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Global Stability Analysis of a Delayed SEIQR Epidemic Model with Quarantine and Latent

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ABSTRACT

In this paper, we study a kind of the delayed SEIQR infectious disease model with the quarantine and latent, and get the threshold value which determines the global dynamics and the outcome of the disease. The model has a disease-free equilibrium which is unstable when the basic reproduction number is greater than unity. At the same time, it has a unique endemic equilibrium when the basic reproduction number is greater than unity. According to the mathematical dynamics analysis, we show that disease-free equilibrium and endemic equilibrium are locally asymptotically stable by using Hurwitz criterion and they are globally asymptotically stable by using suitable Lyapunov functions for any τ. Besides, the SEIQR model with nonlinear incidence rate is studied, and the τ_0 that the basic reproduction number is a unity can be found out. Finally, numerical simulations are performed to illustrate and verify the conclusions that will be useful for us to control the spread of infectious diseases. Meanwhile, the k_1, k_3 will effect changing trends of S, E, I, Q, R in system (1), which is obvious in simulations. Here, we take k_3 as an example to explain that.

Keywords: SEIQR Model; Lyapunov Function; Delay; Global Stability; Nonlinear Incidence Rate; Simulations

1. Introduction

Many people have been paying attention to the study of some epidemics, and have accumulated a lot of experience. By establishing reasonable mathematical models, they put forward the measures which controlled the spread of epidemics effectively. And many scholars researched specific diseases and considered the diseases with incubation period, recovery time, quarantine and so on [1-6]. So many epidemics were controlled. Generally speaking, when epidemics spread, there are many kinds of delays, which include immunity period delay [7-9], infectious period delay, incubation period delay. In [10], Enatsu *et al.* studied stability analysis of delayed SIR epidemic models with a class of nonlinear incidence rates, at the same time, they proved disease-free equilibrium was globally asymptotically stable and endemic equilibrium was permanent under certain conditions. At the same time, global stability of an SIR (where S, I, R denote the number of susceptible individuals, infectious individuals, recovery individuals) epidemic model with constant infectious period was studied by Zhang *et al.* [11], they showed the endemic equilibrium was globally asymptotically stable with appropriate Lyapunov functions. And in [12], Gao *et al.* discussed pulse vaccination of an SEIR (E denote the number of exposed individuals) epidemic model with delay and bilinear incidence. Meanwhile, impulsive vaccination of SEIR epidemic model with time delay and nonlinear incidence rate was researched by Zhao *et al.* [13], and showed the pulse system that was similar to the pulse system with bilinear incidence rate. Besides, on the basis of [13], Xu and Ma introduced the saturated incidence rate. Meanwhile, they showed disease-free equilibrium and endemic equilibrium were globally asymptotically stable under certain condition in [14]. However, in addition to the bilinear incidence rate, nonlinear incidence rate and saturated incidence rate, there were some scholars who studied the non-monotone incidence. For example, an SIRS epidemic model with pulse vaccination and non-monotonic incidence rate was discussed by Zhang *et al.* [15], and they proved the disease-free equilibrium and endemic equilibrium were asymptotically stable under certain conditions. Besides, some scholars studied a delayed SEIQR (Q denote the number of quarantined individuals) epidemic

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model with pulse vaccination and the quarantine measure, and they showed that the disease-free equilibrium of the system was globally attractive and endemic equilibrium was permanent under certain conditions. In this paper, we study a delayed SEIQR epidemic model without pulse on the basis of [14,16].

The organization of this paper is as follows: In Section 2, SIQR epidemic model and its basic reproduction number and existence of equilibrium are given. In Section 3, the local stability of endemic equilibrium and disease-free equilibrium is showed by using Hurwitz criterion. By using suitable Lyapunov functions and LaSalle's invariance principle, we prove the disease-free equilibrium is globally asymptotically stable when the basic reproduction number is less than unity and the endemic equilibrium is globally asymptotically stable when the basic reproduction number is greater than unity. At the same time, the system with the nonlinear incidence rate is discussed in Section 3. In Section 4, presents the numerical simulations of the system followed by a conclusion in Section 3. At last, a brief discussion is given in Section 5 to conclude this work.

2. Establishment of the Model

We establish the following SEIQR epidemic model, Here $S(t)$ represents the number of individuals who are susceptible to disease, that is, who are not yet infected at time *t*. $E(t)$ is the number of individuals who are infected but hardly infectious. So we think they can't infect other people, but they need to be quarantined. $I(t)$ represents the number of infected individuals who are infectious and are able to spread the disease by contact with susceptible individuals. $Q(t)$ is the number of infectious individuals who are quarantined at time *t*. $R(t)$ represents the number of recovered individuals at time *t*.

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = A - \mu_1 S - \beta SI, \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = \beta SI - \beta \mathrm{e}^{-\mu_2 \tau} S(t-\tau) I(t-\tau) - (\mu_2 + k_1 + k_3) E, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta \mathrm{e}^{-\mu_2 \tau} S(t-\tau) I(t-\tau) - (\mu_3 + \alpha + \nu + k_2) I, \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = k_1 E + k_2 I - (\mu_4 + \gamma) Q, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \gamma Q + \nu I + k_3 E - \mu_5 R. \end{cases} \tag{1}$$

The initial conditions for system (1) are

$$\begin{aligned} &S(\theta) = \phi_1(\theta), E(\theta) = \phi_2(\theta), I(\theta) = \phi_3(\theta), \\ &Q(\theta) = \phi_4(\theta), R(\theta) = \phi_5(\theta), \phi_i \geq 0, \theta \in [-\tau, 0], \\ &\phi_i(0) > 0, \phi_i \in C\left([-h, 0], \mathbb{R}_+^5\right), i = 1,2,3,4,5. \\ &\mathbb{R}_+^5 = \{x \in \mathbb{R} : x \geq 0\}. \end{aligned}$$

And the feasible region of the model with the initial conditions above is

$$\Omega_1 = \left\{ (S,E,I,Q,R) \in \mathbb{R}_+^5 \middle| S + E + I + Q + R \leq \frac{A}{\mu} \right\}.$$

Here, we presume that

$$\mu \leq \min\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}.$$

It is easy to show that Ω_1 is positively invariant with respect to system (1).

