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A Closed-Form Approximation for Pricing Temperature-Based Weather Derivatives

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ABSTRACT

This paper develops analytical distributions of temperature indices on which temperature derivatives are written. If the deviations of daily temperatures from their expected values are modelled as an Ornstein-Uhlenbeck process with time-varying variance, then the distributions of the temperature index on which the derivative is written is the sum of truncated, correlated Gaussian deviates. The key result of this paper is to provide an analytical approximation to the distribution of this sum, thus allowing the accurate computation of payoffs without the need for any simulation. A data set comprising average daily temperature spanning over a hundred years for four Australian cities is used to demonstrate the efficacy of this approach for estimating the payoffs to temperature derivatives. It is demonstrated that expected payoffs computed directly from historical records are a particularly poor approach to the problem when there are trends in underlying average daily temperature. It is shown that the proposed analytical approach is superior to historical pricing.

Keywords: Weather Derivatives; Temperature Models; Cooling-Degree Days; Distributions for Correlated Variables

1. Introduction

A weather derivative takes its value from an underlying measure of weather, such as temperature, rainfall or snowfall over a particular period of time, and permits the financial risk associated with climatic conditions to be managed. Major participants in this market include utilities and insurance companies along with other firms with costs or revenues that are dependent upon the weather. For example, an electricity supplier normally provides its customers with electricity at a fixed price irrespective of the wholesale price. On the other hand the wholesale price of electricity can fluctuate wildly with extreme temperatures, and so temperature-based derivatives can provide a hedging tool for fluctuations in wholesale electricity prices. The first weather derivative was transacted in the US in 1996 and the size of the market now exceeds US$ 8 billion. Almost all weather derivatives are based on temperature indices such as heating degree days and cooling degree days and consequently the focus of this paper will be exclusively on developing closed-form approximations to the distribution of the temperature indices on which temperature-based derivatives are written which in turn affects their valuation[1].

Traditionally, the valuation of options discounts the expected payoff at the risk-free force of interest based on a zero-arbitrage argument involving the formation of a portfolio consisting of a risk-free combination of an option and the underlying asset [3]. Because temperature cannot be traded, there is no arbitrage-free pricing framework available to price this kind of option. The generally accepted way to value temperature derivatives is the *actuarial method* in which the fair price is taken to be the expected value of the payoff ignoring discounting and any volatility premium. The crucial element of this valuation strategy is the accurate calculation of the distribution of the relevant temperature index on which the weather derivative is written.

The most direct way to compute the distribution of temperature indices is from historical records [4,5]. A more elaborate method is to fit a model to the time-series

[1]The first recorded activity was an over-the-counter heating degree day swap option between Entergy-Koch and Enron for the winter of 1997 in Milwaukee, Wisconsin [1]. Garmen *et al*. [2] posit that 98% - 99% of all weather derivatives currently traded are based on temperature. Currently temperature-based derivatives are traded in several US, European and Japanese cities.

of average daily temperature so as to capture seasonal variations in both temperature and its volatility [5,6]. The model is then used to simulate temperature outcomes over the period of the contract in order to construct the distribution of the temperature-based index on which the derivative is written. Note that widely-available meteorological forecasts are not suitable for this purpose because these forecasts are made over relatively short horizons, such as 7 days, whereas temperature derivatives are often traded well before the contracts generate any payoffs [6-8].

This paper makes two contributions to the existing literature on pricing temperature derivatives. First, it builds on the early work of Benth and Šaltytė-Benth [9] by developing closed-form approximations to the distribution of the indices on which temperature-based derivatives are written with particular emphasis on obtaining good estimates of the variance of relevant index. Second, two methods are provided for estimating the parameters of the model underpinning the behaviour of temperature that are required to implement the pricing strategy. There are respectively a two-step least-squares based approach and a more comprehensive maximum-likelihood procedure.

The ideas developed in this paper are applied to data comprising average daily temperatures for over a century in four Australian cities, namely, Brisbane (BNE), Melbourne (MEL), Perth (PER) and Sydney (SYD), where accurate temperature records of long-duration are available at single weather stations. This is a quality data set which represents a substantial improvement on what appears to be the current standard used in the literature. The empirical results based on this data set, demonstrate that the closed-form pricing strategy performs substantially better that using historical pricing.

