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# Algorithms for Computing Some Invariants for Discrete Knots

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## ABSTRACT

Given a cubic knot *K*, there exists a projection $p:\mathbb{R}^3 \to P$ of the Euclidean space $\mathbb{R}^3$ onto a suitable plane $P \subset \mathbb{R}^3$ such that *p*(*K*) is a knot diagram and it can be described in a discrete way as a cycle permutation. Using this fact, we develop an algorithm for computing some invariants for *K*: its fundamental group, the genus of its Seifert surface and its Jones polynomial.

**Keywords:** Cubic Knots; Discrete Knots; Algorithms

## 1. Introduction

Considering the set $S \subset \mathbb{R}^3$ consists of the lattice $\mathbb{Z}^3$ and all the straight lines parallel to the coordinate axis and passing through points in $\mathbb{Z}^3$, we say that a knot $K \subset \mathbb{R}^3$ is a *cubic knot* if it is contained in $S$. In [1] it was shown that any classical knot is isotopic to a cubic knot and by [2] we know that there exists a generic projection *p* of any cubic knot into a suitable plane. If we combine these two results, we have that *p*(*K*) is a diagram of *K* and it can be described in a discrete way as a cyclic permutation of points $(w_1, w_2, \cdots, w_n)$ (with some restrictions). This allows us to develop an algorithm for computing the fundamental group of *K*, the genus of its Seifert surface and its Jones polynomial.

## 2. Discrete Knots and Some Invariants

Consider an oriented cubic knot *K*. In [2] it was proved that we can associate to *K* a unique sequence of points $(v_1, v_2, \cdots, v_n)$ such that $v_i \in \mathbb{Z}^3$, $v_i \neq v_j$, $1 \le i, j \le n$, $v_i$ is joined to $v_{i+1}$ by a unit edge, $v_n$ is likewise joined to $v_1$ by a unit edge, and the numbering of the $v_i$'s is compatible with the orientation of *K*. Henceforth, we will assume that all the coordinates of the points in *K* are positive.

An advantage of cubic knots is that there exists a canonical generic projection *p* (for details see [2]). In fact, let $N = (1, \pi, \pi^2)$, where $\pi$ is the well-known transcendental number. Let *P* be the plane through the origin in $\mathbb{R}^3$ orthogonal to *N* and consider the orthogonal projection $p:\mathbb{R}^3 \to P$. Then $p|_{\mathbb{Z}^3}$ is injective. Let $\hat{K} = p(K)$ be its projection into the plane *P*. Thus $\hat{K}$ is a polygonal curve contained in *P* with some self-intersections called *inessential vertices* or *crossings*. The crossings are not contained in $p(\mathbb{Z}^3) := \Lambda_P = p(K)$ and are transverse, hence *p* is regular. The projections of the vertices of *K* are contained in $\Lambda_P$, and are called *vertices*. Hence we can write $\hat{K}$ as a *cyclic permutation of points* $(w_1, w_2, \cdots, w_n)$ where $w_i \in \Lambda_P$, $w_i \neq w_j$, $1 \le i, j \le n$ and $w_i$ is joined to $w_{i+1}$ by a straight line segment whose preimage is a unit edge and in the same way $w_n$ is likewise joined to $w_1$ (for details see [2]).

**Definition 2.1.** A *discrete knot* $\hat{K}$ is the equivalence class of the *n* cyclic permutations of *n* points $(w_1, w_2, \cdots, w_n)$ in $\Lambda_P \subset P$ such that the $w_i$'s satisfy the above assumptions.

Next we will describe the crossings of $\hat{K}$. Consider an orthonormal basis *β* of the plane $P \subset \mathbb{R}^3$, given by

$$\beta = \left\{ \frac{1}{A}(\pi, -1, 0), \frac{1}{B}(\pi^2, \pi^3, -1-\pi^2) \right\}$$

where $A = \sqrt{\pi^2 + 1}$ and $B = \sqrt{2\pi^2 + 2\pi^4 + \pi^6 + 1}$. Consider four points $w_{i_1}$, $w_{i_2}$, $w_{i_3}$ and $w_{i_4} \in \hat{K}$ whose coordinates with respect to the basis *β* are $w_{i_j} = (x_j, y_j)$. The following lemma gives us a criteria to know when the line segment $\overline{w_{i_1} w_{i_2}}$ intersects the line segment $\overline{w_{i_3} w_{i_4}}$. Notice that for the computing algorithm purpose, we just need to consider only the quadruples of points where $i_2 = i_1 + 1$ and $i_4 = i_3 + 1$ (see [2]).