Where all the parameters are positive constants, A is the recruitment rate of the susceptible population, μ_1, μ_2, μ_3, μ_4, μ_5 are the natural death rate of the susceptible, exposed, infectious, quarantine and recovered respectively, β is the disease transmission coefficient, α_1 is the death rate due to disease without quarantine, α_2 is the death rate due to disease after quarantine, γ is the recovery rate after quarantine, ν is the recovery rate without quarantine, k_1, k_2 are quarantine rate of E, I respectively, k_3 is the recovery rate of E and τ is the latent period of the epidemic.

Because the variables *R* and *Q* do not appear in the first three equations in system (1), we further simplify system (1) and then obtain the following model

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = A - \mu_1 S - \beta SI, \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = \beta SI - \beta \mathrm{e}^{-\mu_2 \tau} S(t-\tau) I(t-\tau) - (\mu_2 + k_1 + k_3) E, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta \mathrm{e}^{-\mu_2 \tau} S(t-\tau) I(t-\tau) - (\mu_3 + \alpha + \nu + k_2) I. \end{cases} \tag{2}$$

In this paper, we are concerned with system (2).

The initial conditions for system (2) are

$$\begin{aligned} &S(\theta) = \phi_1(\theta), E(\theta) = \phi_2(\theta), I(\theta) = \phi_3(\theta), \phi_i \geq 0, \\ &\theta \in [-\tau, 0], \phi_i(0) > 0, \phi_i \in C\left([-h, 0], \mathbb{R}_+^3\right), i = 1,2,3. \\ &\mathbb{R}_+^3 = \{x \in \mathbb{R} : x \geq 0\}. \end{aligned}$$

And the feasible region of the model with the initial condition above is

$$\Omega_2 = \left\{ (S,E,I) \in \mathbb{R}_+^3 \middle| S + E + I \leq \frac{A}{\mu} \right\}.$$

Here, we presume that

$$\mu \le \min\{\mu_1, \mu_2, \mu_3\}.$$

It is easy to show that Ω_2 is positively invariant with respect to system (2).

According to the practical significance of the epidemic model, system (2) always has a disease-free equilibrium

$$E_0\left(\frac{A}{\mu_1}, 0, 0\right).$$

Denote the basic reproduction number of system (2)

$$R_0 = \frac{\beta A \mathrm{e}^{-\mu_2 \tau}}{\mu_1(\mu_3 + \alpha_1 + k_2 + \nu)}.$$

Define $a = \mu_3 + \alpha_1 + k_2 + \nu$. If the basic reproductive number $R_0 > 1$, system (2) has an unique endemic equilibrium

$$E^*\left(\frac{a}{\beta \mathrm{e}^{-\mu_2\tau}}, \frac{a(1-\mathrm{e}^{-\mu_2\tau})I^*}{\mathrm{e}^{-\mu_2\tau}(\mu_2 + k_1 + k_3)}, \frac{\beta \mathrm{e}^{-\mu_2\tau}A - a\mu_1}{a\beta}\right).$$

3. The Stability of Equilibrium

In this section, we discuss the local stability of endemic equilibrium and disease-free equilibrium of system (2) by analyzing the corresponding characteristic equations respectively. By defining reasonable Lyapunov functions, we resolve the global dynamics of equilibriums without requiring any extra conditions. In addition, system (2) with nonlinear incidence is studied.

3.1. Stability of Disease-Free Equilibrium

Theorem 3.1.1. If $R_0 < 1$, the disease-free equilibrium E_0 of system (2) is locally asymptotically stable for any τ in Ω_2. If $R_0 < 1$, it is unstable for any τ in Ω_2.

Proof. The characteristic matrix at the disease-free equilibrium E_0

$$J_{E_0} = \begin{pmatrix} \lambda + \mu_1 & 0 & \frac{\beta A}{\mu_1} \\ 0 & \lambda + \mu_2 + k_1 & \frac{\beta A \mathrm{e}^{-\mu_2\tau}\mathrm{e}^{-\lambda\tau}}{\mu_1} - \frac{\beta A}{\mu_1} \\ 0 & 0 & \lambda + a - \frac{\beta A \mathrm{e}^{-\mu_2\tau}\mathrm{e}^{-\lambda\tau}}{\mu_1} \end{pmatrix}.$$

When $\tau \neq 0$, the characteristic equation at the disease-free equilibrium E_0 of system (2) takes the form

$$(\lambda + \mu_1)(\lambda + \mu_2 + k_1)\left(\lambda + a - \frac{\beta A \mathrm{e}^{-\mu_2\tau}\mathrm{e}^{-\lambda\tau}}{\mu_1}\right) = 0.$$

Clearly, system (2) always has two negative real roots

$$\lambda_1 = -\mu_1, \lambda_2 = -(\mu_2 + k_1).$$

All other roots are given by the roots of equation

$$\lambda + a - \frac{\beta A \mathrm{e}^{-\mu_2\tau}\mathrm{e}^{-\lambda\tau}}{\mu_1} = 0,$$

$$\mathrm{Re}(\lambda) = \frac{\beta A \mathrm{e}^{-\mu_2\tau}\mathrm{e}^{-\mathrm{Re}(\lambda)}\cos(\tau \mathrm{Im}\lambda)}{\mu_1} - a.$$

Assume $\mathrm{Re}(\lambda) \ge 0$, $\mathrm{Re}(\lambda) \le \frac{\beta A \mathrm{e}^{-\mu_2\tau}}{\mu_1} - a.$

That is, $\mathrm{Re}(\lambda) \le a(R_0 - 1)$.