2. A Model of Daily Temperature

The first step in pricing any temperature-based option must be a model of the underlying index from which the option derives its value, which in the case of temperature derivatives is average daily temperature. Let average daily temperature be expressed as the sum of the seasonal mean temperature $\bar{T}(t)$ at time t and the deviation $\theta(t)$ of the average daily temperature from its seasonal mean. Suppose that $\theta(t)$ is modelled by the Ornstein-Uhlenbeck process[2]

$$\mathrm{d}\theta = -\alpha\theta\mathrm{d}t + \sigma(t)\mathrm{d}W,\ \alpha > 0, \tag{1.1}$$

where dW is the increment in the Wiener process. The parameter α and the volatility $\sigma(t)$ are to be determined from observations of average daily temperature. Equation (1.1) has solution

$$\theta(t) = \int_{-\infty}^{t} \mathrm{e}^{-\alpha(t-s)}\sigma(s)\mathrm{d}W(s), \tag{1.2}$$

with autocorrelation function at lag u given by

$$\begin{aligned} E\left[\theta(t)\theta(t+u)\right] &= \mathrm{e}^{-\alpha u}S(t), \\ S(t) &= \int_{-\infty}^{t} \mathrm{e}^{-2\alpha(t-s)}\sigma^2(s)\mathrm{d}s, \end{aligned} \tag{1.3}$$

where $S(t)$ is the variance of daily average temperature. It is straightforward to show that $\sigma^2(t)$ and $S(t)$ satisfy

$$\sigma^2(t) = \frac{\mathrm{d}S(t)}{\mathrm{d}t} + 2\alpha S(t).$$

The joint distribution of the average daily temperatures T_t and T_{t+s} at the respective calender times t and $(t+s)$ $(s>0)$ is given by the product

$$f(T_t, T_{t+s}) = f(T_t, t)\, f(T_{t+s}, t+s \,|\, T_t, t) \tag{1.4}$$

where $f(T_t, t)$ is the marginal distribution of T_t, namely

$$f(T_t, t) = \frac{1}{\sqrt{2\pi S_t}} \exp\left[-\frac{(T_t - \bar{T}_t)^2}{2S_t}\right]$$

and

$$f(T_{t+s}, t+s \,|\, T_t, t) = \frac{1}{\sqrt{2\pi(S_{t+s} - \mathrm{e}^{-2\alpha s}S_t)}} \times \exp\left[-\frac{\left(T_{t+s} - \bar{T}_{t+s} - (T_t - \bar{T}_t)\mathrm{e}^{-\alpha s}\right)^2}{2(S_{t+s} - \mathrm{e}^{-2\alpha s}S_t)}\right]$$

is the transitional probability density function from T_t to T_{t+s}. Consequently the joint probability density function, $f(T_t, T_{t+s})$, in Equation (1.4) becomes

$$\frac{\mathrm{e}^{-\phi}}{2\pi\sqrt{S_t(S_{t+s} - \beta^2 S_t)}}$$

where

$$\phi = \frac{S_{t+s}(T_t - \bar{T}_t)^2 - 2\beta S_t(T_t - \bar{T}_t)(T_{t+s} - \bar{T}_{t+s}) + S_t(T_{t+s} - \bar{T}_{t+s})^2}{2S_t(S_{t+s} - \beta^2 S_t)}$$

and $\beta = \mathrm{e}^{-\alpha s}$. Thus the joint probability density function of (T_t, T_{t+s}) is multivariate Gaussian with mean value $\boldsymbol{\mu} = (\bar{T}_t, \bar{T}_{t+s})$ and covariance matrix

$$\Sigma = \begin{bmatrix} S_t & \mathrm{e}^{-\alpha s}S_t \\ \mathrm{e}^{-\alpha s}S_t & S_{t+s} \end{bmatrix}.$$

This model of average daily temperature is now used to develop a closed-form approximation to the distributions of the underlying temperature indices on which

[2]This specification is consistent with previous work [9-11].

vanilla European options[3] are written, namely cumulative heating degree days (HDDs) and cumulative cooling degree days (CDDs).

3. Distribution of Temperature Indices

Let T_{ave} denote the average temperatures in degrees Celsius measured on a particular day at a specific weather station. The HDD and CDD indices at that station on that day are defined respectively by

$$\begin{aligned} HDD &= \max\left(T - T_{\text{ave}}, 0\right), \\ CDD &= \max\left(T_{\text{ave}} - T, 0\right), \end{aligned} \tag{1.5}$$

where T˚C is a threshold temperature. The choice of threshold, in this instance 18˚C, is set by market convention and is the standard used in the US. In the southern (northern) hemisphere the HDD (CDD) season would be from May to September, while the CDD (HDD) season would be from November to March. Without loss of generality, the analysis of this paper will be limited to considering European call options written on cumulative CDDs.

