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Approximation by Splines of Hermite Type

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ABSTRACT

The approximation evaluations by polynomial splines are well-known. They are obtained by the similarity principle; in the case of non-polynomial splines the implementation of this principle is difficult. Another method for obtaining of the evaluations was discussed earlier (see [1]) in the case of nonpolynomial splines of Lagrange type. The aim of this paper is to obtain the evaluations of approximation by non-polynomial splines of Hermite type. Considering a linearly independent system of column-vectors $\{\boldsymbol{a}_j\}_{j\in\{0,1,\cdots,m\}}$, $\boldsymbol{a}_j \in R^{m+1}$. Let $A \overset{\text{def}}{=} (\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_m)$ be square matrix. Supposing that $\varphi \overset{\text{def}}{=} (\varphi_0, \varphi_1, \cdots, \varphi_m)^{\mathrm{T}}$ and $\omega \overset{\text{def}}{=} (\omega_0, \omega_1, \cdots, \omega_m)^{\mathrm{T}}$ are columns with components from the linear space $\mathcal{F}$ such that $A\omega = \varphi$. Let $g \overset{\text{def}}{=} (g_0, g_1, \cdots, g_m)^{\mathrm{T}}$ be vector with components g_j belonging to conjugate space $\mathcal{F}^*$. For an element $u \in \mathcal{F}$ we consider a linear combination of elements $\{\omega_j\}_{j\in\{0,1,\cdots,m\}}$: $\tilde{u} \overset{\text{def}}{=} \sum_{j=0}^{m} \langle g_j, u\rangle \omega_j$. By definition, put $\langle g, u\rangle \overset{\text{def}}{=} (\langle g_0, u\rangle, \langle g_1, u\rangle, \cdots, \langle g_m, u\rangle)^{\mathrm{T}}$. The discussions are based on the next assertion. *The following relation holds*: $u - \tilde{u} = \det A^{-1} \det \begin{pmatrix} A & \varphi \\ \langle g, u\rangle^{\mathrm{T}} & u \end{pmatrix}$, where the second factor on the right-hand side is the determinant of a block-matrix of order $m + 2$. Using this assertion, we get the representation of residual of approximation by minimal splines of Hermite type. Taking into account the representation, we get evaluations of the residual and calculate relevant constants. As a result the obtained evaluations are exact ones for components of generated vector-function $\varphi(t)$.

Keywords: Splines; Errors of Approximations

1. Representation of Approximation Residual

For convenience we shall give scheme of representation of the approximation residual in general situation (see also [1]).

We consider a linearly independent system of column-vectors $\{\boldsymbol{a}_j\}_{j\in\{0,1,\cdots,m\}}$ (where m is a natural number) in the space R^{m+1}. The matrix A composed of these columns is denoted by

$$A \overset{\text{def}}{=} (\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_m). \tag{1}$$

Let $\mathcal{F}$ be linear space.

Suppose that $\varphi \overset{\text{def}}{=} (\varphi_0, \varphi_1, \cdots, \varphi_m)^{\mathrm{T}}$ and $\omega \overset{\text{def}}{=} (\omega_0, \omega_1, \cdots, \omega_m)^{\mathrm{T}}$ are columns with components belonging to the space $\mathcal{F}$; assume the relation

$$A\omega = \varphi \tag{2}$$

is valid; matrix A is defined by (1).

Let $g \overset{\text{def}}{=} (g_0, g_1, \cdots, g_m)^{\mathrm{T}}$ be vector with components g_j belonging to conjugate space $\mathcal{F}^*$.

For an element $u \in \mathcal{F}$ we consider a linear combination of elements $\{\omega_j\}_{j\in\{0,1,\cdots,m\}}$:

$$\tilde{u} \overset{\text{def}}{=} \sum_{j=0}^{m} \langle g_j, u \rangle \omega_j. \quad (3)$$

From (2) and (3) it follows that

$$\tilde{u} = \left(\langle g, u \rangle, A^{-1}\varphi \right) = \left(\left(A^{\mathrm{T}} \right)^{-1} \langle g, u \rangle, \varphi \right), \quad (4)$$

where $\langle g, u \rangle$ denotes the column-vector in $\mathbb{R}^{m+1}$, namely, $\langle g, u \rangle \overset{\text{def}}{=} \left(\langle g_0, u \rangle, \langle g_1, u \rangle, \cdots, \langle g_m, u \rangle \right)^{\mathrm{T}}$. The outer round brackets in (4) mean the inner product of $m+1$-dimensional vectors.

Theorem 1 *The following relation holds*:

$$u - \tilde{u} = \det A^{-1} \det \begin{pmatrix} A & \varphi \\ \langle g, u \rangle^{\mathrm{T}} & u \end{pmatrix}, \quad (5)$$

where the second factor on the right-hand side is the determinant of a block-matrix of order $m+2$.

