

# Controlling Unstable Discrete Chaos and Hyperchaos Systems\*

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## ABSTRACT

A method is introduced to stabilize unstable discrete systems, which does not require any adjustable control parameters of the system. 2-dimension discrete Fold system and 3-dimension discrete hyperchaotic system are stabilized to fixed points respectively. Numerical simulations are then provided to show the effectiveness and feasibility of the proposed chaos and hyperchaos controlling scheme.

**Keywords:** 2-Dimension Discrete Fold System; 3-Dimension Discrete Hyperchaotic System; Lyapunov Stability Theory; Controlling Chaos

## 1. Introduction

In many engineering and other practical problems, chaos is undesirable and therefore needs to be controlled. Thus, a large number of control methods have been developed and are being applied to real systems [1-10]. The method given by Ott, Grebogi and Yorke (OGY) [1] is to stabilize an unstable orbit in the neighborhood of a hyperbolic fixed point by forcing the orbit onto the stable manifold. The method proposed by Romeiras, Grebogi, Ott and Dayawansa (RGOD) [2] is not yet suitable for controlling hyperchaos since the method changes the stability property of the fixed point completely. However, the method proposed by Yang, Liu and Jian-min Mao [11] gives a new idea to stabilize unstable orbits even if there is no preexisting stable manifold nearby. For a finite-dimensional dynamical system, whose governing equations may or may not be analytically available, Yang, Liu and Mao show how to stabilize an unstable orbit in a neighborhood of a “fully” unstable fixed point. The advantage of this method is: only one of the unstable directions is to be stabilized via time-dependent adjustments of control parameters. The parameter adjustments can be optimized. Recently, Bu [12] and Li [13] stabilized unstable discrete systems by a method which does not require any adjustable control parameters of the system.

Consider an  $n$ -dimensional dynamical system defined by

$$x_{k+1} = F(x_k), \quad (1)$$

where  $x \in R^n$  is an  $n$ -dimensional vector,  $F$  is a nonlinear vector valued function. Let  $x_f$  be the fixed point of the map (1). To stabilize a chaotic orbit to this fixed point, we take a variable feedback control described by

$$x_{k+1} = F(x_k) + M(F(x_k) - x_k), \quad (2)$$

Define an infinitesimal deviation of  $x_k$  from  $x_f$  as  $\delta x_k = x_k - x_f$ . Then from Equation (2), one has

$$\delta x_{k+1} \approx J \delta x_k + M(J - I) \delta x_k, \quad (3)$$

where  $J = (\partial F / \partial x_k) \big|_{x_k = x_f}$  is the Jacobian matrix of the original system  $F$  evaluated at the fixed point  $x_f$  and  $I$  is the  $n \times n$  identity matrix. The goal of controlling here is to make  $\lim_{k \rightarrow \infty} |\delta x_k| = 0$ . For this aim, one requires

$$\delta x_{k+1} = Q \delta x_k, \quad (4)$$

where  $Q$  is an  $n \times n$  matrix and takes the form

$$Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \quad (5)$$

where are constants. Substituting Equation (4) and Equation (5) into Equation (3) and eliminating  $\delta x_k$ , choosing one special form of the matrix  $Q = qI, q \in (-1, 1)$ , one have

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$$M = (qI - J)(J - I)^{-1}. \quad (6)$$

This needs to use numeric computation to do. Therefore the above scheme based on the symbolic numeric computation is summarized as follows.

**Input:**

- 1) The unstable system (1);
- 2) The system (2) with a variable feedback controller;
- 3) Choose the initial values of systems (2).

**Output:**

- 1)  $J = (\partial F / \partial x_k) \Big|_{x_k = x_f}$ ;
- 2)  $M$  in (6);
- 3) Deduce the system (2) according to the results (6);
- 4) Numerical simulations of the states  $x_k$  when  $k \rightarrow \infty$ .