Because $R_0 < 1$, $\mathrm{Re}(\lambda) < 0$, which is contradictory. So $\mathrm{Re}(\lambda) < 0$, Therefore the disease-free equilibrium E_0 of system (2) is locally asymptotically stable.

If $R_0 > 1$, let

$$f(\lambda) = \lambda + a - \frac{\beta A \mathrm{e}^{-\mu_2\tau}\mathrm{e}^{-\lambda\tau}}{\mu_1},$$
$$f(0) < 0, f(+\infty) > 0,$$

so there is a positive real root at least. The disease-free equilibrium E_0 of system (2) is unstable.

When $\tau = 0$, it is easy for us to prove the disease-free equilibrium E_0 of system (2) is locally asymptotically stable.

Theorem 3.1.2. If $R_0 < 1$, the disease-free equilibrium E_0 of system (2) is globally asymptotically stable for any τ in Ω_2.

Proof. For $t > 0$, define a differentiable Lyapunov function

$$V = V_1 + V_2.$$

$$\begin{cases} V_1 = I, \\ V_2 = \int_{t-\tau}^{t} \beta \mathrm{e}^{-\mu_2\tau} S(\theta) I(\theta) \mathrm{d}\theta. \end{cases}$$

Obviously, $V = V_1 + V_2 \ge 0$.

Calculating the derivative of $V(t)$ along positive solutions of system (2), it follows that

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = \beta \mathrm{e}^{-\mu_2\tau} SI - aI.$$

According to the feasible region, $S \le \frac{A}{\mu}$.

So

$$\dot{V} \le \left(\frac{\beta \mathrm{e}^{-\mu_2} A}{\mu_1} - a\right) I.$$

That is,

$$\dot{V} \le a(R_0 - 1) I.$$

And when $R_0 < 1, \dot{V} \le 0$.

While $\dot{V} = 0$, if and only if,

$$S = A/\mu,\ E = 0, I = 0.$$

For all t, it is easy to show that E_0 is the largest invariant subset of the set $\{(S,E,I): \dot{V} = 0\}$. Because of LaSalle's invariance principle, disease-free equilibrium E_0 of system (2) is globally asymptotically stable. This completes the proof.

3.2. The Stability of Endemic Equilibrium

Theorem 3.2.1. For any τ, if $R_0 > 1$, the endemic equilibrium E^* of system (2) is locally asymptotically stable in Ω_2.

Proof. The characteristic matrix at the endemic equilibrium E^*

$$J_{E^*} = \begin{pmatrix} \lambda+\mu_1+\beta I^* & 0 & \beta S^* \\ \beta I^* e^{-\mu_2\tau}e^{-\lambda\tau} - \beta I^* & \lambda+\mu_2+k_1 & \beta S^* e^{-\mu_2\tau}e^{-\lambda\tau} - \beta S^* \\ -\beta I^* e^{-\mu_2\tau}e^{-\lambda\tau} & 0 & \lambda + a - \beta S^* e^{-\mu_2\tau}e^{-\lambda\tau} \end{pmatrix}.$$

Order $\beta I^* = b, \beta S^* = c$.

The characteristic equation at the endemic equilibrium E^* is

$$(\lambda+\mu_2+k_1)\left((\lambda+\mu_1+b)\left(\lambda - ce^{-\mu_2\tau}e^{-\lambda\tau} + a\right) + bce^{-\mu_2\tau}e^{-\lambda\tau}\right) = 0.$$

Clearly, system (2) always has a negative real root

$$\lambda = -(\mu_2 + k_1).$$

When $\tau = 0$, all other roots are given by the roots of equation

$$(\lambda+\mu_1+b)(\lambda - c + a) + bc = 0.$$

$a = c$, so

$$a + b + \mu_1 - c > 0, bc + (b+\mu_1)(a-c) > 0.$$

So according to Hurwitz criterion, the endemic equilibrium E^* of system (2) is locally asymptotically stable.

When $\tau \neq 0$, all other roots are given by the roots of equation

$$(\lambda+\mu_1+b)\left(\lambda - ce^{-\mu_2\tau}e^{-\lambda\tau} + a\right) + bce^{-\mu_2\tau}e^{-\lambda\tau} = 0.$$

Simplify, we can get

$$\lambda^2 + (a+\mu_1+b)\lambda - (\lambda+\mu_1)ce^{-\mu_2\tau}e^{-\lambda\tau} + a(\mu_1+b) = 0.$$

Let

$$\begin{cases} m_1 = a+b+\mu_1, \\ m_2 = a(\mu_1+b). \end{cases}$$

Then

$$\lambda^2 + m_1\lambda + m_2 - ce^{-\mu_2\tau}e^{-\lambda\tau}(\lambda+\mu_1) = 0. \qquad (3)$$

Let $\lambda = i\omega$ is the root of Equation (3), on substituting to $\lambda = i\omega$ Equation (3), we derive that

$$-\omega^2 + m_1 i\omega + m_2 - ce^{-\mu_2\tau}(i\omega+\mu_1) \cdot (\cos(\omega\tau) - i\sin(\omega\tau)) = 0.$$