Proof *By* (4), *we have* $\tilde{u} = \left(\left\langle \left(A^T \right)^{-1} g, u \right\rangle, \varphi \right)$. *Hence*

$$\tilde{u} = \det A^{-1} \sum_{i=0}^{m} \sum_{j=0}^{m} A_{ij} \langle g_j, u \rangle \varphi_i, \quad (6)$$

where A_{ij} is the cofactor of an entry a_{ij} of the matrix A. By (6), we can represent the difference $\tilde{u} - u$ as the product of determinants, written as

$$u - \tilde{u} = \det A^{-1} \begin{vmatrix} \boldsymbol{a}_0 & \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_m & \varphi \\ \langle g_0, u \rangle & \langle g_1, u \rangle & \cdots & \langle g_m, u \rangle & u \end{vmatrix}. \quad (7)$$

The equality (7) is equivalent to the equality (5).

2. Representation of the Remainder of Approximation by Elementary Hermite Type Splines

On (α, β) we consider a grid of the form

$$X : \cdots < x_{-1} < x_0 < x_1 < \cdots,$$

$$\lim_{j \to -\infty} x_j = \alpha, \quad \lim_{j \to +\infty} x_j = \beta.$$

We set

$$G \overset{\text{def}}{=} \bigcup_{j \in \mathbb{Z}} \left(x_j, x_{j+1} \right), \quad h \overset{\text{def}}{=} \sup_{j \in \mathbb{Z}} \left(x_{j+1} - x_j \right).$$

Let $\varphi(t) = \left(\varphi_0(t), \varphi_1(t), \cdots, \varphi_m(t) \right)^{\mathrm{T}}$ be $m+1$-component vector-function with components in $C^m(\alpha, \beta)$. We assume that Wronskian of the components is separated from zero.

Consider function $u(t)$, $u \in C^m(\alpha, \beta)$, and introduce notation

$$\varphi_j^{(i)} \overset{\text{def}}{=} \varphi^{(i)}(x_j), \quad u_j^{(i)} \overset{\text{def}}{=} u^{(i)}(x_j). \quad (8)$$

Let symbol $|\mathcal{M}|$ denote the number of elements of a set $\mathcal{M}$.

We assume that natural numbers l, m, q, s, s_i, l_i comply with relations $l = l_0 \ge l_1 \ge \cdots \ge l_{q-1} \ge 1$, $s = s_0 \ge s_1 \ge \cdots \ge s_{q-1} \ge 1$, $\sum_{i=0}^{q-1} (s_i + l_i) = m+1$.

By definition, put

$$q_r \overset{\text{def}}{=} \left| \left\{ i \middle| 1 - l_i \le r \le s_i \right\} \right|,$$

where $r \in \{1-l, 2-l, \cdots, s\}$. Obviously $1 \le q_r \le q$.

We introduce the functions $\omega_{j,i}(t)$ by the approximate relations

$$\sum_{j=k+1-l}^{k+s} \sum_{i=0}^{q_{j-k}-1} \varphi_j^{(i)} \omega_{j,i}(t) = \varphi(t) \quad (9)$$

$$\forall t \in (x_k, x_{k+1}) \quad \forall k \in \mathbb{Z}, \operatorname{supp} \omega_{j,i} = \left[x_{j-s_i}, x_{j+l_i} \right].$$

Consider square matrix A_k of the order $m+1$ (see notation (8)),

$$A_k \overset{\text{def}}{=} \Big(\varphi_{k+1-l}, \varphi'_{k+1-l}, \cdots, \varphi_{k+1-l}^{(q_{1-l}-1)}, \varphi_{k+2-l}, \varphi'_{k+2-l}, \cdots, \varphi_{k+2-l}^{(q_{2-l}-1)}, \cdots, \varphi_{k+s}, \varphi'_{k+s}, \cdots, \varphi_{k+s}^{(q_s-1)} \Big),$$

and vector-function

$$\omega_{(k)}(t) \overset{\text{def}}{=} \Big(\omega_{k+1-l,0}, \omega_{k+1-l,1}, \cdots, \omega_{k+1-l,q_{1-l}-1}, \omega_{k+2-l,0}, \omega_{k+2-l,1}, \cdots, \omega_{k+2-l,q_{2-l}-1}, \cdots, \omega_{k+s,0}, \omega_{k+s,1}, \cdots, \omega_{k+s,q_s-1} \Big)^{\mathrm{T}};$$

then the relations (9) may be rewritten as

$$A_k \, \omega(t)_{(k)} = \varphi(t) \quad \forall t \in (x_k, x_{k+1}) \quad \forall k \in \mathbb{Z},$$

