

On a Generalized Integral Operator

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ABSTRACT

We have considered several integral operators from literature and we have made a generalization of them. It can be easily seen that their properties are also preserved. Therefore, we use known results concerning the starlike functions (see [1,2]) and we unify some known integral operators (see [3]) into one single integral operator, called $I^\lambda(z)$, in Section 3 of this paper.

Keywords: Univalent Function; Integral Operator; Generalization; Particular Cases

1. Introduction

Let $H(U)$ be the set of functions which are regular in the unit disc U ,

$$\mathcal{A} = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and

$$S = \{f \in \mathcal{A} : f \text{ is univalent in } U\}.$$

Prof. G. S. Sălăgean has introduced the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, a_j \geq 0, j = 2, 3, \dots, z \in U \quad (1)$$

in 1999.

Thus, we have the subfamily $S-T$ consisting of functions f of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, a_j \geq 0, j = 2, 3, \dots, z \in U \quad (2)$$

The purpose of this paper is to give a generalization with respect to several integral operators that exist in literature. In order to do so, we have firstly considered the univalent function f and the family S of analytic and univalent functions, notions that we use further.

2. Preliminary Results

Remark 2.1. In [1], we have introduced the following operator concerning the functions of Form (1):

$$D_\lambda^\beta : A \rightarrow A; D_\lambda^\beta f(z) = z - \sum_{j=n+1}^{\infty} [1+(j-1)\lambda]^\beta a_j z^j \quad (3)$$

Definition 2.1. [2] Let $\beta, \lambda \in R, \beta \geq 0, \lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. We denote by D_λ^β the linear operator defined by

$$D^0 f(z) = f(z); D^1 f(z) = Df(z) = zf'(z), \quad (4)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

Professor Breaz ([3]) has introduced the following integral operators on univalent function spaces:

$$J(z) = \left\{ \beta \int_0^z [f_1'(t)]^{\gamma_1} \cdots [f_p'(t)]^{\gamma_p} dt \right\}^{\frac{1}{\beta}}, \quad (5)$$

$$H(z) = \left\{ \beta \int_0^z t^{\beta-1} [f_1'(t)]^{\gamma_1} \cdots [f_p'(t)]^{\gamma_p} dt \right\}^{\frac{1}{\beta}}, \quad (6)$$

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\gamma_1} \cdots \left(\frac{f_p(t)}{t} \right)^{\gamma_p} dt, \quad (7)$$

$$G(z) = \left[\beta \int_0^z \left(\frac{f_1(t)}{t} \right)^{\gamma_1} \cdots \left(\frac{f_p(t)}{t} \right)^{\gamma_p} dt \right]^{\frac{1}{\beta}}, \quad (8)$$

$$F_{\gamma,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t} \right)^{\frac{1}{\gamma_1}} \cdots \left(\frac{f_p(t)}{t} \right)^{\frac{1}{\gamma_p}} dt \right\}^{\frac{1}{\beta}} \quad (9)$$

and

$$G_{\gamma,p}(z) = \left\{ [p(\gamma-1)+1] \int_0^z g_1^{\gamma-1}(t) \cdots g_p^{\gamma-1}(t) dt \right\}^{\frac{1}{p(\gamma-1)+1}}, \quad (10)$$

where

$$\gamma_i, \gamma, \beta \in \mathbb{C}, \forall i = \overline{1, p}, p \in \mathbb{N} - \{0\}, n \in \mathbb{N} - \{0, 1\}.$$

Let D^n be the Sălăgean differential operator, $D^n : \mathcal{A} \rightarrow \mathcal{A}, n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z), D^1 f(z) = Df(z) = zf'(z), \quad (11)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

and $D^k, D^k : \mathcal{A} \rightarrow \mathcal{A}, k \in \mathbb{N} \cup \{0\}$ of form:

$$D^0 f(z) = f(z), \dots, \quad (12)$$

$$D^k f(z) = D(D^{k-1} f(z)) = z + \sum_{n=2}^{\infty} n^k a_n z^n.$$

The following integral operator is studied in [4], where $f_i, i = 1, \dots, n, n \in \mathbb{N}$, are considered to be of Form (2):