Separating real and imaginary parts, it follows that

$$\begin{cases} -\omega^2 + m_2 - ce^{-\mu_2\tau}\mu_1\cos(\omega\tau) - ce^{-\mu_2\tau}\omega\sin(\omega\tau) = 0, \\ m_1\omega - ce^{-\mu_2\tau}\omega\cos(\omega\tau) + ce^{-\mu_2\tau}\mu_1\sin(\omega\tau) = 0. \end{cases}$$

$$\begin{cases} \sin(\omega\tau) = \dfrac{-\omega^3 + (m_2 - m_1\mu_1)\omega}{ce^{-\mu_2\tau}(\omega^2+\mu_1^2)}, \\ \cos(\omega\tau) = \dfrac{(m_1-\mu_1)\omega^2 + \mu_1 m_2}{ce^{-\mu_2\tau}(\omega^2+\mu_1^2)}. \end{cases}$$

Then we can get

$$\omega^6 + \left((m_1-\mu_1)^2 - 2(m_2 - m_1\mu_1) - c^2e^{-2\mu_2\tau}\right)\omega^4 + \left((m_2 - m_1\mu_1)^2 + 2m_2\mu_1(m_1-\mu_1) - 2c^2e^{-2\mu_2\tau}\mu_1^2\right)\omega^2 + m_2^2\mu_1^2 - c^2e^{-2\mu_2\tau}\mu_1^4 = 0.$$

Order

$$\begin{cases} f_1 = (m_1-\mu_1)^2 - 2(m_2 - m_1\mu_1) - c^2e^{-2\mu_2\tau}, \\ f_2 = (m_2 - m_1\mu_1)^2 + 2m_2\mu_1(m_1-\mu_1) - 2c^2e^{-2\mu_2\tau}\mu_1^2, \\ f_3 = m_2^2\mu_1^2 - c^2e^{-2\mu_2\tau}\mu_1^4. \end{cases}$$

Letting $z = \omega^2$, then Equation (4) becomes

$$z^3 + f_1 z^2 + f_2 z + f_3 = 0.$$

Here

$$\begin{cases} f_1 = b^2 + 2\mu_1^2 + 2b\mu_1 > 0, \\ f_2 = b^2c^2e^{-2\mu_2\tau} + 2b\mu_1c^2e^{-2\mu_2\tau} + (a^2 + b^2 + \mu_1^2 + 2b\mu_1)\mu_1^2 > 0, \\ f_3 = \mu_1^2(m_2 + ce^{-\mu_2\tau}\mu_1)bce^{-\mu_2\tau} > 0. \end{cases}$$

Application of the conclusions of [17], we can know that positive z doesn't exist. Hence ω also doesn't exist. There are not pure imaginary roots in system (2). Therefore all the roots have negative real component. So endemic equilibrium E^* of system (2) is locally asymptotically stable.

Theorem 3.2.2. If $R_0 > 1$, when $\tau = 0$, the endemic equilibrium E^* of system (2) is globally asymptotically stable in Ω_2.

Proof. Define a differentiable Lyapunov function

$$V = V_1 + V_2.$$

$$\begin{cases} V_1 = \omega_1\left(S - S^* - S^* \ln\dfrac{S}{S^*}\right), \\ V_2 = \omega_2\left(I - I^* - I^* \ln\dfrac{I}{I^*}\right). \end{cases}$$

$\omega_1 > 0, \omega_2 > 0,$ both of them are real numbers. The function is positive definite. Calculating the derivative of V along positive solutions of system (2), it follows that

$$\dot{V} = \dot{V}_1 + \dot{V}_2.$$

On substituting $A = \mu_1 S^* + \beta S^* I^*, a = \beta S^*,$ we have

$$\dot{V} = \omega_1\mu_1 S^* + \omega_1\beta S^* I^* - \omega_1\mu_1 S - \omega_1\beta SI - \omega_1\frac{\mu_1\left(S^*\right)^2}{S}$$
$$-\omega_1\frac{\beta\left(S^*\right)^2 I^*}{S} + \omega_1\mu_1 S^* + \omega_1\beta S^* I + \omega_2\beta SI - \omega_2\beta S^* I$$
$$-\omega_2\beta SI^* + \omega_2\beta S^* I^*.$$

Let $\omega_1 = \omega_2$.

So $\dot{V} \le 0$. In addition, when $\dot{V} = 0$, if and only if $S = S^*, E = E^*, I = I^*$.

It is easy to show that E^* is the largest invariant subset of the set $\left\{(S,E,I) : \dot{V} = 0\right\}$. Because of LaSalle's invariance principle, the endemic equilibrium E^* of system (2) is globally asymptotically stable when $\tau = 0$. This completes the proof.

Theorem 3.2.3. If $R_0 > 1$, when $\tau \ne 0$, the endemic equilibrium E^* of system (2) is globally asymptotically stable in Ω_2.

Proof. For $t > 0$, define a differentiable Lyapunov function

$$V = V_1 + V_2.$$

Order

$$\begin{cases} V_1 = \omega_1\left(S - S^* - S^* \ln\dfrac{S}{S^*}\right) + \omega_2 \mathrm{e}^{\mu_2\tau}\left(I - I^* - I^* \ln\dfrac{I}{I^*}\right), \\ V_2 = \omega_2\beta\int_{t-\tau}^{t}\left(S(\theta)I(\theta) - S^*I^* - S^*I^* \ln\dfrac{S(\theta)I(\theta)}{S^*I^*}\right)\mathrm{d}\theta. \end{cases}$$

$\omega_1 > 0, \omega_2 > 0,$ both of them are real numbers. Let $A = \mu_1 S^* + \beta S^* I^*, a = \beta S^*$.