**Lemma 2.2.** Let $w_{i_1}$, $w_{i_2}$, $w_{i_3}$ and $w_{i_4} \in \hat{K}$, whose coordinates are $w_{i_j} = (x_j, y_j)$. Let $u_{r,s} = w_{i_r} - w_{i_s}$ and consider the 2 × 2 matrices, $A = \begin{bmatrix} u_{2,3} & u_{4,3} \end{bmatrix}$, $B = \begin{bmatrix} u_{1,3} & u_{4,3} \end{bmatrix}$, $C = \begin{bmatrix} u_{3,1} & u_{2,1} \end{bmatrix}$ and $D = \begin{bmatrix} u_{4,1} & u_{2,1} \end{bmatrix}$. Then the line segment $\overline{w_{i_1} w_{i_2}}$ intersects the line segment $\overline{w_{i_3} w_{i_4}}$ if and only if $\det(A)\det(B) < 0$ and $\det(C)\det(D) < 0$.

**Algorithm 1.** Projection and crossings

**Require:** List of points at $\mathbb{R}^3$, $L[v_1, v_2, \cdots, v_n]$, where $v_i = (a_i, b_i, c_i)$, list of points at *P*, $L_1[w_1, w_2, \cdots, w_n]$, an empty set *I*, the constant numbers *A* and *B* given above.

**for all** $v_s \in L$ **do**

$$w_s \leftarrow \left( \frac{a_s\pi - b_s}{A}, \frac{a_s\pi^2 + b_s\pi^3 - c_s - c_s\pi^2}{B} \right)$$

**for all** $w_i, w_k \in L_1$ **do**

Create matrices

$M = \left[ (w_{i+1} - w_k) \quad (w_{k+1} - w_k) \right]$,

$N = \left[ (w_k - w_j) \quad (w_{k+1} - w_k) \right]$,

$O = \left[ (w_k - w_i) \quad (w_{i+1} - w_k) \right]$,

$P = \left[ (w_{k+1} - w_k) \quad (w_{i+1} - w_k) \right]$

where $w_s = (x_s, y_s)$ and $w_s - w_r = (x_s - x_r, y_s - y_r)$

**if** $(\det(M)\det(N) < 0)$ and $(\det(O)\det(P) < 0)$ **then**

**if** $(i,j) \notin I$ and $(j,i) \notin I$ **then** add $(i,j)$ to *I*

Suppose that the line segment $l_i = \overline{w_{i+1} w_i}$ intersects the line segment $l_k = \overline{w_{k+1} w_k}$. We will determine which crosses "over" the other. Since, the line segments $l_i$ and $l_k$ are the image under the projection *p* of two segments whose endpoints are $v_{i+1}$, $v_i$ and $v_{k+1}$, $v_k$, respectively, we have that these both segments are parallel to two different canonical coordinate vectors. Let $u_i = v_{i+1} - v_i = (x_i, y_i, z_i)$ y $u_k = v_{k+1} - v_k = (x_k, y_k, z_k)$. If $x_i = x_k$, then we compare the first coordinate of the vectors $v_i = (a_i, b_i, c_i)$ and $v_k = (a_k, b_k, c_k)$, *i.e.*, we compare $a_i$ and $a_k$. Thus,

- If $a_i < a_k$ we say that $l_k$ *crosses over* $l_i$ (over crossing).
- If $a_k < a_i$ we say that $l_k$ *crosses under* $l_i$ (under crossing).

If $y_i = y_k$, then we compare the second coordinate of the vectors $v_i$ *y* $v_k$, and we have the same criteria of the previous case changing *a* by *b*. The last case $z_i = z_k$ is analogous to the previous one.

Let *c* be a crossing point of the segment $l_k$ over the segment $l_i$. Consider the vectors $u_i = w_{i+1} - w_i$, $u_k = w_{k+1} - w_k$ and construct the 2 × 2-matrix $M = \begin{bmatrix} u_k & u_i \end{bmatrix}$. Thus we have two possible configurations: If det(*M*) > 0, we say that *c* is a *positive crossing*; If det(*M*) < 0, then *c* is a *negative crossing*.