$$\operatorname{supp} \omega_{j,i} = \left[x_{j-s_i}, x_{j+l_i} \right].$$

It can be proved (for example, see [2]) that the matrix A_k is invertible. Hence the functions $\omega_{j,i}(t)$ are defined uniquely and they are linear independent. If $q = q_r$, $r = 1-l, 2-l, \cdots, s$, then the functions $\omega_{j,i}(t)$ belong to $C^{q-1}(\alpha, \beta)$, and functional system $\{ g_{j,i} \}$ defined by formula

$$\langle g_{j,i}, u \rangle \overset{\text{def}}{=} u^{(i)}(x_j),$$

is biorthogonal to the system $\{ \omega_{j',i'}(t) \}$ so that

$$\langle g_{j,i}, \omega_{j',i'} \rangle = \delta_{i,i'} \delta_{j,j'} \quad \forall i, i' \in \mathbb{Z}, \ j, j' \in \{0, 1, \cdots, q-1\}.$$

Rewrite the system (9) in the form

$$\sum_{r=1-l}^{s} \sum_{i=0}^{q_r-1} \varphi_{k+r}^{(i)} \omega_{k+r,i}(t) = \varphi(t) \quad \forall t \in (x_k, x_{k+1}) \quad \forall k \in \mathbb{Z}. \quad (10)$$

Under condition $t \to x_k + 0$ we have

$$\omega_{k+r,i'}^{(i)}(x_k + 0) = \delta_{i,i'} \delta_{r,0} \quad i, i' = 0, 1, \cdots, q_r - 1, \ 1 - l \le r \le s.$$

Analogously on the adjacent interval we get

$$\omega^{(i)}_{k+r-1,i'}\left(x_k-0\right)=\delta_{i,i'}\delta_{r,1}\quad i=0,1,\cdots,q_r-1,\quad 2-l\le r\le s+1.$$

Discuss the linear space

$$\mathbb{S}^H_{X,\varphi}\overset{\text{def}}{=}Cl_p\mathcal{L}\left(\left\{\omega_{j,i}\right\}_{i\in\mathbb{Z},\,j\in\{0,1,\cdots,q-1\}}\right),$$

where $\mathcal{L}(\dots)$ is the linear hull of the elements in the curly brackets and Cl_p means the closure of the linear hull in the topology of pointwise convergence.

We call $\mathbb{S}^H_{X,\varphi}$ *the space of elementary Hermite type* (X,φ)*-splines*.

By definition, put

$$\langle g,u\rangle_k\overset{\text{def}}{=}\Big(u_{k+1-l},u'_{k+1-l},\cdots,u^{(q_{1-l}-1)}_{k+1-l},u_{k+2-l},u'_{k+2-l},\cdots,$$
$$u^{(q_{2-l}-1)}_{k+2-l},\cdots,u_{k+s},u'_{k+s},\cdots,u^{(q_s-1)}_{k+s}\Big).$$

We consider the function $\tilde u(t)$ defined by

$$\tilde u(t)\overset{\text{def}}{=}\sum_{r=1-l}^{s}\sum_{i=0}^{q_r-1}u^{(i)}_{k+r}\omega_{k+r,i}(t)\quad\forall t\in\left(x_k,x_{k+1}\right)\quad\forall k\in\mathbb{Z}.\tag{11}$$

Theorem 2 *For* $t\in\left[x_k,x_{k+1}\right]$, $k\in\mathbb{Z}$,

$$u(t)-\tilde u(t)=\det A_k^{-1}\begin{vmatrix}A_k & \varphi(t)\\ \langle g,u\rangle_k & u(t)\end{vmatrix},\tag{12}$$

where the second factor on the right-hand side is the determinant of the square matrix of order $m+2$ written in the block form.

Proof *We can obtain the identity* (12) *by expanding the second determinant of right part of* (12) *and by usage of the relations* (10)-(11) (*cf.* [1]).

3. Some Auxiliary Assertions

Let m,n, $p_0,p_1,\cdots,p_n$ be natural numbers with property $p_0+p_1+\cdots+p_n=m+1$; let $a,b,z_0,\cdots,z_n$ be real numbers, which comply with inequalities $a\le z_0<z_1<\cdots<z_{n-1}<z_n\le b$. Let us put $p\overset{\text{def}}{=}\left(p_0,p_1,\cdots,p_n\right)$, $z\overset{\text{def}}{=}\left(z_0,z_1,\cdots,z_n\right)$.