Then the derivative of V along the solution of system (2) satisfies

$$\dot{V} = \omega_1\left(2\mu_1 S^* - \mu_1 S - \frac{\mu_1 S^*}{S}\right) + \omega_1\beta S^* I^* - \omega_1\beta SI$$
$$-\frac{\omega_1\beta\left(S^*\right)^2 I^*}{S} + \omega_1\beta S^* I + \omega_2\beta S(t-\tau)I(t-\tau)$$
$$-\omega_2\beta S^* I - \frac{\omega_2\beta S(t-\tau)I(t-\tau)I^*}{I} + \omega_2\beta S^* I^*$$
$$+\omega_2\beta SI - \omega_2\beta S(t-\tau)I(t-\tau)$$
$$+\omega_2\beta S^* I^* \ln\frac{S(t-\tau)I(t-\tau)}{SI}.$$

$$2\mu_1 S^* - \mu_1 S - \frac{\mu_1 S^*}{S} \le 0.$$

Then

$$\dot{V} \le \omega_1\beta S^* I^* - \omega_1\beta SI - \frac{\omega_1\beta\left(S^*\right)^2 I^*}{S}$$
$$+\omega_1\beta S^* I + \omega_2\beta SI + \omega_2\beta S^* I^* \ln\frac{S(t-\tau)I(t-\tau)}{S^* I}$$
$$+\omega_2\beta S^* I^* - \frac{\omega_2\beta I^* S(t-\tau)I(t-\tau)}{I}$$
$$+\omega_2\beta S^* I^* \ln\frac{S^*}{I^*} + \omega_2\beta S^* I^* - \frac{\omega_2\beta\left(S^*\right)^2 I^*}{S}$$
$$-2\omega_2\beta S^* I^* + \frac{\omega_2\beta\left(S^*\right)^2 I^*}{S}.$$

Simplify, we can get

$$\dot{V} \le \left(\omega_2\beta S - \omega_2\beta S^* + \omega_1\beta S^* - \omega_1\beta S\right)I$$
$$+\frac{\omega_2\beta\left(S^*\right)^2 I^*}{S} - \omega_2\beta S^* I^*$$
$$+\omega_1\beta S^* I^* - \frac{\omega_1\beta\left(S^*\right)^2 I^*}{S}.$$

Order $\omega_2 = \omega_1$, $\dot{V} \le 0$. Besides, when $\dot{V} = 0$, if and only if $S = S^*, E = E^*, I = I^*$.

It is easy to show that E^* is the largest invariant subset of the set $\left\{(S,E,I) : \dot{V} = 0\right\}$. Because of LaSalle's invariance principle, the endemic equilibrium E^* of system (2) is globally asymptotically stable when $\tau \ne 0$. This completes the proof.

3.3. The SEIQR Epidemic Model with Nonlinear Incidence Rate

Zhao *et al.* studied delay SEIR epidemic model with the nonlinear incidence rate like $\beta S^p I$ in the case of pulse. In this paper, the model without pulse is discussed.

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = A - \mu_1 S - \beta S^p I, \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = \beta S^p I - \beta \mathrm{e}^{-\mu_2 \tau} S^p (t-\tau) I (t-\tau) - (\mu_2 + k_1 + k_3) E, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta \mathrm{e}^{-\mu_2 \tau} S^p (t-\tau) I (t-\tau) - (\mu_3 + \alpha + \nu + k_2) I, \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = k_1 E + k_2 I - (\mu_4 + \gamma) Q, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \gamma Q + \nu I + k_3 E - \mu_5 R. \end{cases}$$

It is easy to show disease-free equilibrium is globally asymptotically stable, endemic equilibrium is locally asymptotically stable. The ways we use are similar to that in system (1), here they are omitted.

4. The Numerical Simulations

In this section, we study system (1) numerically. According to the different datas that can reflect the actual situation, we get the different simulation images to prove our conclusions obviously (**Figures 1-9**).

Here, according to the different actual situations, while take different parameters, we can get different simulation diagrams of the disease-free equilibrium. At the same time, we find out the disease will die out after much more time when R_0 increases. For example,

$\beta = 0.2$, $A = 1$, $\mu_1 = 0.1$, $\mu_2 = 0.15$, $\mu_3 = 0.2$, $\mu_4 = 0.05$, $\mu_5 = 0.06$, $\alpha_1 = 0.4$, $\alpha_2 = 0.2$, $k_1 = 0.3$, $k_2 = 0.8$, $k_3 = 0.2$, $\gamma = 0.5$, $\nu = 0.3$, $\tau = 3$.

Here $R_0 = 0.7502$, see **Figure 1**.

$\beta = 0.2$, $A = 1$, $\mu_1 = 0.05$, $\mu_2 = 0.1$, $\mu_3 = 0.2$, $\mu_4 = 0.05$, $\mu_5 = 0.06$, $\alpha_1 = 0.4$, $\alpha_2 = 0.2$, $k_1 = 0.3$, $k_2 = 0.8$, $k_3 = 0.2$, $\gamma = 0.5$, $\nu = 0.3$, $\tau = 10$.

