta_0 + \gamma_0 \cos\theta_0}{\mu_0 \cos\theta_0 + \gamma_0 \sin\theta_0},$$

in the above formula, choosing $\mu_0 = 1, \gamma_0 = 0$ and $\mu_0 = 0, \gamma_0 = 1$, respectively, we get

$$\sigma_{0t} = -a_0 \tan\theta_0 \quad (9)$$

and

$$\sigma_{0c} = a_0 \cot\theta_0, \quad (10)$$

respectively, with (4), the periodic wave solutions

$$u_{1-1} = u_0 - 2a_0^2 \sec^2\theta_0$$

and

$$u_{1-2} = u_0 - 2a_0^2 \csc^2\theta_0$$

are obtained.

Now we construct the doubly periodic solutions generated from u_1, thanks to (4), we see that

$$u_2 = 2\lambda_1 - \left(2\lambda_0 - u_0 - 2\sigma_0^2\right) - 2\sigma_1^2, \quad (11)$$

we first give σ_1, then substitute σ_0 and σ_1 into (11). Again according to [11], we can obtain the fundamental solution matrix to the lax pair associated with the known periodic wave solution u_1 in the following manner

$$\Phi_1(x,t,\lambda) = \begin{pmatrix} -\sigma_0 & 1 \\ \lambda - \lambda_0 + \sigma_0^2 & -\sigma_0 \end{pmatrix} \Phi_0(x,t,\lambda)$$
$$= \begin{pmatrix} -\sigma_0 \cos\theta - a\sin\theta & -\sigma_0 \sin\theta + a\cos\theta \\ P & Q \end{pmatrix}, \quad (12)$$

where $P = \left(\lambda - \lambda_0 + \sigma_0^2\right)\cos\theta + \sigma_0 a \sin\theta$, $Q = \left(\lambda - \lambda_0 + \sigma_0^2\right)\sin\theta - \sigma_0 a\cos\theta$. After combining (5) and (12), choosing $\mu_1 = 1, \gamma_1 = 0$, we get

$$\sigma_{1t} = -\frac{\left(\lambda_1 - \lambda_0 + \sigma_0^2\right) + \sigma_0 a_1 \tan\theta_1}{\sigma_0 + a_1 \tan\theta_1}. \quad (13)$$

Substituting (9) and (13) into (11), we have new doubly periodic solution

$$u_{2-1} = u_0 + \frac{2\Gamma_{01}\left(a_0^2 \sec^2\theta_0 - a_1^2 \sec^2\theta_1\right)}{\left(a_0 \tan\theta_0 - a_1 \tan\theta_1\right)^2}. \quad (14)$$

Again substituting (10) and (13) into (11), we obtain another new doubly periodic solution

$$u_{2-2} = u_0 + \frac{2\Gamma_{01}\left(a_0^2 \csc^2\theta_0 - a_1^2 \sec^2\theta_1\right)}{\left(a_0 \cot\theta_0 + a_1 \tan\theta_1\right)^2}. \quad (15)$$

Similarly, choosing $\mu_1 = 0, \gamma_1 = 1$, we have

$$\sigma_{1c} = -\frac{\left(\lambda_1 - \lambda_0 + \sigma_0^2\right) - \sigma_0 a_1 \cot\theta_1}{\sigma_0 - a_1 \cot\theta_1}, \quad (16)$$

which implies the doubly periodic solutions

$$u_{2-3} = u_0 + \frac{2\Gamma_{01}\left(a_0^2 \sec^2\theta_0 - a_1^2 \csc^2\theta_1\right)}{\left(a_0 \tan\theta_0 + a_1 \cot\theta_1\right)^2} \quad (17)$$

and

$$u_{2-4} = u_0 + \frac{2\Gamma_{01}\left(a_0^2 \csc^2\theta_0 - a_0^2 \csc^2\theta_1\right)}{\left(a_0 \cot\theta_0 - a_1 \cot\theta_1\right)^2}. \quad (18)$$

Specially, although u_{2-3} is a doubly periodic solution, its structure is very similar to a given two-soliton solution in [1].

3. Triply Periodic Solutions

As shown in [11], the fundamental solution matrix to the lax pair associated with the doubly periodic wave solution u_2 can be given by

$$\Phi_2(x,t,\lambda) = \begin{pmatrix} -\sigma_1 & 1 \\ \lambda - \lambda_1 + \sigma_1^2 & -\sigma_1 \end{pmatrix} \Phi_1(x,t,\lambda), \quad (19)$$

substituting (12) into (19), in exactly the same manner as in Section 2, we get

$$\sigma_{2t}=\frac{(\lambda_1-\lambda_2)(\sigma_0+a_2\tan\theta_2)}{\lambda_2-\lambda_0+(\sigma_0+\sigma_1)(\sigma_0+a_2\tan\theta_2)}-\sigma_1$$

and

$$\sigma_{2c}=\frac{(\lambda_1-\lambda_2)(\sigma_0-a_2\cot\theta_2)}{\lambda_2-\lambda_0+(\sigma_0+\sigma_1)(\sigma_0-a_2\cot\theta_2)}-\sigma_1.$$

Owing to (4) and (11), we have

$$u_3=2(\lambda_0-u_0-\sigma_0^2)+2(\lambda_2-\lambda_1+\sigma_1^2-\sigma_2^2)+u_0. \quad (20)$$