**Algorithm 2.** Crossing criteria

**Require:** The list of indexes of intersection points $I[(i_1,k_1),(i_2,k_2),\cdots,(i_r,k_r)]$ and the list of points in $\mathbb{R}^3$ $L[v_1, v_2, \cdots, v_n]$, where $v_i = (a_i, b_i, c_i)$ and $L_1 = [w_1, w_2, \cdots, w_n]$.

**for all** $(i,j) \in I$ **do**

where $u_s = v_{s+1} - v_s = (x_s, y_s, z_s)$

$u_i \leftarrow v_{i+1} - v_i$

$u_k \leftarrow v_{k+1} - v_k$

**if** $x_i = x_k$ **then**

**if** $a_i < a_k$ **then**

**print** $\overline{w_{i+1} w_i}$ crosses under

**else**

**print** $\overline{w_{k+1} w_k}$ crosses under

**if** $y_i = y_k$ **then**

**if** $b_i < b_k$ **then**

**print** $\overline{w_{i+1} w_i}$ crosses under

**else**

**print** $\overline{w_{k+1} w_k}$ crosses under

**if** $z_i = z_k$ **then**

**if** $c_i < c_k$ **then**

**print** $\overline{w_{i+1} w_i}$ crosses under

**else**

**print** $\overline{w_{k+1} w_k}$ crosses under

## 2.1. Fundamental Group

Let $\hat{K} = (w_1, w_2, \cdots, w_n)$ be an oriented discrete knot and $c_1, c_2, \cdots, c_r$ be its crossings. We will compute the fundamental group *K* denoted by $\Pi_1(K)$, using the Wirtinger presentation (see [3,4]). We will start describing the set of generators of $\Pi_1(K)$ (see [2]).

Suppose that $c_j$ is the crossing point of the linear segment $l_{k_j} = \overline{w_{k_j+1} w_{k_j}}$ over the linear segment $l_{i_j} = \overline{w_{i_j+1} w_{i_j}}$. Now we are going to rearrange the crossings $c_j$ in such a way that $i_1 < i_2 < \cdots < i_r$. Let $\gamma_i$ be the segment of $\hat{K}$ whose endpoints are $c_i$ and $c_{i+1}$ (where $c_{r+1} = c_1$). Thus $\gamma_1 = (i_1 + 1, i_1 + 2, \cdots, i_2)$, $\gamma_2 = (i_2 + 1, i_2 + 2, \cdots, i_3), \cdots, \gamma_r = (i_r + 1, i_r + 2, \cdots, i_1)$, where each index $i_j$ is considered mod *n*. We know by Wirtinger presentation that there exists a bijection between the set of segments $\gamma_i, i = 1, \cdots, r$ and the set of generators of $\Pi_1(K)$, so the set of generators of $\Pi_1(K)$ is $\{\alpha_1, \cdots, \alpha_r\}$.

Again, by the Wirtinger presentation we know that for each $c_j = l_{i_j} \cap l_{k_j}$ corresponds a relation among the generators $\alpha_l$, $\alpha_{l+1}$ and $\alpha_s$, where the indexes *s* and *l* satisfy that $k_j \in \gamma_s$ and $i_j \in \gamma_l$. So

- If $\det\left[ \left( w_{i_j+1} - w_{i_j} \right) \left( w_{k_j+1} - w_{k_j} \right) \right] > 0$, then we have the relation $R_l$ given by $\alpha_l \alpha_s = \alpha_s \alpha_{l+1}$.
- If $\det\left[ \left( w_{i_j+1} - w_{i_j} \right) \left( w_{k_j+1} - w_{k_j} \right) \right] < 0$, then we write the relation $R_l$ given by $\alpha_s \alpha_l = \alpha_{l+1} \alpha_s$.

Therefore $\Pi_1(K)=\{\alpha_1,\cdots,\alpha_r|R_1,\cdots,R_r\}$.

**Algorithm 3.** Fundamental group

**Require:** The list of indexes of intersection points $I[(i_1,k_1),(i_2,k_2),\cdots,(i_r,k_r)]$.

Create lists

$\gamma_1=(i_1+1,i_1+2,\cdots,i_2)$

$\gamma_2=(i_2+1,i_2+2,\cdots,i_3)$

$\vdots$

$\gamma_r=(i_r+1,i_r+2,\cdots,i_1)$

**for all** $i_j,k_j\in L$ **do**

Search $r$ and $l$ such that $k_j\in\gamma_s$ and $i_j\in\gamma_l$.