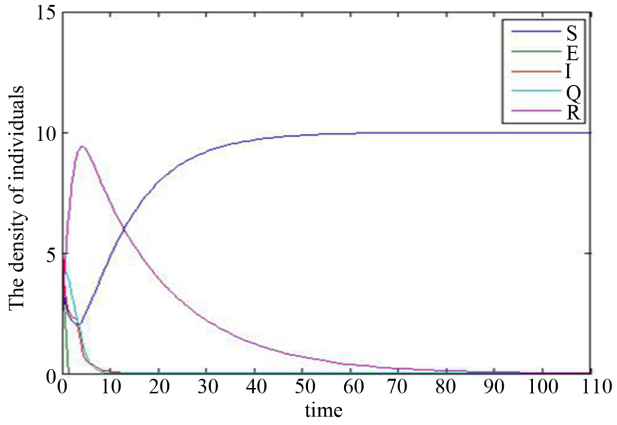

Figure 1. Simulation diagram of the disease-free equilibrium when $R_0 = 0.7502$.

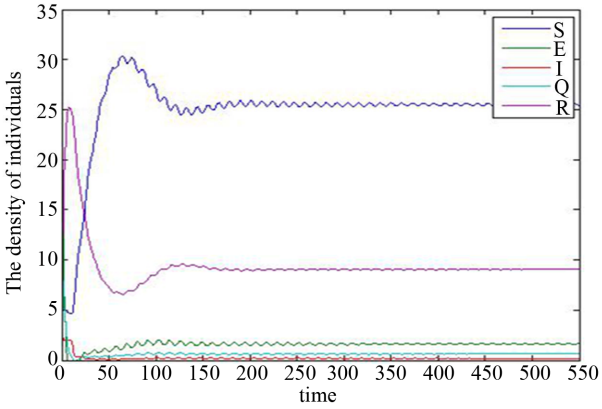

Figure 3. Simulation diagram of the endemic equilibrium when $k_3 = 0.001$.

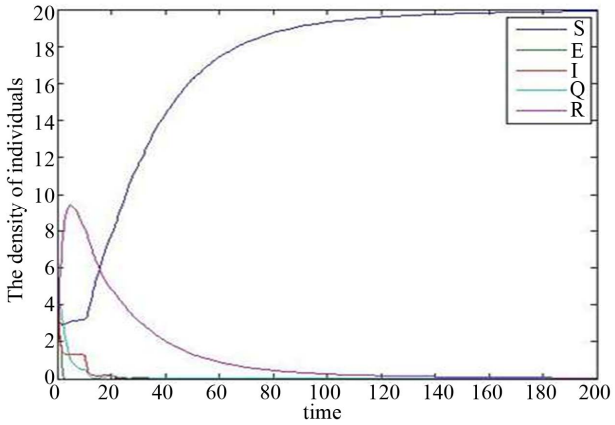

Figure 2. Simulation diagram of the disease-free equilibrium when $R_0 = 0.8656$.

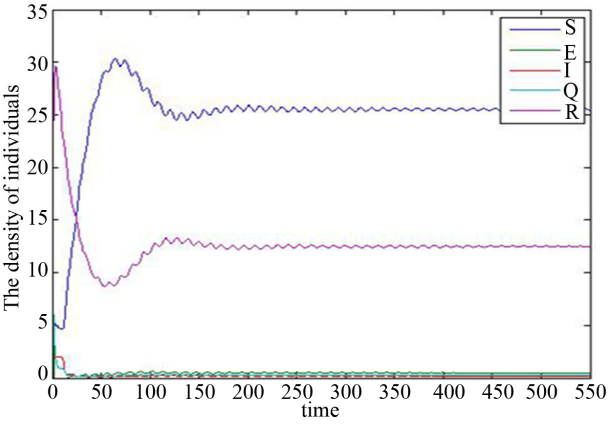

Figure 4. Simulation diagram of the endemic equilibrium when $k_3 = 0.95$.

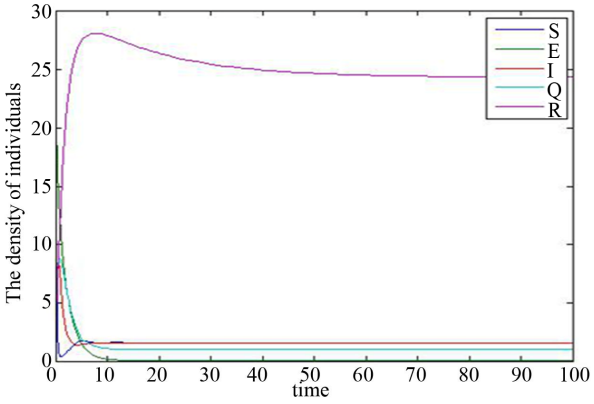

Figure 5. Simulation diagram of the endemic equilibrium when $R_0 = 31.6816$.

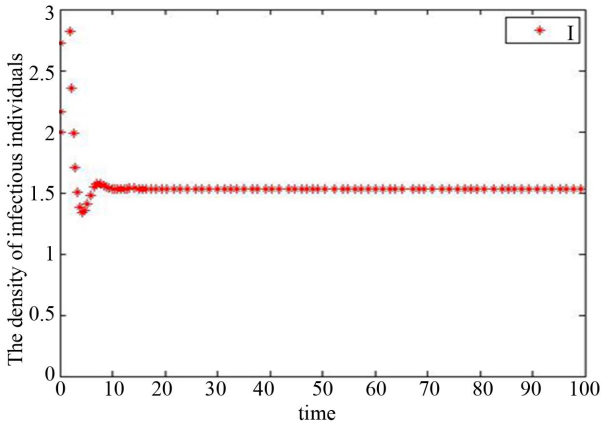

Figure 6. Simulation diagram of I when $R_0 = 31.6816$.