In this paper, we use the method to stabilize 2-dimension discrete Fold system [14] and 3-dimension discrete hyperchaotic system due to Wang [15] to fixed points respectively.

## 2. Stabilizing 2-Dimensional Discrete Fold System

Using the above method, we stabilize 2-dimension discrete Fold system presented as:

$$\begin{cases} x_1(k+1) = x_2(k) + \alpha x_1(k) \\ x_2(k+1) = \beta + x_1^2(k) \end{cases}, \quad (7)$$

where  $\alpha, \beta$  are the parameters, and we choose  $\alpha = -0.1$ ,

$$\begin{cases} x_1(k+1) = x_2(k) + \alpha x_1(k) - 1.76666313[x_2(k) + \alpha x_1(k) - x_1(k)] \\ \quad - 0.1766663133[\beta + x_1^2(k) - x_2(k)], \\ x_2(k+1) = \beta + x_1^2(k) - 0.6943329445[x_2(k) + \alpha x_1(k) - x_1(k)] \\ \quad - 1.194332944[\beta + x_1^2(k) - x_2(k)], \end{cases} \quad (11)$$

and

$$\begin{cases} x_1(k+1) = x_2(k) + \alpha x_1(k) - 1.247332838[x_2(k) + \alpha x_1(k) - x_1(k)] \\ \quad - 0.2473328387[\beta + x_1^2(k) - x_2(k)], \\ x_2(k+1) = \beta + x_1^2(k) - 0.9720661223[x_2(k) + \alpha x_1(k) - x_1(k)] \\ \quad - 1.272066122[\beta + x_1^2(k) - x_2(k)]. \end{cases} \quad (12)$$

In the following, we give the orbit of 2-dimension discrete Fold system before being stabilized in **Figure 1(a)**. And in **Figure 1(b)**, three orbits starting from different initial points are stabilized to the fixed point (1.965097170, 2.161606887). It is shown that the unstable orbit is stabilized to the desired fixed point monotonically. Then the orbits stabilized of  $x_k$  and  $y_k$  versus  $t_k$

$\beta = -1.7$ .

In the following based on the method mentioned above, we will make the Fold system stabilize at the fixed point. It is easy to get the two fixed points (1.965097170, 2.161606887), and (-0.8650971698, -0.9516068868) of Equation (7). The Jacobian matrix corresponding the fixed point  $(x_{1f}, x_{2f})$  is

$$J = \begin{pmatrix} \alpha & 1 \\ 2x_{1f} & 0 \end{pmatrix}. \quad (8)$$

From (6) one can have

$$M = \begin{pmatrix} \frac{q - \alpha - 2x_{1f}}{\alpha - 1 - 2x_{1f}} & \frac{-1 + q}{\alpha - 1 - 2x_{1f}} \\ \frac{2x_{1f}(q - 1)}{\alpha - 1 - 2x_{1f}} & \frac{2x_{1f} + q\alpha - q}{\alpha - 1 - 2x_{1f}} \end{pmatrix}, \quad (9)$$

here we choose

$(x_{1f}, x_{2f}) = (1.965097170, 2.161606887)$  as our research object. Choosing the parameter  $q = 0.5$ , and  $q = 0.3$  respectively, one gets