Create a matrix,

$A=\left[\begin{pmatrix}w_{i_j+1}-w_{i_j}\end{pmatrix}\ \begin{pmatrix}w_{k_j+1}-w_{k_j}\end{pmatrix}\right]$

**if** $\det(A)<0$ **then**

**print** $\alpha_s\alpha_l=\alpha_{l+1}\alpha_s$

**else**

**print** $\alpha_l\alpha_s=\alpha_s\alpha_{l+1}$

## 2.2. Seifert Surface

Given a knot $K$ there exists an algorithm to construct its Seifert surface via an oriented diagram of it (for details see [3,4]). Roughly speaking, suppose that the corresponding diagram has $r$ crossings, then the crossings are replaced by two disjoint arcs respecting the orientation. At the end, we obtain a collection of $s$ simple closed curves called *Seifert curves*. We construct a *Seifert surface F* for $K$ considering each Seifert curve as the boundary of a disk. The disks are connected at each crossing by a twisted band (so we need $r$ bands). The genus of $F$ is $\frac{1-s+r}{2}$. The *Seifert genus* of a knot is the minimal genus possible for a Seifert surface of that knot.

Next, we apply the above algorithm to our case. As in the previous section, $\hat{K}=(w_1,w_2,\cdots,w_n)$ denotes an oriented discrete knot and $c_1,c_2,\cdots,c_r$ are its crossings, where $c_j$ is as above. Let $A=(i_1,\cdots,i_r)$ and $B=(k_1,\cdots,k_r)$. In [2] it was defined the bijective map $\sigma:(w_1,w_2,\cdots,w_n)\to(w_1,w_2,\cdots,w_n)$, given by $\sigma(w_l)=w_{l+1}$ if $l\notin A$ and $l\notin B$, $\sigma(w_l)=w_{k_s+1}$ if $l=i_s$, or $\sigma(w_l)=w_{i_s+1}$ if $l=k_s$.

This permutation can be expressed as a product of $s$ disjoint cycles, where each cycle represents a Seifert curve. Hence we can compute the Seifert genus $g$.

**Algorithm 4.** Seifert surface

**Require:** The set or indexes $A=(i_1,\cdots,i_r)$ and $B=(k_1,\cdots,k_r)$ such that $w_{i_j+1}w_{i_j}$ crosses under $w_{k_j+1}w_{k_j}$. The empty sets $L_1,L_2,\cdots,L_n$, and $r\leftarrow 1$ and the genus of the knot $g\leftarrow 0$.

Create a function $\sigma\in\mathbb{Z}$ where $l$ is an index

$\sigma(w_l)=w_{l+1}$ if $l\notin A$ and $l\notin B$

If $l\in A\Rightarrow l=i_s$ so that $\sigma(w_l)=w_{k_s+1}$

If $l\in B\Rightarrow l=k_s$ so that $\sigma(w_l)=w_{i_s+1}$

Now we will form cycles

**for** $i_s\notin L_r\cup L_{r-1}\cup\cdots\cup L_1$ **do**

Add $i_s$ to $L_r$ and $m\leftarrow i_s$

**while** $i_s\neq\sigma(m)$ **do**

Add $\sigma(m)$ to $L_r$ and $m\leftarrow\sigma(m)$

$r$++

**print** $(L_1)(L_2)\cdots(L_r)$

$g\leftarrow\frac{1-s+r}{2}$

**print** $g$

## 2.3. Jones Polynomial

The Jones polynomial is a very important invariant of an oriented knot $K$. We compute the Jones polynomial of a cubic knot $K$ using the method described on "The knot atlas website" ([5]) applied to our case.

Let $\hat{K}=(w_1,w_2,\cdots,w_n)$ be as above. Let $(i_j,k_j)$ be pairs of indexes such that $l_{k_j}$ crosses over $l_{i_j}$, $j=1,\cdots,r$. Consider the sequence $c=(i_1,i_1+1,k_1,k_1+1,\cdots,i_r,i_r+1,k_r,k_r+1)$ and up to rearrangement, we can assume that $C=(l_1,l_1+1,l_2,l_2+1,\cdots,l_{2k},l_{2k}+1)$ is an increasing sequence. Consider the segments of curves

$C_1=(l_1+1,l_1+2,\cdots,l_2)$, $C_2=(l_2+1,l_2+2,\cdots,l_3)$, $\cdots$, $C_{2k}=(l_{2k}+1,l_{2k}+2,\cdots,l_1)$, where the index $n+1$ is equal to 1.