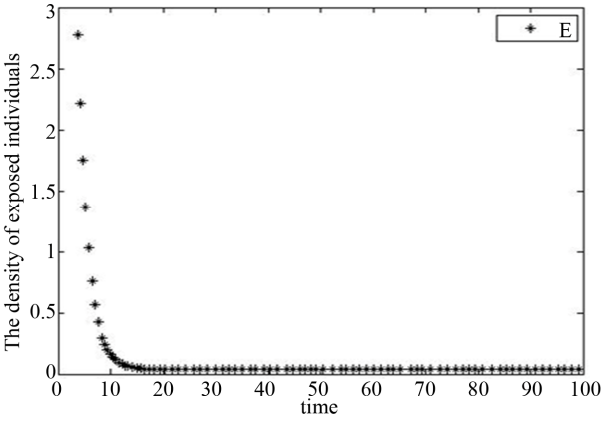

Figure 7. Simulation diagram of E when $R_0 = 31.6816$.

Here $R_0 = 0.8656$, see **Figure 2**.

When take different k_3, we can get different simulation images. In other words, the R increases when k_3 increases, which is obvious in **Figures 3** and **4**. And then it is easy for us to find that how k_3 effects changing trends of S, E, I, Q, R.

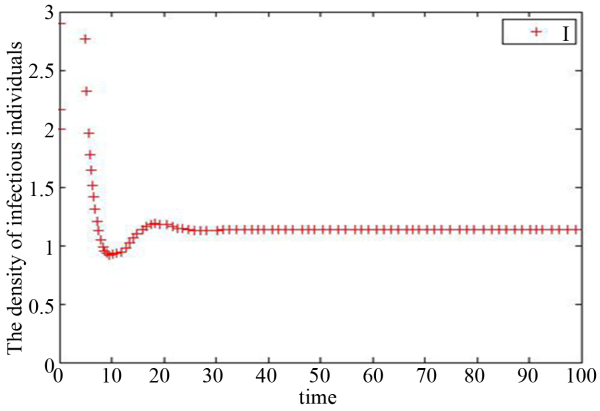

Figure 8. Simulation diagram of I when $\tau = 3$.

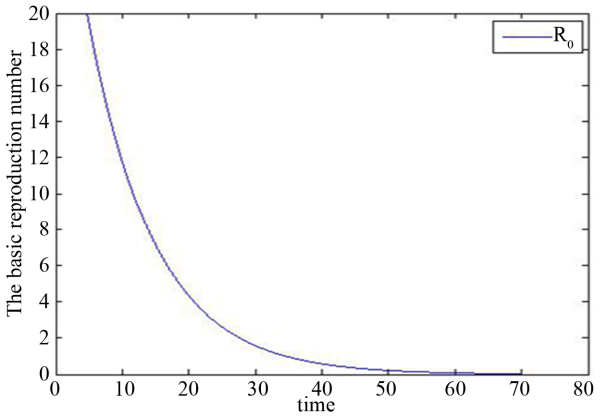

Figure 9. Simulation diagram of basic reproduction number.

$\beta = 0.2,\ A = 2,\ \mu_1 = 0.04,\ \mu_2 = 0.1,\ \mu_3 = 0.2,$
$\mu_4 = 0.05,\ \mu_5 = 0.06,\ \alpha_1 = 0.4,\ \alpha_2 = 0.2,\ k_1 = 0.3,$
$k_2 = 0.6,\ k_3 = 0.001,\ \gamma = 0.7,\ \nu = 0.5,\ \tau = 11.$

Here $R_0 = 1.9581$, see **Figure 3**.

$\beta = 0.2,\ A = 2,\ \mu_1 = 0.04,\ \mu_2 = 0.1,\ \mu_3 = 0.2,$
$\mu_4 = 0.05,\ \mu_5 = 0.06,\ \alpha_1 = 0.4,\ \alpha_2 = 0.2,\ k_1 = 0.3,$
$k_2 = 0.6,\ k_3 = 0.95,\ \gamma = 0.7,\ \nu = 0.5,\ \tau = 11.$

Here $R_0 = 1.9581$, see **Figure 4**.

At last, if the basic reproduction number is much larger and we will get new diagrams. For example, let

$\beta = 0.8,\ A = 2,\ \mu_1 = 0.04,\ \mu_2 = 0.1,\ \mu_3 = 0.1,$
$\mu_4 = 0.05,\ \mu_5 = 0.06,\ \alpha_1 = 0.05,\ \alpha_2 = 0.2,\ k_1 = 0.3,$
$k_2 = 0.6,\ k_3 = 0.1,\ \gamma = 0.7,\ \nu = 0.5,\ \tau = 0.1.$

Here $R_0 = 31.6816$, see **Figure 5**.

At the same time, the changing trends of I and E are shown in **Figures 6** and **7**. And the time which I comes peak will become large as τ increases, the I^* will decrease. For example, $\tau = 3$, see **Figure 8**. In

addition, when τ changes, R_0 will change. And we can find out that when $\tau = 34.657, R_0 = 1$. That is, disease will be endemic disease while $\tau < 34.657$, see **Figure 9**.

5. Discussions

In this paper, a kind of a delayed SEIQR epidemic model with the quarantine and latent is studied. Using Hurwitz criterion, the local stability of the disease-free equilibrium and endemic equilibrium of system (2) is proved. For any time delay τ, we prove the disease-free equilibrium is globally asymptotically stable when the basic reproduction number is less than unity and the endemic equilibrium is globally asymptotically stable when the basic reproduction number is greater than unity by means of suitable Lyapunov functions and LaSalle's invariance principle. So the delay is harmless to system (2). From the biological point of view, the delay here has no influence on the transmission of diseases. However, in [16], the disease-free equilibrium is periodic and globally attractive. At the same time, the disease will be endemic after some period of time. Above all, we consider that E is quarantined and can recover in this model, which will effect changing trends of S, E, I, Q, R. Here, we take k_3 as an example to explain that. Meanwhile, the simulation image which R_0 changes as τ can be obtained and we can find out τ_0 which the basic reproduction number is a unity. Those are useful for us to control epidemics. At last, the conclusions above are verified by numerical simulations.