Here, we set $F_i=a_i\tan\theta_i$, $G_i=a_i\cot\theta_i$, $i=0,1,2$. Substituting σ_{0t},σ_{1t} and σ_{2t} into (20), we obtain triply periodic solution

$$u_{3-1}=u_0+2a_0^2\sec^2\theta_0+\frac{2\Gamma_{12}\Gamma_{01}(a_0^2\sec^2\theta_0-a_1^2\sec^2\theta_1)(F_0-F_2)^2}{[\Gamma_{02}(F_1-F_0)+\Gamma_{10}(F_0-F_2)]^2}+\frac{2\Gamma_{12}\Gamma_{02}(a_2^2\sec^2\theta_2-a_0^2\sec^2\theta_0)(F_1-F_0)^2}{[\Gamma_{02}(F_1-F_0)+\Gamma_{10}(F_0-F_2)]^2}.$$

Similarly, we have

$$u_{3-2}=u_0+2a_0^2\csc^2\theta_0+\frac{2\Gamma_{12}\Gamma_{01}(a_0^2\csc^2\theta_0-a_1^2\sec^2\theta_1)(G_0+F_2)^2}{[\Gamma_{02}(F_1+G_0)-\Gamma_{10}(G_0+F_2)]^2}+\frac{2\Gamma_{12}\Gamma_{02}(a_2^2\sec^2\theta_2-a_0^2\csc^2\theta_0)(F_1+G_0)^2}{[\Gamma_{02}(F_1+G_0)-\Gamma_{10}(G_0+F_2)]^2},$$

$$u_{3-3}=u_0+2a_0^2\sec^2\theta_0+\frac{2\Gamma_{12}\Gamma_{01}(a_0^2\sec^2\theta_0-a_1^2\csc^2\theta_1)(F_0-F_2)^2}{[\Gamma_{02}(G_1+F_0)-\Gamma_{10}(F_0-F_2)]^2}+\frac{2\Gamma_{12}\Gamma_{02}(a_2^2\sec^2\theta_2-a_0^2\sec^2\theta_0)(G_1+F_0)^2}{[\Gamma_{02}(G_1+F_0)-\Gamma_{10}(F_0-F_2)]^2},$$

$$u_{3-4}=u_0+2a_0^2\csc^2\theta_0+\frac{2\Gamma_{12}\Gamma_{01}(a_0^2\csc^2\theta_0-a_1^2\csc^2\theta_1)(G_0+F_2)^2}{[\Gamma_{02}(G_1-G_0)+\Gamma_{10}(G_0+F_2)]^2}+\frac{2\Gamma_{12}\Gamma_{02}(a_2^2\sec^2\theta_2-a_0^2\csc^2\theta_0)(G_1-G_0)^2}{[\Gamma_{02}(G_1-G_0)+\Gamma_{10}(G_0+F_2)]^2},$$

$$u_{3-5}=u_0+2a_0^2\sec^2\theta_0+\frac{2\Gamma_{12}\Gamma_{01}(a_0^2\sec^2\theta_0-a_1^2\sec^2\theta_1)(F_0+G_2)^2}{[\Gamma_{02}(F_1-F_0)+\Gamma_{10}(F_0+G_2)]^2}+\frac{2\Gamma_{12}\Gamma_{02}(a_2^2\csc^2\theta_2-a_0^2\sec^2\theta_0)(F_1-F_0)^2}{[\Gamma_{02}(F_1-F_0)+\Gamma_{10}(F_0+G_2)]^2},$$

$$u_{3-6}=u_0+2a_0^2\csc^2\theta_0+\frac{2\Gamma_{12}\Gamma_{01}(a_0^2\csc^2\theta_0-a_1^2\sec^2\theta_1)(G_0-G_2)^2}{[\Gamma_{02}(F_1+G_0)-\Gamma_{10}(G_0-G_2)]^2}+\frac{2\Gamma_{12}\Gamma_{02}(a_2^2\csc^2\theta_2-a_0^2\csc^2\theta_0)(F_1+G_0)^2}{[\Gamma_{02}(F_1+G_0)-\Gamma_{10}(G_0-G_2)]^2},$$

$$u_{3-7}=u_0+2a_0^2\sec^2\theta_0+\frac{2\Gamma_{12}\Gamma_{01}(a_0^2\sec^2\theta_0-a_1^2\csc^2\theta_1)(F_0+G_2)^2}{[\Gamma_{02}(G_1+F_0)-\Gamma_{10}(F_0+G_2)]^2}+\frac{2\Gamma_{12}\Gamma_{02}(a_2^2\csc^2\theta_2-a_0^2\sec^2\theta_0)(G_1+F_0)^2}{[\Gamma_{02}(G_1+F_0)-\Gamma_{10}(F_0+G_2)]^2},$$

and

$$u_{3-8}=u_0+2a_0^2\csc^2\theta_0+\frac{2\Gamma_{12}\Gamma_{01}(a_0^2\csc^2\theta_0-a_1^2\csc^2\theta_1)(G_0-G_2)^2}{[\Gamma_{02}(G_1-G_0)+\Gamma_{10}(G_0-G_2)]^2}+\frac{2\Gamma_{12}\Gamma_{02}(a_2^2\csc^2\theta_2-a_0^2\csc^2\theta_0)(G_1-G_0)^2}{[\Gamma_{02}(G_1-G_0)+\Gamma_{10}(G_0-G_2)]^2}.$$

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