For each pair $(i_s,k_s)$, consider the segments $C_{as}$, $C_{bs}$, $C_{cs}$ and $C_{ds}$ such that $i_s\in C_{cs}$, $i_s+1\in C_{as}$, $k_s\in C_{ds}$ and $k_s+1\in C_{bs}$. Now we take the following expressions

- If $\det\left[\begin{pmatrix}w_{i_s+1}-w_{i_s}\end{pmatrix}\ \begin{pmatrix}w_{k_s+1}-w_{k_s}\end{pmatrix}\right]<0$, then we consider $A[as,ds][bs,cs]+A^{-1}[as,bs][cs,ds]$,
- if $\det\left[\begin{pmatrix}w_{i_s+1}-w_{i_s}\end{pmatrix}\ \begin{pmatrix}w_{k_s+1}-w_{k_s}\end{pmatrix}\right]>0$, then we consider $A[as,ds][cs,ds]+A^{-1}[as,ds][bs,cs]$,

as formal sums, where $A$ denotes a variable and $s=1,\cdots,r$. Notice that in the above expressions the order does not matter; for instance, the expressions $[as, ds]$ and $[ds, as]$ are equal. Now, we compute the formal product of all the above expressions to obtain a new expression $Q$.

We calculate *the Kauffman bracket*, denoted by $t(A)$ from $Q$ replacing first $[as,bs][bs,cs]$ by $[as,cs]$ and afterward replace $[as,as]$ by $-A^2-A^{-2}$. Next we compute *the writhe number* denoted by $w$, which is equal to the number of positive crossings minus the number of negative crossings.

Finally, the Jones polynomial $J(q)$ is equal to

$$J(q)=\frac{\left(-q^{\frac{1}{4}}\right)^{3w}t\left(q^{\frac{1}{4}}\right)}{-q^{\frac{1}{2}}-q^{-\frac{1}{2}}},$$

where $q$ denotes a variable, $t\left(q^{\frac{1}{4}}\right)$ is the Kauffman bracket evaluated at $q^{\frac{1}{4}}$ and $w$ is the writhe number.

**Algorithm 5.** Jones polynomial

**Require:** The list of indexes of intersection points $I\left[\left(i_1,k_1\right),\left(i_2,k_2\right),\cdots,\left(i_r,k_r\right)\right]$, *bracketKauffman*, *polynomJones* and *writhe* ← 0.

Create array
$c=\left(i_1,i_1+1,k_1,k_1+1,\cdots,i_r,i_r+1,k_r,k_r+1\right)$
Sort $C$ and produces $C\left[l_1,l_2,\cdots,l_n\right]$ where $l_1 \le l_2 \le \cdots \le l_n$
Take curve segments
$C_1=\left[l_1+1,l_1+2,\cdots,l_2\right]$
$C_2=\left[l_2+1,l_2+2,\cdots,l_3\right]$
$\vdots$
$C_n=\left[l_n+1,l_n+2,\cdots,l_1\right]$

**for all** $i_s,k_s \in L$ **do**
  Take $i_s \in C_l$, $k_s \in C_m$, $i_{s+1} \in C_n$, $k_{s+1} \in C_p$
  such that *l*, *m*, *n*, *p* are the labels of the edges around that crossing, starting from the incoming lower edge *l* and proceeding counterclockwise direction.
  Example:
  $[l,m,n,p]$ such that *m* is next to *l* in counterclockwise direction, *n* is next to *m* in counterclockwise direc tion, etc.
  Save $[l,m,n,p]$
Replace each $[l,m,n,p] \to A[l,p][m,n]+A^{-1}[l,m][n,p]$
*bracketKauffman* ← Multiply all replacements
*bracketKauffman*←Replace $[as,bs][bs,cs] \to [as,cs]$ and $[as,bs]^2 \to [as,as]$
*bracketKauffman* → Replace and simplify $[as,as] \to -A^2-A^{-2}$
**print** $t(A)$ = *bracketKauffman*
**for all** $i_s,k_s \in L$ **do**
  Create matrix $U=\left[\left(w_{i_s+1}-w_{i_s}\right)\ \left(w_{k_s+1}-w_{k_s}\right)\right]$
  **if** $\det(U)<0$ **then**
    *writhe++*
  **else**
    *writhe--*

$$PolynomJones \leftarrow \frac{(-A)^{3writhe}\, t(A)}{-A^2-A^{-2}}$$

*PolynomJones* ← replace and simplify $A \to q^{\frac{1}{4}}$
**print** $J(q)$ = *PolynomJones*

# 3. Examples

## 3.1. Left-Handed Trefoil Knot

Considering the left-handed trefoil knot as a cubic knot, see **Figure 1**, where you can see the corresponding vectors $v_i$'s; $i=1,\cdots,24$.