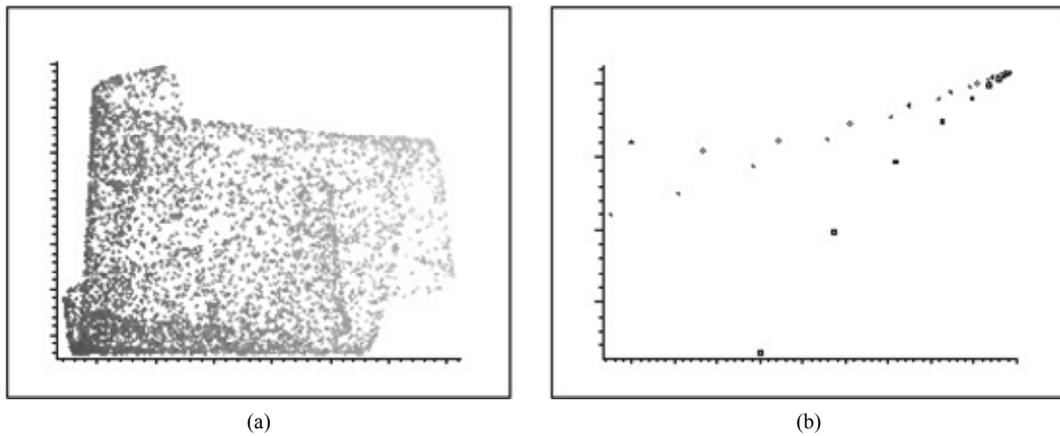
$$\begin{aligned} M_1 &= \begin{pmatrix} -1.176666313 & -0.1766663133 \\ -0.6943329445 & -1.194332944 \end{pmatrix}, \\ M_2 &= \begin{pmatrix} -1.247332838 & -0.2473328387 \\ -0.9720661223 & -1.272066122 \end{pmatrix}. \end{aligned} \quad (10)$$

From (2), respectively substitute (10) into (7), we can obtain

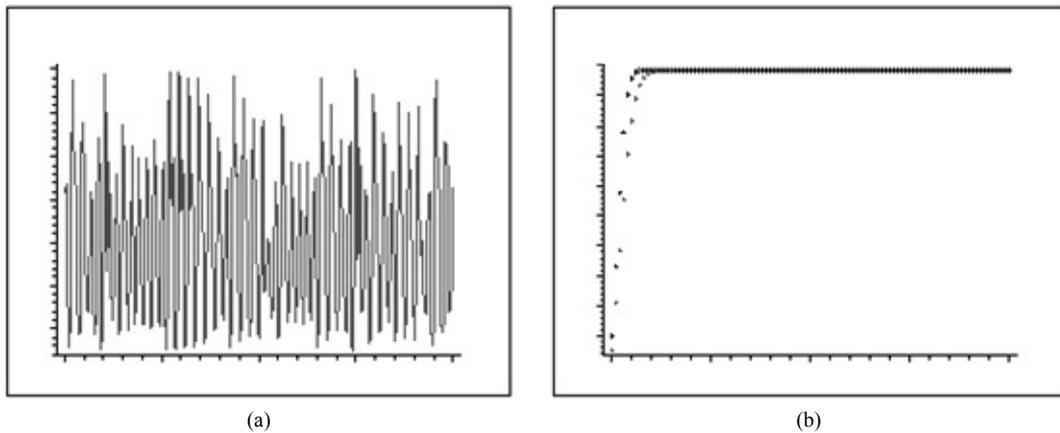
are depicted contrasting with the ones before being stabilized in **Figures 2** and **3**, respectively.

## 3. Stabilizing 3-Dimension Discrete Hyperchaotic System

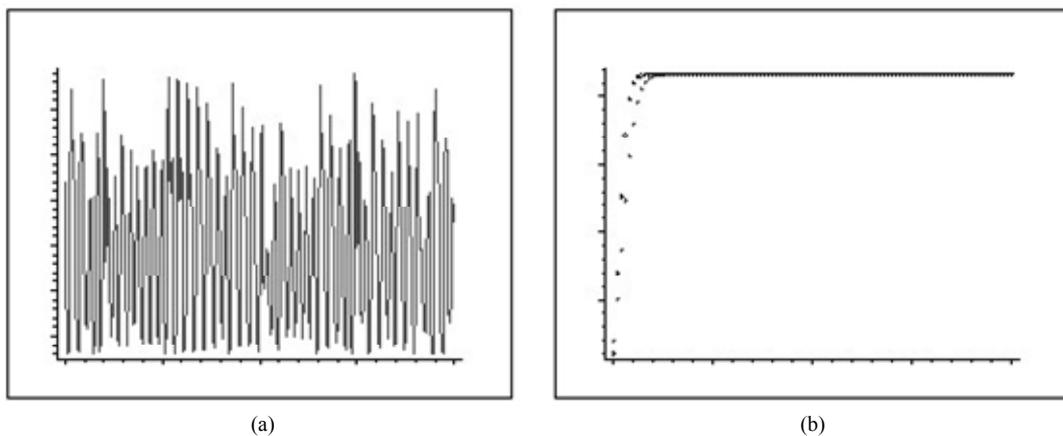
In this section, we consider 3-dimension discrete hyperchaotic system