Lemma 1 *For arbitrary* $m+1$*-component vector-function* $\psi(t)\in C^m(a,b)$ *the representation*

$$\det\Big(\psi(z_0),\psi'(z_0),\cdots,\psi^{(p_0-1)}(z_0),\psi(z_1),\psi'(z_1),\cdots,$$
$$\psi^{(p_1-1)}(z_1),\cdots,\psi(z_n),\psi'(z_n),\cdots,\psi^{(p_n-1)}(z_n)\Big)$$
$$=\mathcal{J}(p,z)\det\Big(\psi(z_0),\psi'(z_0),\cdots,\psi^{(p_0-1)}(z_0),\psi^{p_0}(\xi_1),$$
$$\cdots,\psi^{(p_0+p_1-1)}(\xi_1),\cdots,\psi^{(p_0+p_1+\cdots+p_{n-1})}(\xi_n),$$
$$\cdots,\psi^{(m)}(\xi_n)\Big)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_n\tag{13}$$

is valid; here $\mathcal{J}(p,z)$ is a linear operator of integration over parallelepiped

$$\Pi_n\overset{\text{def}}{=}\left\{\left(\xi_1,\xi_2,\cdots,\xi_n\right)\middle|\,\forall\xi_i\in\left[z_{i-1},z_i\right],i=1,2,\cdots,n\right\}$$

with nonnegative kernel.

Proof *We consider the case* $n=3$, $p_0=2$, $p_1=p_2=1$, $m=3$. *Introduce value* y_1 *with property* $z_0<y_1<z_1$ *and use notation*

$$y_0=z_0,\quad y_2=z_1,\quad y_3=z_2\tag{14}$$

so that $a<y_0<y_1<y_2<y_3<b$.

Using the additivity property of determinants and integrals and applying the Newton?-Leibnitz formula, we find

$$\det\left(\psi(y_0),\psi(y_1),\psi(y_2),\psi(y_3)\right)$$
$$=\mathcal{J}_1(y)\det\left(\psi(y_0),\psi'(\xi_1),\psi'(\eta),\psi'(\theta)\right)\mathrm{d}\xi_1\mathrm{d}\eta\mathrm{d}\theta,$$

where $y=\left(y_0,y_1,y_2,y_3\right)$,

$$\mathcal{J}_1(y)f\left(\xi_1,\eta,\theta\right)\mathrm{d}\xi_1\mathrm{d}\eta\mathrm{d}\theta\overset{\text{def}}{=}\int_{y_0}^{y_1}\mathrm{d}\xi_1\int_{y_1}^{y_2}\mathrm{d}\eta\int_{y_2}^{y_3}f\left(\xi_1,\eta,\theta\right)\mathrm{d}\theta.\tag{15}$$

Similarly,

$$\det\left(\psi(y_0),\psi(y_1),\psi(y_2),\psi(y_3)\right)$$
$$=\mathcal{J}_2(y)\det\left(\psi(y_0),\psi'(\xi_1),\psi''(\xi_2),\psi''(\zeta)\right)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\zeta,$$

where

$$\mathcal{J}_2(y)f\left(\xi_1,\xi_2,\zeta\right)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\zeta=\int_{y_0}^{y_1}\mathrm{d}\xi_1\int_{y_1}^{y_2}\mathrm{d}\eta\int_{y_2}^{y_3}\mathrm{d}\theta\int_{\xi_1}^{\eta}\mathrm{d}\xi_2\int_{\eta}^{\theta}f\left(\xi_1,\xi_2,\zeta\right)\mathrm{d}\zeta.\tag{16}$$

Finally

$$\det\left(\psi(y_0),\psi(y_1),\psi(y_2),\psi(y_3)\right)$$
$$=\mathcal{J}_3(y)\det\left(\psi(y_0),\psi'(\xi_1),\psi''(\xi_2),\psi'''(\xi_3)\right)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3,$$

where

$$\mathcal{J}_3(y)f\left(\xi_1,\xi_2,\xi_3\right)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3=\int_{y_0}^{y_1}\mathrm{d}\xi_1\int_{y_1}^{y_2}\mathrm{d}\eta\int_{y_2}^{y_3}\mathrm{d}\theta\int_{\xi_1}^{\eta}\mathrm{d}\xi_2\int_{\eta}^{\theta}\mathrm{d}\zeta\int_{\xi_2}^{\zeta}f\left(\xi_1,\xi_2,\xi_3\right)\mathrm{d}\xi_3.\tag{17}$$

Integral operators $\mathcal{J}_i$ can be rewritten in the form

$$\mathcal{J}_i(y)f(\xi)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\overset{\text{def}}{=}\int_{\Pi_3}\mathcal{K}_i(y,\xi)f(\xi)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3,\ i=1,2,3,$$

where $\xi=\left(\xi_1,\xi_2,\xi_3\right)$, and

$$\Pi_3\overset{\text{def}}{=}\left\{\left(\xi_1,\xi_2,\xi_3\right)\middle|\,\forall\xi_i\in\left[y_{i-1},y_i\right],i=1,2,3\right\}.$$