Definition 2.2. [4] We define the general integral operator $I_{k,n,\lambda,\mu} : \mathcal{A}_n \rightarrow \mathcal{A}$ by

$$I_{k,n,\lambda,\mu}(f_1, \dots, f_n) = F, \quad (13)$$

$$D^k F(z) = \int_0^z \left(\frac{D_1^\lambda f_1(t)}{t} \right)^{\mu_1} \cdots \left(\frac{D_n^\lambda f_n(t)}{t} \right)^{\mu_n} dt$$

where

$$f_i \in \mathcal{A}, i \in \mathbb{N} - \{0\}, \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}_0^n,$$

$$\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n, n \in \mathbb{N}$$

and $k \in \mathbb{N}_0$.

Lemma 2.1. [5] (Schwarz Lemma) Let the function $f(z)$ be regular in U , with $f(0) = 0$. If $|f(z)| \leq 1 (z \in U)$, then $|f(z)| \leq |z|$ for all $z \in \#U$, where the equality holds only if $f(z) = Kz, |K| = 1$.

Some integral operators and the related properties of them are also studied in [6,7].

The neighborhoods of the class of functions defined using the operator (3) is studied in [8].

Remark 2.2. In [9], we have introduced and studied the following operator of the functions $f \in \#S$, $S = \{f \in \mathcal{A} : f \text{ is univalent in } U\}$:

$$D_{\lambda_1, \lambda_2}^{n, \beta} f(z) = (h * \psi_1 * f)(z) = z \pm \sum_{k \geq 2} \frac{[1 - \lambda_1(k-1)]^{\beta-1}}{[1 - \lambda_2(k-1)]^\beta} \cdot \frac{1+c}{k+c} \cdot C(n, k) \cdot a_k \cdot z^k \quad (14)$$

where $C(n, k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$; $(n)_k$ is the Pochhammer symbol; $k \geq 2, c \geq 0$ and $\text{Re}\{c\} \geq 0; z \in \#U$.

In [10], Prof. G. S. Sălăgean has introduced the subfamily T of S consisting of functions f of the Form (1); see Section 1.

Remark 2.3. If we denote by $(x)_k$ the Pochhammer symbol, we define it as follows:

$$(x)_k = \begin{cases} 1 & \text{for } k = 0, x \in \mathbb{C} \setminus \{0\} \\ x(x+1)(x+2) \cdots (x+k-1) & \text{for } k \in \mathbb{N} - \{0\} \text{ and } x \in \mathbb{C}. \end{cases}$$

Following, we introduce a new general integral operator in Theorem 3.1, $I^1(z)$, giving also several examples which prove its relevance. We derive it as well in order to show its applicability.

Considering the univalent functions with negative coefficients (see [11]), we develop, $I^2(z)$ from Corollary 3.1.

3. Main Results

Theorem 3.1. Let $\alpha, \gamma_1, \gamma_2, \beta \in \mathbb{C}, \text{Re } \alpha = a > 0$ and

$$D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(z) \in \mathcal{A}, \lambda_1, \lambda_2, \kappa \geq 0, \sigma \in \mathbb{R}, j = \overline{1, p}, p \in \mathbb{N}; D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(z^n)$$

of Form (14). If $\left| \frac{(D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(z^n))^n}{(D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(z^n))'} \right| \leq \frac{1}{n}$ and

$$\left| \frac{(D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(z^n))'}{(D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(z^n))} \right| \leq \frac{1}{n}, \forall z \in U, j = \overline{1, p}, \quad (15)$$

$$\frac{\sum_{j=1}^p [|\delta_j^1| \cdot (|2\gamma_1 - 1| - |\sigma|) + |\delta_j^2| \cdot (|2\gamma_2 - 1| - |\sigma|)]}{\left| \sigma \cdot (2\gamma_1 - 1) \cdot (2\gamma_2 - 1) \cdot \left(\prod_{j=1}^p \delta_j^1 \cdot \delta_j^2 \right) \right|} \leq 1 \quad (16)$$

and

$$\left| \sigma \cdot (2\gamma_1 - 1) \cdot (2\gamma_2 - 1) \cdot \left(\prod_{j=1}^p \delta_j^1 \cdot \delta_j^2 \right) \right| \leq \frac{n+2a}{2} \cdot \left(\frac{n+2a}{n} \right)^{\frac{1}{n+2a}}, \quad (17)$$