Now, we apply our program to compute its fundamental group, genus Seifert and Jones polynomial. See **Figure 2**. In this case, its fundamental group has 3 generators a1, a2, a3; and relations: a3a2 = a2a1, a1a3 = a3a2, a2a1 = a1a3. Its genus surface is one and its Jones polynomial is $J(q)=-q^4+q^3+q$.

## 3.2. Figure Eight Knot

Considering the figure eight knot as a cubic knot, in this case, we have 40 vertices. See **Figure 3**.

We now compute its fundamental group, its genus surface and its Jones polynomial. Thus, its fundamental group has 4 generators a1, a2, a3, a4; and relations: a2a4 = a1a2, a1a3 = a3a2, a4a2 = a3a4, a3a1 = a1a4. Its genus surface is one and its Jones polynomial is
$J(q)=q^2-q+1-q^{-1}+q^{-2}$. See **Figure 4**.

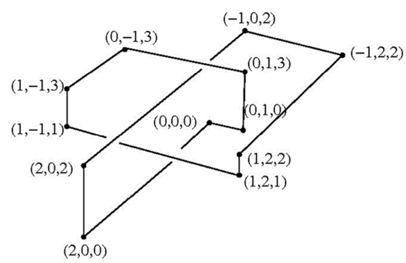


**Figure 1. Cubic left-handed trefoil knot.**

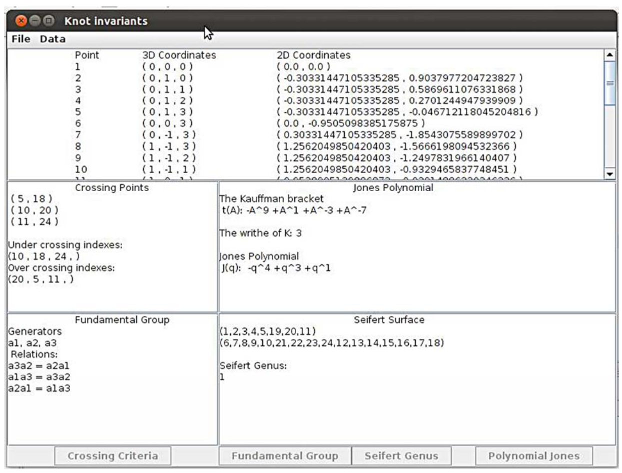


**Figure 2. Left-handed trefoil knot invariants.**

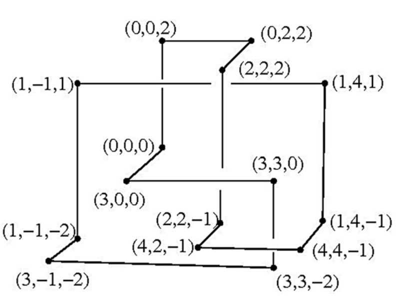


**Figure 3. Cubic eight knot.**

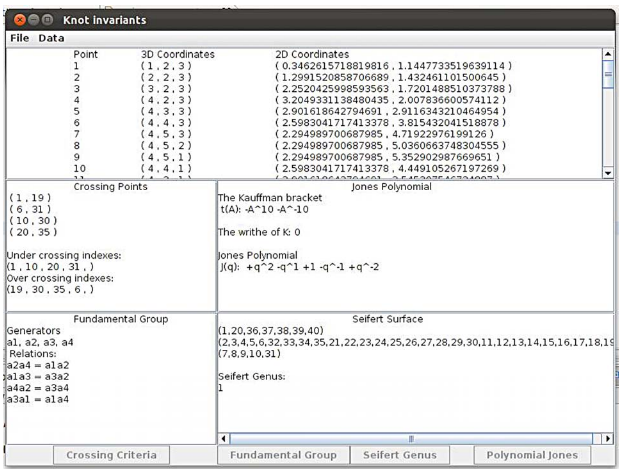


**Figure 4. Figure eight knot invariants.**

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