It is obvious that

$$\begin{aligned}&y_0\le\xi_1\le y_1\le\eta\le y_2\le\theta\le y_3,\\&\xi_1\le\xi_2\le\eta\le\zeta\le\theta.\end{aligned}\tag{18}$$

Since the lower limit is no more than the upper one in the integrals in (15)-(17), the result of integration is nonnegative for any nonnegative continuous function $f(\xi)$. Hence the integral operations $\mathcal{J}_i, i=1,2,3$, have nonnegative kernels

By (17) we have

$$\det\left(\psi(y_0),\frac{\psi(y_1)-\psi(y_0)}{y_1-y_0},\psi(y_2),\psi(y_3)\right)$$
$$=\frac{1}{y_1-y_0}\int_{y_0}^{y_1}\mathrm{d}\xi_1\int_{y_1}^{y_2}\mathrm{d}\eta\int_{y_2}^{y_3}\mathrm{d}\theta\int_{\xi_1}^{\eta}\mathrm{d}\xi_2\int_{\eta}^{\theta}\mathrm{d}\zeta \tag{19}$$
$$\cdot\int_{\xi_2}^{\zeta}\det\left(\psi(y_0),\psi'(\xi_1),\psi''(\xi_2),\psi'''(\xi_3)\right)\mathrm{d}\xi_3.$$

Recall that vector-function $\psi(t)$ is continuously differentiable in neighborhood of the point y_0, and passaging to limit as $y_1\to y_0+0$, we get

$$\det\left(\psi(y_0),\psi'(y_0),\psi(y_2),\psi(y_3)\right)$$
$$=\int_{y_0}^{y_2}\mathrm{d}\eta\int_{y_2}^{y_3}\mathrm{d}\theta\int_{y_0}^{\eta}\mathrm{d}\xi_2\int_{\eta}^{\theta}\mathrm{d}\zeta \tag{20}$$
$$\cdot\int_{\xi_2}^{\zeta}\det\left(\psi(y_0),\psi'(y_0),\psi''(\xi_2),\psi'''(\xi_3)\right)\mathrm{d}\xi_3.$$

It follows easily that relation (20) can be written in the form

$$\det\left(\psi(y_0),\psi'(y_0),\psi(y_2),\psi(y_3)\right)$$
$$=\mathcal{J}_{(1)}(\bar{y})\det\left(\psi(y_0),\psi'(y_0),\psi''(\xi_2),\psi'(\xi_3)\right)\mathrm{d}\xi_2\mathrm{d}\xi_3,$$

where $\bar{y}=(y_0,y_2,y_3)$, and the operator $\mathcal{J}_{(1)}(\bar{y})$ is defined by identity

$$\mathcal{J}_{(1)}(\bar{y})\,f(y_0,\xi_2,\xi_3)\mathrm{d}\xi_2\mathrm{d}\xi_3$$
$$\stackrel{\text{def}}{=}\int_{y_0}^{y_2}\mathrm{d}\eta\int_{y_2}^{y_3}\mathrm{d}\theta\int_{y_0}^{\eta}\mathrm{d}\xi_2\int_{\eta}^{\theta}\mathrm{d}\zeta\int_{\xi_2}^{\zeta}f(y_0,\xi_2,\xi_3)\mathrm{d}\xi_3. \tag{21}$$

By relations (18) and (21) we see that the integral operator $\mathcal{J}_{(1)}(\bar{y})$ may be represented in the form

$$\mathcal{J}_{(1)}(\bar{y})\,f(y_0,\xi_2,\xi_3)\mathrm{d}\xi_2\mathrm{d}\xi_3$$
$$=\int_{\Pi_2}\mathcal{K}_{(1)}(\bar{y},\xi_2,\xi_3)\,f(y_0,\xi_2,\xi_3)\mathrm{d}\xi_2\mathrm{d}\xi_3,$$

where $\Pi_2\stackrel{\text{def}}{=}\left\{(\xi_1,\xi_2)\,\middle|\,\forall\xi_i\in[y_{i-1},y_i],i=2,3\right\}$, and $\mathcal{K}_{(1)}(\xi_2,\xi_3)$ is nonnegative function

Taking into account (14), we obtain $\mathcal{J}(p,z)=\mathcal{J}_{(1)}(\bar{y})$, where $p=(2,1,1)$, $z=(z_0,z_1,z_2)$, $\bar{y}=(y_0,y_2,y_3)$. Thus the assertion is true in discussed case.

Now consider the case of $n=1$, $p_0=3$, $p_1=1$.

Let y_2 is new variable, $z_0<y_2<z_1$; by definition put

$$y_0=z_0,\quad y_3=z_1, \tag{22}$$

so that $a<y_0<y_2<y_3<b$.