Then, for $\forall \delta, \delta_j^1, \delta_j^2 \in \mathbb{C}, j = 1, \dots, p, \operatorname{Re}(\beta) \geq a, \operatorname{Re}(\beta\delta) \geq a$, the function

$$I^1(z) = \left\{ \beta \int_0^z t^{\beta\delta-1} \cdot \prod_{j=1}^p \left[\frac{(D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(t^n))^{2\gamma_1-1}}{t^\sigma} \right]^{\delta_j^1} \cdot \left[\frac{(D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(t^n))^{2\gamma_2-1}}{t^\sigma} \right]^{\delta_j^2} dt \right\}^{\frac{1}{\beta}} \tag{18}$$

is univalent for all $n \in \mathbb{N} - \{0\}$.

Proof.

Let

$$\begin{aligned} h(z) &= \int_0^z \prod_{j=1}^p \left[\frac{(D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(t^n))^{2\gamma_1-1}}{t^\sigma} \right]^{\delta_j^1} \cdot \left[\frac{(D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(t^n))^{2\gamma_2-1}}{t^\sigma} \right]^{\delta_j^2} dt \\ &= \int_0^z \prod_{j=1}^p [h_j^1(t^n)]^{\delta_j^1} \cdot [h_j^2(t^n)]^{\delta_j^2}, \end{aligned}$$

from where we obtain the following derivatives:

$$\begin{aligned} h'(z) &= \prod_{j=1}^p [h_j^1(z^n)]^{\delta_j^1} \cdot [h_j^2(z^n)]^{\delta_j^2}, \\ h''(z) &= \sum_{j=1}^p \delta_j^a \cdot (n \cdot z^{n-1}) \cdot [h_j^a(z^n)]^{\delta_j^a-1} \\ &\quad \cdot [h_j^a(z^n)]' \cdot \prod_{l=1, l \neq j}^p [h_l^b(z^n)]^{\delta_l^b}, \end{aligned}$$

$a, b \in \{1, 2\}, a \neq b$.

Furthermore, we have that

$$\frac{h''(z)}{h'(z)} = \sum_{j=1}^p \frac{[h_j^a(z^n)]^{\delta_j^a-1}}{[h_j^a(z^n)]^{\delta_j^a}}, a \in \{1, 2\}, \text{ which means}$$

$$\begin{aligned} \frac{zh''(z)}{h'(z)} &= \sum_{j=1}^p \left\{ \delta_j^1 \cdot \left[(2\gamma_1-1) \cdot \frac{(D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(t^n))'}{D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(t^n)} - \sigma \right] \right. \\ &\quad \left. + \delta_j^2 \cdot \left[(2\gamma_2-1) \cdot \frac{D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(t^n)'}{D_{\lambda_1, \lambda_2}^{n, \kappa} f_j(t^n)} - \sigma \right] \right\} \end{aligned}$$

$a \in \{1, 2\}$. We consider

$$p(z) = \frac{1}{\left| \sigma \cdot (2\gamma_1-1) \cdot (2\gamma_2-1) \cdot \left(\prod_{j=1}^p \delta_j^1 \cdot \delta_j^2 \right) \right|} \cdot \frac{h''(z)}{h'(z)}$$

and we obtain

$$\begin{aligned} |p(z)| &= \left| \frac{1}{\left| \sigma \cdot (2\gamma_1-1) \cdot (2\gamma_2-1) \cdot \left(\prod_{j=1}^p \delta_j^1 \cdot \delta_j^2 \right) \right|} \cdot \frac{h''(z)}{h'(z)} \right| \\ &\leq \frac{\sum_{j=1}^p \left[|\delta_j^1| \cdot (|2\gamma_1-1| - |\sigma|) + |\delta_j^2| \cdot (|2\gamma_2-1| - |\sigma|) \right]}{\left| \sigma \cdot (2\gamma_1-1) \cdot (2\gamma_2-1) \cdot \left(\prod_{j=1}^p \delta_j^1 \cdot \delta_j^2 \right) \right|}. \end{aligned}$$