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REFERENCES

[1] X. B. Liu and L. J. Yang, "Stability Analysis of an SEIQV Epidemic Model with Saturated Incidence Rate," *Nonlinear Analysis*: *Real World Applications*, Vol. 13, No. 6, 2012, pp. 2671-2979.

[2] M. Y. Li, J. R. Graef, L. C. Wang and J. Karsai, "Global Dynamics of a SEIR Model with Varying Total Population Size," *Mathematical Biosciences*, Vol. 160, No. 2, 1999, pp. 191-213. http://dx.doi.org/10.1016/S0025-5564(99)00030-9

[3] J. Zhang and Z. E. Ma, "Global Dynamics of an SEIR Epidemic Model with Saturating Contact Rate," *Mathematical Biosciences*, Vol. 185, No. 1, 2003, pp. 15-32. http://dx.doi.org/10.1016/S0025-5564(03)00087-7

[4] A. Onofrio, "Stability Properties of Pulse Vaccination Strategy in SEIR Epidemic Model," *Mathematical Biosciences*, Vol. 179, No. 1, 2002, pp. 57-72. http://dx.doi.org/10.1016/S0025-5564(02)00095-0

[5] G. H. Li and Z. Jin, "Global Stability of a SEIR Epidemic Model with Infectious Force in Latent Infected and Immune Period," *Chaos, Solitons and Fractals*, Vol. 25, No. 5, 2005, pp. 1177-1184. http://dx.doi.org/10.1016/j.chaos.2004.11.062

[6] B. K. Mishra and N. Jha, "SEIQRS Model for the Transmission of Malicious Objects in Computer Network," *Applied Mathematical Modelling*, Vol. 34, No. 3, 2010, pp. 710-715. http://dx.doi.org/10.1016/j.apm.2009.06.011

[7] Y. N. Kyrychkoa and K. B. Blyussb, "Global Properties of a Delayed SIR Model with Temporary Immunity and Nonlinear Incidence Rate," *Nonlinear Analysis*: *Real World Applications*, Vol. 6, No. 3, 2005, pp. 495-507. http://dx.doi.org/10.1016/j.nonrwa.2004.10.001

[8] K. L. Cooke and P. van den Driessche, "Analysis of an SEIRS Epidemic Model with Two Delays," *Journal of Mathematical Biology*, Vol. 35, No. 2, 1996, pp. 240-260. http://dx.doi.org/10.1007/s002850050051

[9] R. Xu, Z. E. Ma and Z. P. Wang, "Global Stability of a Delayed SIRS Epidemic Model with Saturation Incidence and Temporary Immunity," *Computers and Mathematics with Applications*, Vol. 59, No. 9, 2010, pp. 3211-3221. http://dx.doi.org/10.1016/j.camwa.2010.03.009

[10] Y. Enatsu, E. Messina, Y. Muroya, Y. Nakata, E. Russo and A. Vecchio, "Stability Analysis of Delayed SIR Epidemic Models with a Class of Nonlinear Incidence Rates," *Applied Mathematics and Computation*, Vol. 218, No. 9, 2012, pp. 5327-5336. http://dx.doi.org/10.1016/j.amc.2011.11.016

[11] F. P. Zhang, Z. Z. Li and F. Zhang, "Global Stability of an SIR Epidemic Model with Constant Infectious Period," *Applied Mathematics and Computation*, Vol. 199, No. 1, 2008, pp. 285-291. http://dx.doi.org/10.1016/j.amc.2007.09.053

[12] S. J. Gao, L. S. Chen and Z. D. Teng, "Pulse Vaccination of an SEIR Epidemic Model with Time Delay," *Nonlinear Analysis*: *Real World Applications*, Vol. 9, No. 2, 2008, pp. 599-607. http://dx.doi.org/10.1016/j.nonrwa.2006.12.004

[13] Z. Zhao, L. S. Chen and X. Y. Song, "Impulsive Vaccination of SEIR Epidemic Model with Time Delay and Nonlinear Incidence Rate," *Mathematics and Computers in Simulation*, Vol. 79, No. 3, 2008, pp. 500-510. http://dx.doi.org/10.1016/j.matcom.2008.02.007

[14] R. Xu and Z. E. Ma, "Global Stability of a Delayed SEIRS Epidemic Model with Saturation Incidence Rate," *Nonlinear Dynamics*, Vol. 61, No. 1-2, 2010, pp. 229-239. http://dx.doi.org/10.1007/s11071-009-9644-3

[15] X. B. Zhang, H. F. Huo, H. Xiang and X. Y. Meng, "Two Profitless Delays for the SEIRS Epidemic Disease Model with Nonlinear Incidence and Pulse Vaccination," *Applied Mathematics and Computation*, Vol. 186, No. 1, 2007, pp. 516-529. http://dx.doi.org/10.1016/j.amc.2006.07.124

[16] Y. Z. Pei, S. Y. Liu, S. J. Gao, S. P. Li and C. G. Li, "A Delayed SEIQR Epidemic Model with Pulse Vaccination and the Quarantine Measure," *Computers and Mathemat-*

ics with Applications, Vol. 58, No. 1, 2009, pp. 135-145.

[17] H. M. Wei, X. Z. Li and M. Martcheva, "An Epidemic Model of a Vector-Borne Disease with Direct Transmission and Time Delay," *Journal of Mathematical Analysis and Applications*, Vol. 342, No. 2, 2008, pp. 895-908.