Under condition $y_2\to y_0$ according to Taylor formula we have

$$\psi(y_2)=\psi(y_0)+\psi'(y_0)(y_2-y_0)$$
$$+\psi''(y_0)(y_2-y_0)^2+O\left((y_2-y_0)^3\right),$$

whence we get

$$\det\left(\psi(y_0),\psi'(y_0),\psi(y_2),\psi(y_3)\right)$$
$$=\frac{(y_2-y_0)^2}{2}\det\left(\psi(y_0),\psi'(y_0),\psi''(y_0)\right.$$
$$\left.O(y_2-y_0),\psi(y_3)\right).$$

Thus by (21) we obtain

$$\det\left(\psi(y_0),\psi'(y_0),\psi''(y_0)+O(y_2-y_0),\psi(y_3)\right)$$
$$=\frac{2}{(y_2-y_0)^2}\int_{y_0}^{y_2}\mathrm{d}\eta\int_{y_0}^{\eta}F(\eta,\xi_2,y_2,y_3)\mathrm{d}\xi_2,$$

where

$$F(\eta,\xi_2,y_2,y_3)$$
$$\stackrel{\text{def}}{=}\int_{y_2}^{y_3}\mathrm{d}\theta\int_{\eta}^{\theta}\mathrm{d}\zeta\int_{\xi_2}^{\zeta}\det\left(\psi(y_0),\psi'(y_0),\psi''(\xi_2),\psi'''(\xi_3)\right)\mathrm{d}\xi_3. \tag{23}$$

It follows in the standard way that

$$\det\left(\psi(y_0),\psi'(y_0),\psi''(y_0)+O(y_2-y_0),\psi(y_3)\right)$$
$$=\frac{2}{(y_2-y_0)^2}\int_{y_0}^{y_2}\mathrm{d}\eta\int_{y_0}^{\eta}\mathrm{d}\xi_2\cdot F\left(\bar{\eta},\bar{\xi}_2,y_2,y_3\right)$$
$$=F\left(\bar{\eta},\bar{\xi}_2,y_2,y_3\right),$$

where $y_0\le\bar{\eta}\le y_2$, $y_0\le\bar{\xi}_2\le\eta\le y_2$.

Passaging to limit under $y_2\to y_0$ we obtain

$$\det\left(\psi(y_0),\psi'(y_0),\psi''(y_0),\psi(y_3)\right)=F(y_0,y_0,y_0,y_3);$$

taking into account (23), we rewrite the formula in the form

$$\det\left(\psi(y_0),\psi'(y_0),\psi''(y_0),\psi(y_3)\right)$$
$$=\int_{y_0}^{y_3}\mathrm{d}\theta\int_{y_0}^{\theta}\mathrm{d}\zeta\int_{y_0}^{\zeta}\det\left(\psi(y_0),\right.$$
$$\left.\psi'(y_0),\psi''(y_0),\psi'''(\xi_3)\right)\mathrm{d}\xi_3.$$

Thus

$$\det\left(\psi(y_0),\psi'(y_0),\psi''(y_0),\psi(y_3)\right)$$
$$=\mathcal{J}_{(2)}(y_0,y_3)\det\left(\psi(y_0),\psi'(y_0),\psi''(y_0),\psi'''(\xi_3)\right)\mathrm{d}\xi_3,$$

where

$$\mathcal{J}_{(2)}(y_0,y_3)\,f(y_0,\xi_3)\mathrm{d}\xi_3$$
$$=\int_{\Pi_1}\mathcal{K}_{(2)}(y_0,y_3,\xi_3)\,f(y_0,\xi_3)\mathrm{d}\xi_3;$$

here $\Pi_1\stackrel{\text{def}}{=}[y_2,y_3]$ and $\mathcal{K}_{(2)}(y_0,y_3,\xi_3)$ is nonnegative function.

Now recall notation (22); we obtain $\mathcal{J}(p,z)=\mathcal{J}_{(2)}(y_0,y_3)$, where $p=(3,1)$, $z=(z_0,z_1)$. This completes the proof in discussed case.

For an arbitrary natural p_0 one can obtain a similar representation via multiple integrals with the lower integration limit less than the upper one. Analogously the assertion is proved for $p_i, i=1,2,\cdots,n$. This completes the proof.

Denote $\theta(t)\overset{\text{def}}{=}(1,t,t^2,\cdots,t^m)^{\mathrm{T}}$ and introduce the function $e(z_0,z_1,\cdots,z_n)\equiv 1$.