Using (16) from hypothesis, we see that $|P(z)| \leq 1$. By applying Schwarz Lemma, we have that

$$\left| \frac{1}{\left| \sigma \cdot (2\gamma_1-1) \cdot (2\gamma_2-1) \cdot \left(\prod_{j=1}^p \delta_j^1 \cdot \delta_j^2 \right) \right|} \cdot \frac{h''(z)}{h'(z)} \right| \leq |z|^{n-1} \leq |z|.$$

If we multiply the last inequality with $\left(\frac{1-|z|^{2a}}{a} \right)$, we obtain:

$$\begin{aligned} \left(\frac{1-|z|^{2a}}{a} \right) \cdot \left| \frac{zh''(z)}{h'(z)} \right| &\leq \left| \sigma \cdot (2\gamma_1-1) \cdot (2\gamma_2-1) \cdot \left(\prod_{j=1}^p \delta_j^1 \cdot \delta_j^2 \right) \right| \cdot \left(\frac{1-|z|^{2a}}{a} \right) \cdot |z|^n. \end{aligned} \tag{19}$$

Let $Q: [0, 1] \rightarrow \mathbb{R}, Q(x) = \frac{1-x^{2a}}{a} \cdot x^n, x = |z|$. We can easily see that $Q(0) = Q(1) = 0, Q(x) \geq 0, \forall x \in [0, 1]$, and $Q\left(\left(\frac{n}{n+2a}\right)^{\frac{1}{2a}}\right) > 0$, where $\left(\frac{n}{n+2a}\right)^{\frac{1}{2a}}$ is the point of maximum for the function Q .

Thus,

$$Q(x) \leq Q\left(\left[\frac{n}{n+2a}\right]^{\frac{1}{2a}}\right) = \frac{2}{n+2a} \cdot \left(\frac{n}{n+2a}\right)^{\frac{n}{2a}}, \tag{20}$$

$\forall x \in [0, 1]$.

By taking into account the inequalities (17), (19) and (20), we obtain the following:

$$\left(\frac{1-|z|^{2a}}{a} \right) \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq 1, \text{ which means that the function}$$

$I^1(z)$ of Form (18) is univalent for all $n \in \mathbb{N} - \{0\}$.

Remark 3.1. *The mapping properties with respect to integral operator $I^1(z)$ of Form (18) are studied in [7].*

If we consider the operator $D^\lambda(z)$ of Form (3) we obtain the following Corollary. It can be proved in a similar way as the Theorem 3.1 is.

Corollary 3.1. *Let $\alpha, \gamma_1, \gamma_2, \beta \in \mathbb{C}, \operatorname{Re} \alpha = a > 0$ and $D_\lambda^\beta f_j z \in \mathcal{A}, \beta \geq 0, \lambda \geq 0, \sigma \in \mathbb{R}, D_\lambda^\beta f(z)$ of Form (3). If*

$$\left| \frac{\left(D_\lambda^\beta f_j(z^n) \right)''}{\left(D_\lambda^\beta f_j(z^n) \right)'} \right| \leq \frac{1}{n} \text{ and } \left| \frac{\left(D_\lambda^\beta f_j(z^n) \right)'}{\left(D_\lambda^\beta f_j(z^n) \right)} \right| \leq \frac{1}{n},$$

$$\forall z \in U, j = \overline{1, p},$$

$$\frac{\left(\sum_{j=1}^p \left[|\delta_j^1| \cdot (|2\gamma_1 - 1| - |\sigma|) + |\delta_j^2| \cdot (|2\gamma_2 - 1| - |\sigma|) \right] \right)}{\left| \sigma \cdot (2\gamma_1 - 1) \cdot (2\gamma_2 - 1) \cdot \left(\prod_{j=1}^p \delta_j^1 \cdot \delta_j^2 \right) \right|} \leq 1$$

and

$$\left| \sigma \cdot (2\gamma_1 - 1) \cdot (2\gamma_2 - 1) \cdot \left(\prod_{j=1}^p \delta_j^1 \cdot \delta_j^2 \right) \right|$$