**Figure 1.** (a) 2-dimension discrete Fold system; (b) Three orbits starting from different initial points are stabilized to the fixed point (1.965097170, 2.161606887), for  $q = 0.5$ .



**Figure 2.** (a)  $x_1(k)$  versus  $k$  before being stabilized; (b)  $x_1(k)$  versus  $k$  after being stabilized for  $q = 0.5$  and  $q = 0.3$ .



**Figure 3.** (a)  $x_2(k)$  versus  $k$  before being stabilized; (b)  $x_2(k)$  versus  $k$  after being stabilized for  $q = 0.5$  and  $q = 0.3$ .

$$\begin{cases} y_1(k+1) = a_3\delta y_2(k) + (a_4\delta + 1)y_1(k), \\ y_2(k+1) = a_2\delta y_3(k) + a_1\delta y_1(k) + y_2(k), \\ y_3(k+1) = a_5\delta + a_6\delta y_2(k)y_3(k) + (a_7\delta + 1)y_3(k), \end{cases} \quad (13)$$

which was derived from the generalized Rössler system

via a first-order difference algorithm [15].

We take the fixed point (0.09610764055, 0.4420951466, 0.9130225853) as our research object, that is  $(y_{1f}, y_{2f}, y_{3f}) = (0.09610764055, 0.4420951466, 0.9130225853)$ .

Following the procedure above, the Jacobian matrix of

map (13) is

$$J = \begin{pmatrix} a_4\delta + 1 & a_3\delta & 0 \\ a_1\delta & 1 & a_2\delta \\ 0 & a_6\delta y_{3f} & a_6\delta y_{2f} + a_7\delta + 1 \end{pmatrix}. \quad (14)$$

Here we let  $a_1 = -1.9, a_2 = 0.2, a_3 = 0.5, a_4 = -2.3, a_5 = 2, a_6 = -0.6, a_7 = -1.9$  and  $\delta = 1$ .

From (6), choosing  $q = 0.5$  and  $q = -0.2$  respectively, the matrix  $M$  at the fixed point  $(0.09610764055, 0.4420951466, 0.9130225853)$  is correspondingly obtained as following

$$M = \begin{pmatrix} -0.9762747382 & 0.2344378413 & 0.02165450397 \\ -0.8908637970 & 0.0784140703 & 0.09961071835 \\ 0.2253899842 & -0.2728405070 & -0.7942822117 \end{pmatrix}, \quad (15)$$

and

$$M = \begin{pmatrix} -0.9430593722 & 0.5626508191 & 0.05197080956 \\ -2.138073113 & 1.588193768 & 0.2390657240 \\ 0.5409359615 & -0.6548172166 & -0.562773086 \end{pmatrix}. \quad (16)$$