Lemma 2 *If suppositions of Lemma* 1 *are fulfilled, then*

$$\mathcal{J}(p,z)e=\prod_{\gamma=1}^{m}\frac{1}{\gamma!}\prod_{j=1}^{n}\prod_{\sigma=0}^{p_j-1}\sigma!\prod_{i=1}^{j-1}(z_j-z_i)^{p_jp_i}. \tag{24}$$

Proof *Substituting vector-function* $\theta(t)$ *for* $\psi(t)$ *in* (13), *we have*

$$\begin{aligned}&\det\Big(\theta(z_0),\theta'(z_0),\cdots,\theta^{(p_0-1)}(z_0),\theta(z_1),\theta'(z_1),\\ &\quad\cdots,\theta^{(p_1-1)}(z_1),\cdots,\theta(z_n),\theta'(z_n),\cdots,\theta^{(p_0-1)}(z_n)\Big)\\ &=\mathcal{J}(p,z)\det\Big(\theta(z_0),\theta'(z_0),\cdots,\theta^{(p_0-1)}(z_0),\theta^{(p_0)}(\xi_1),\\ &\quad\cdots,\theta^{(p_0+p_1-1)}(\xi_1),\cdots,\theta^{(p_0+p_1+\cdots+p_{n-1})}(\xi_n),\\ &\quad\cdots,\theta^{(m)}(\xi_n)\Big)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_n.\end{aligned} \tag{25}$$

The determinant on the right-hand side of (25) contains a lower triangular matrix with entries $0!,1!,2!,\cdots,m!$ at the main diagonal so that right-hand side is equal to

$$\prod_{\gamma=1}^{m}\gamma!\cdot\mathcal{J}(p,z)e. \tag{26}$$

The left-hand side contains the determinant of matrix, which appears in Hermite interpolation problem

$$P^{(i)}(z_j)=b_j^{(i)},\quad i=1,2,\cdots,p_j-1,\ j=1,2,\cdots,n,$$

where $b_j^{(i)}$ are prescribed numbers and $P(t)=\sum_{j=0}^{m}a_jt^j$. Value of the mentioned determinant is known (see [3], p. 43); it is equal to

$$\prod_{j=1}^{n}\prod_{\sigma=0}^{p_j-1}\sigma!\prod_{i=1}^{j-1}(z_j-z_i)^{p_jp_i} \tag{27}$$

Equating of (26) to (27) gives (24). It completes the proof.

4. Evaluations of Approximation by Splines of Hermite Type

We assume that $u,\varphi\in C^{m+1}[\alpha,\beta]$ and

$$\left|\det\left(\varphi,\varphi',\varphi'',\cdots,\varphi^{(m)}\right)(t)\right|\ge c>0. \tag{28}$$

By the uniform continuity of the function under consideration on [a,b], from (28) we conclude that for any $\varepsilon\in(0,c)$ there exists $h_0(\varepsilon)$ such that for $h\in(0,h_0(\varepsilon)]$ and $\xi_1,\xi_2,\cdots,\xi_m\in[x_{k-l+1},x_{k+s}]$

$$\left|\det\left(\varphi(x_k),\varphi'(\xi_1),\varphi''(\xi_2),\cdots,\varphi^{(m)}(\xi_m)\right)\right|\ge c_\varepsilon>0, \tag{29}$$

where $c_\varepsilon=c-\varepsilon$.

By definition, put

$$\mathcal{D}(m,n,p,z)\overset{\text{def}}{=}\left[\prod_{\gamma=1}^{m}\gamma!\right]^{-1}\prod_{j=1}^{n}\prod_{\sigma=0}^{p_j-1}\sigma!\prod_{i=1}^{j-1}(z_j-z_i)^{p_jp_i}.$$

Lemma 3 *Under the assumption* (29), *for* $h\in(0,h_0(\varepsilon)]$ *the inequality*

$$|\det A_k|\ge c_\varepsilon\mathcal{D}(m,n,p,z) \tag{30}$$

is true; here $n=l+s-1$, $z_j=x_{k+1-l+j}$, $p_j=q_{1-l+j}$, $j=0,1,\cdots,n$.

Proof *We use Lemma* 1 *and represent* $\det A_k$ *in the form* (13) *for* $n=s+l-1$, $z_j=x_{k+1-l+j}$, $p_j=q_{1-l+j}$, $j=0,1,\cdots,n$, $\psi(t)=\varphi(t)$. *As a result, we find*

$$\begin{aligned}\det A_k&=J(p,z)\det\Big(\varphi(x_{k-l+1}),\varphi'(x_{k-l+1}),\cdots,\\ &\varphi^{(q_{k-l+1})}(x_{k-l+1}),\varphi(\xi_1),\varphi'(\xi_1),\cdots,\varphi^{(q_{k-l+1})}(\xi_1),\\ &\cdots,\varphi(\xi_n),\cdots,\varphi^{(m)}(\xi_n)\Big)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_n.\end{aligned}$$

Using the estimate (29), the positiveness of the kernel of the integral operation $\mathcal{J}(p,z)$, and the relation (24) obtained in Lemma 2, we derive the estimate (4.3) for $h\in(0,h_0(\varepsilon)]$.