$$\leq \frac{n+2a}{2} \cdot \left(\frac{n+2a}{n} \right)^{\frac{1}{n+2a}},$$

then for all

$$\delta, \delta_j^1, \delta_j^2 \in \mathbb{C}, j = 1, \dots, p, \operatorname{Re}(\chi) \geq a, \operatorname{Re}(\chi\delta) \geq a,$$

the function

$$I^2(z) = \left\{ \chi \int_0^z t^{\chi\delta-1} \prod_{j=1}^p \left[\frac{\left(D_\lambda^\beta f_j(t^n) \right)'}{t^\sigma} \right]^{2\gamma_j-1} \right\}^{\delta_j^1}$$

$$\cdot \left\{ \frac{\left(D_\lambda^\beta f_j(t^n) \right)^{2\gamma_j-1}}{t^\sigma} \right\}^{\delta_j^2} dt \Bigg\}^{\frac{1}{\chi}}$$

is univalent for all $n \in \mathbb{N} - \{0\}$.

Remark 3.2. *The Corollary is true also for $D^\lambda(z)$ of Form (3), $\forall j \in \overline{1, p}, p \in \mathbb{N}$.*

Next we give few examples of particular cases which can be found in literature.

Let $\beta = 0$ in $D^\lambda(z)$ of Form (3) or (4). So we have

that $D_\lambda^0 f(z) = f(z), \forall \lambda \geq 0$. We will use this form of the integral operator, where the function f is of Form (2) with respect to the operator (21). For further simplification, we consider that $\gamma_1 = \gamma_2 = 1$ and $\delta = 1$ (except of Example 3.5).

For the first four examples we consider

$$\delta_j^1 = 0, j = \overline{1, p}, p \in \mathbb{Z} - \{0\}, n = 1.$$

Example 3.1. *If $\sigma = 1, \chi = 1$ and we use the notation $\delta_j^2 = \gamma_j, j = \overline{1, p}, p \in \mathbb{N} - \{0\}$ we obtain the operator $F(z)$ of Form (7).*

Example 3.2. *If $\sigma = 1$ we obtain the operator $G(z)$ of Form (8) for $\delta_j^2 = 1/\gamma_j, j = \overline{1, p}, p \in \mathbb{N} - \{0\}$.*

Example 3.3. *If $\sigma = 1$ and we use the notation $\delta_j^2 = 1/\gamma_j, j = \overline{1, p}, p \in \mathbb{N} - \{0\}$ we obtain the operator $F_{\gamma, \chi}(z)$ of Form (9).*

Example 3.4. *If $\sigma = 0$ we obtain the operator $G_{\gamma, p}(z)$ of Form (10) for $\chi = [p(\gamma - 1) + 1]$ and*

$$\delta_j^2 = \gamma - 1, \delta = \frac{1}{\chi}.$$

For the next two examples we consider $\delta_j^2 = 0, j = \overline{1, p}, p \in \mathbb{N} - \{0\}$ and $\sigma = 0$.

Example 3.5. *a) If $\chi = 1, \delta = 1$, we obtain a particular case of the function $J(z)$ of Form (5), in which $\beta = 1, \forall n \in \mathbb{N} - \{0\}$.*

b) If $\delta = \frac{1}{\chi}, \delta_j^1 = \gamma_j, j = \overline{1, p}, p \in \mathbb{N} - \{0\}$ we obtain the operator $J(z)$ of Form (5).

Example 3.6. *If $n = 1$ we obtain the operator $H(z)$ of Form (6) for $\delta_j^1 = \gamma_j, j = \overline{1, p}, p \in \mathbb{N} - \{0\}$.*

Remark 3.3. *It can be easily proved that the examples above are also true for $f(z)$ of Form (1).*

4. Conclusions

The integral operator which is introduced and studied in Theorem 3.1 of this paper, called $I^1(z)$, is a generalization of several integral operators that are taken from literature (for e.g., see [3]). Looking to the examples from above, we can easily see that our integral operator can be used further without any concern, because it preserves the initial state of the already known integral operators. Therefore, the integral operator $I^1(z)$ covers the integral operators from [3] without loss of generality.

Open Problem 3.1. *It would be interesting to prove that the integral operator (18) or (21) covers also the integral operator from [5].*

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