From (2), respectively substitute (15) and (16) into (13), one can obtain

$$\begin{cases} y_1(k+1) = 0.0237252618[a_3\delta y_2(k) + a_4\delta y_1(k) + y_1(k)] + 0.9762747382y_1(k) + 0.2344378412 \\ \quad \times [a_2\delta y_3(k) + a_1\delta y_1(k)] + 0.02165450397[a_5\delta + a_6\delta y_2(k)y_3(k) + a_7\delta y_3(k)], \\ y_2(k+1) = 1.07841407[a_2\delta y_3(k) + a_1\delta y_1(k)] + y_2(k) - 0.890863797[a_3\delta y_2(k) + a_4\delta y_1(k)] \\ \quad + 0.09961071835[a_5\delta + a_6\delta y_2(k)y_3(k) + a_7\delta y_3(k)], \\ y_3(k+1) = 0.2057177883[a_5\delta + a_6\delta y_2(k)y_3(k) + (a_7\delta + 1)y_3(k)] + 0.2253899842[a_3\delta y_2(k) \\ \quad + a_4\delta y_1(k)] - 0.272840507[a_2\delta y_3(k) + a_1\delta y_1(k)] + 0.7942822117y_3(k), \end{cases} \quad (17)$$

and

$$\begin{cases} y_1(k+1) = 0.0569406278[a_3\delta y_2(k) + a_4\delta y_1(k) + y_1(k)] + 0.9430593722y_1(k) + 0.5626508191 \\ \quad \times [a_2\delta y_3(k) + a_1\delta y_1(k)] + 0.05197080956[a_5\delta + a_6\delta y_2(k)y_3(k) + a_7\delta y_3(k)], \\ y_2(k+1) = 2.588193768[a_2\delta y_3(k) + a_1\delta y_1(k)] + y_2(k) - 2.138073113[a_3\delta y_2(k) + a_4\delta y_1(k)] \\ \quad + 0.239065724[a_5\delta + a_6\delta y_2(k)y_3(k) + a_7\delta y_3(k)], \\ y_3(k+1) = 0.4937226914[a_5\delta + a_6\delta y_2(k)y_3(k) + (a_7\delta + 1)y_3(k)] + 0.5409359615[a_3\delta y_2(k) \\ \quad + a_4\delta y_1(k)] - 0.6548172166[a_2\delta y_3(k) + a_1\delta y_1(k)] + 0.5062773086y_3(k), \end{cases} \quad (18)$$

The numerical results are shown in the followed figures. The orbit of 3-dimension discrete time hyperchaotic system is given by **Figure 4(a)**. In **Figure 4(b)**, three orbits starting from different initial points are stabilized to the fixed point  $(0.09610764055, 0.4420951466, 0.9130225853)$ .

We can also get the result that 3-dimension discrete time hyperchaotic system is stabilized. In **Figures 5-7**, the stabilized orbits of  $y_1(k), y_2(k), y_3(k)$  versus  $t_k$  are plotted contrasting with the ones before being stabilized, respectively.

### 4. Conclusion

In summary, we have introduced a method to stabilize unstable discrete systems, which does not require any adjustable control parameters of the system. 2-dimension discrete Fold system and 3-dimension discrete hyperchaotic system are stabilized to fixed points respectively. From the process we finish, it is shown that stabilizing the unstable discrete systems neither requires a prior analytical knowledge of the underlying system nor any adjustable control parameters in advance. Numerical simulations are then provided to show the effectiveness

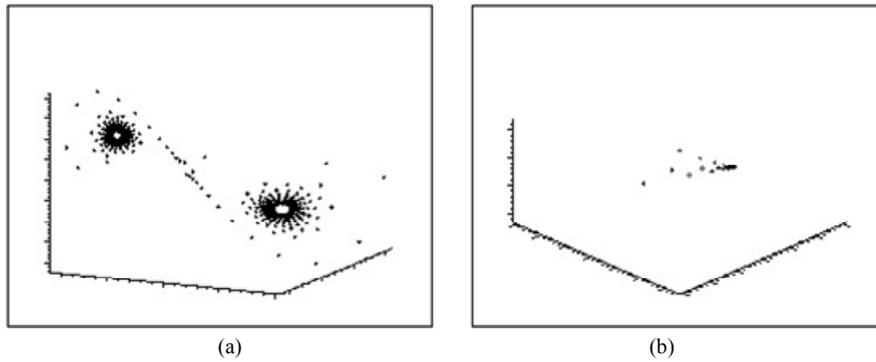


Figure 4. (a) 3-dimension discrete time hyperchaotic system; (b) Three orbits starting from different initial points are stabilized to the fixed point  $(0.09610764055, 0.4420951466, 0.9130225853)$ , for  $q = 0.5$ .