Now we set

$$\psi(t)=(\varphi(t),u(t))^{\mathrm{T}},\quad n=l+s,\ z_l=t,\ p_l=1, \tag{31}$$

$$z_j=x_{k+1-l+j},\ p_j=q_{l-l+j}\ \text{ for } 0\le j\le l-1, \tag{32}$$

$$z_{j'}=x_{k-l+j'},\ p_{j'}=q_{j'-l}\ \text{ for } l+1\le j'\le n. \tag{33}$$

Lemma 4 *If* $u,\varphi\in C^{m+1}(\alpha,\beta)$, *then for* $t\in(x_k,x_{k+1})$ *the following inequality holds*:

$$\left|\det\begin{pmatrix}A_k & \varphi(t)\\ \langle g,u\rangle_k & u(t)\end{pmatrix}\right|\le D(m+1,s+l,\bar{p},\bar{z})D_{m+1}(\varphi,u), \tag{34}$$

where

$$\begin{aligned}D_{m+1}(\varphi,u)&\overset{\text{def}}{=}\max\Big|\det\Big(\psi(x_{k+1-l}),\psi'(x_{k+1-l}),\cdots,\\ &\psi^{(p_0-1)}(x_{k+1-l}),\psi^{p_0}(\xi_1),\cdots,\psi^{(p_0+p_1-1)}(\xi_1),\\ &\cdots,\psi^{(p_0+p_1+\cdots+p_{n-1})}(\xi_n),\cdots,\psi^{(m+1)}(\xi_n)\Big)\Big|,\end{aligned} \tag{35}$$

and the maximum is taken over

$$\xi_1,\xi_2,\cdots,\xi_n\in[x_{k-l+1},x_{k+s}]$$

Proof *By* (31)-(33) *the relation* (13) *may be written in the form*

$$\det\begin{pmatrix} \varphi_{k+1-l} & \cdots & \varphi_k^{(q_0-1)} & \varphi(t) & \varphi_{k+1} & \cdots & \varphi_{k+s}^{(q_s-1)} \\ u_{k+1-l} & \cdots & u_k^{(q_0-1)} & u(t) & u_{k+1} & \cdots & u_{k+s}^{(q_s-1)} \end{pmatrix}$$

$$= \mathcal{J}(p,z)\det\Big(\psi(x_{k+1-l}),\psi'(x_{k+1-l}),\cdots,\psi^{(p_0-1)}(x_{k+1-l}),$$
$$\psi^{(p_0)}(\xi_1),\cdots,\psi^{(p_0+p_1-1)}(\xi_1),$$
$$\psi^{(p_0+p_1+\cdots+p_{n-1})}(\xi_n),\cdots,\psi^{(m+1)}(\xi_n)\Big)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_n.$$

It is clear that conditions of Lemma 1 and Lemma 2 are fulfilled, and therefore the kernel of integral operator $\mathcal{J}(p,z)$ is nonnegative. By Lemma 2 we get evaluation (34)-(35).

Theorem 3 *If* $u,\varphi\in C^{m+1}(\alpha,\beta)$ *and* (29) *holds, then for* $t\in(x_k,x_{k+1})$

$$\left|u(t)-\tilde{u}(t)\right|\le\frac{1}{c_\varepsilon(m+1)!}\cdot\left|\prod_{j=k-l+1}^{k+s}(x_j-t)^{q_{j-k}}\right|\cdot D_{m+1}(\varphi,u), \quad (36)$$

where $D_{m+1}(\varphi,u)$ is defined by (35)

Proof *Usage* (34)-(35) *in* (12) *gives the evaluation* (36).

Corollary 1 *Under the assumptions of Theorem* 3, *the interpolation* $\tilde{u}(t)$ *of a function* $u(t)$ *is exact on elements of the space* $\Phi \overset{\text{def}}{=} L\{\varphi_0,\varphi_1,\cdots,\varphi_m\}$, *i.e.,*

$$\tilde{u}(t)\equiv u(t)\quad\forall t\in(\alpha,\beta)\quad\forall u\in\Phi. \quad (37)$$

Proof *If identity* $u(t)\equiv\varphi_j(t)$ *is fulfilled for a number* j, $j=0,1,\cdots,m$, *then in* (33) *the determinant* D_{m+1} *includes two identical rows*; *therefore* $D_{m+1}(\varphi,\varphi_j)=0$. *Thus the relation* (37) *is true.*

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