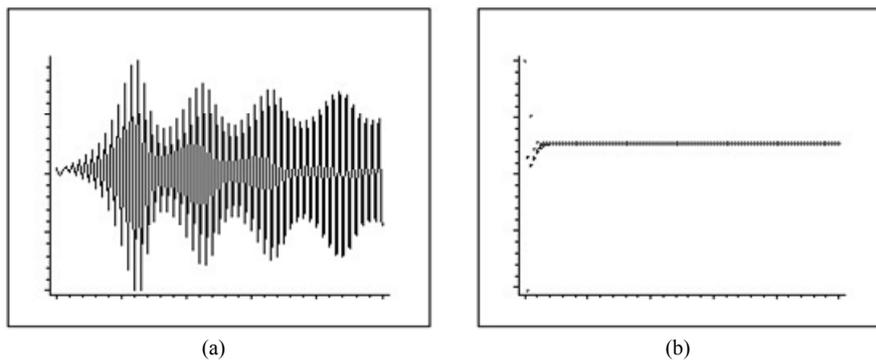


Figure 5. (a)  $y_1(k)$  versus  $k$  before being stabilized; (b)  $y_1(k)$  versus  $k$  after being stabilized.

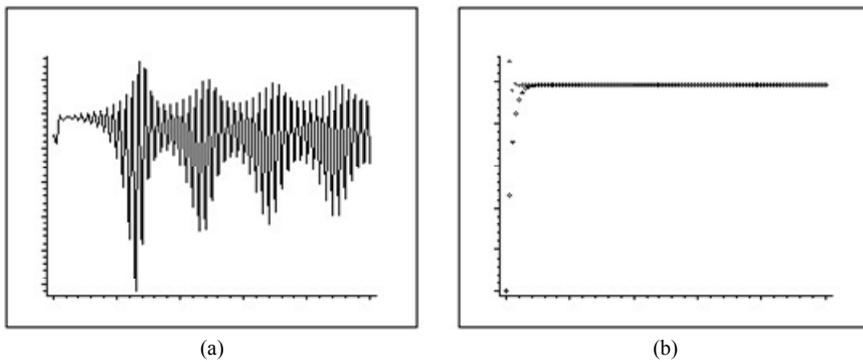


Figure 6. (a)  $y_2(k)$  versus  $k$  before being stabilized; (b)  $y_2(k)$  versus  $k$  after being stabilized.

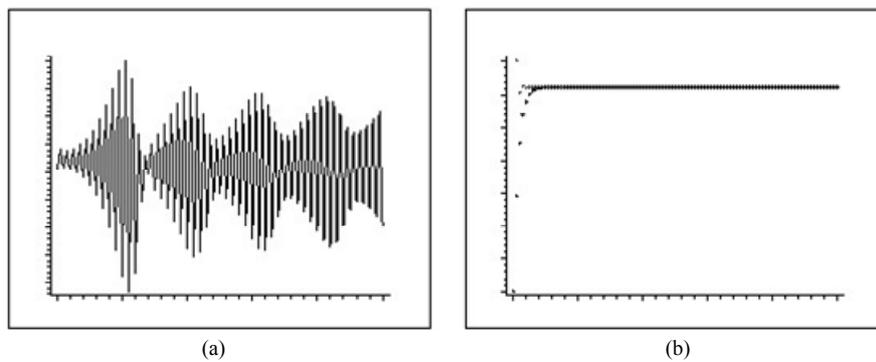


Figure 7. (a)  $y_3(k)$  versus  $k$  before being stabilized; (b)  $y_3(k)$  versus  $k$  after being stabilized.

and feasibility of the proposed chaos and hyperchaos controlling Scheme.

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