The CDD index over a period of N consecutive days is defined by

$$C_N = \sum_{k=1}^{N} \mathcal{T}_k, \; \mathcal{T}_k = \max\left(T_k - T, 0\right) \tag{1.6}$$

where T_k is the average daily temperature on the k^{th} day of the derivative.

Let D be the strike of a call option defined as a particular value of the CDD index. The buyer of this option pays an up-front premium and receives a payout if the value of the CDD index exceeds D at the maturity of the option. The tick value of a cumulative CDD call option with strike D and duration N days is therefore

$$\mathcal{T}_N = \max\left(C_N - D, 0\right). \tag{1.7}$$

The per-unit monetary payoff from the contract is its expected tick value

$$E\left[\mathcal{T}_N\right] = \int_D^{\infty} \left(x - D\right) f_N\left(x\right) \mathrm{d}x, \tag{1.8}$$

where $f_N\left(x\right)$ is the probability density function of C_N and therefore the efficacy of this pricing strategy relies upon the accurate estimation of $f_N\left(x\right)$. The idea pursued here is that although the daily contributions to C_N are truncated correlated random variables in which the degree of truncation is nontrivial, nevertheless C_N will behave as a Gaussian random variable provided N is suitably large. The central theoretical result of the paper is summarized in Proposition 1.

Proposition 1

The tick value C_N of a European option defined on cumulative cooling degree days is approximately Gaussian distributed with mean value

$$\mathbb{E}\left[C_N\right] = \sum_{k=1}^{N} \sqrt{S_k}\left[z_k \Phi\left(z_k\right) + \phi\left(z_k\right)\right],$$

and variance $\text{Var}\left[C_N\right]$ with expression

$$\begin{aligned} &\sum_{k=1}^{N} S_k \left[\Phi\left(z_k\right) - \left(\phi\left(z_k\right) + z_k \Phi\left(z_k\right)\right) \times \left(\phi\left(-z_k\right) - z_k \Phi\left(-z_k\right)\right)\right] \\ &+ 2\sum_{k=1}^{N-1}\Bigg(\sum_{j=k+1}^{N} \sqrt{S_j S_k}\left(z_k \phi\left(z_j\right)\Phi\left(\chi_{j,k}\right) + z_j \phi\left(z_k\right)\Phi\left(\eta_{j,k}\right)\right) - \mathbb{E}\left[\mathcal{T}\right]_k \mathbb{E}\left[\mathcal{T}\right]_j \\ &+ \left(z_k z_j \sqrt{S_j S_k} + \beta_{j,k} S_k\right)\Phi\left(z_k\right) + \sqrt{S_k\left(S_j - \beta_{j,k}^2 S_k\right)}\phi\left(z_k\right)\phi\left(\eta_{j,k}\right) \\ &- \frac{\left(z_k z_j \sqrt{S_j S_k} + \beta_{j,k} S_k\right)}{\sqrt{q+1}}\Phi\left(-\eta_{j,k}\right) \times \Phi\left(\frac{p + z_k}{\sqrt{1+q}}\right)\exp\left[\frac{1}{2}\left(\frac{\left(p + z_k\right)^2}{1+q} - z_k^2\right)\right]\Bigg), \end{aligned}$$

where $z_k = \left(\bar{T}_k - T\right)\big/\sqrt{S_k}$, $\beta_{j,k} = \mathrm{e}^{-\alpha(j-k)}$ and the constants $\eta_{j,k}$, $\chi_{j,k}$, p and q are defined respectively by

$$\eta_{j,k} = \frac{z_j \sqrt{S_j} - \beta_{j,k} z_k \sqrt{S_k}}{\sqrt{S_j - \beta_{j,k}^2 S_k}}, \chi_{j,k} = \frac{z_k \sqrt{S_j} - \beta_{j,k} z_j \sqrt{S_k}}{\sqrt{S_j - \beta_{j,k}^2 S_k}}, \; p = -\frac{\beta_{j,k}\sqrt{S_k}}{\sqrt{S_j - \beta_{j,k}^2 S_k}} \frac{\phi\left(\eta_{j,k}\right)}{\Phi\left(-\eta_{j,k}\right)}, \; q = p^2\left(1 - \frac{\eta_{j,k}\Phi\left(-\eta_{j,k}\right)}{\phi\left(\eta_{j,k}\right)}\right). \tag{1.9}$$

Proposition 1 establishes that accurate closed-form expressions for the mean and the variance of C_N are available in terms of the density function and distribution function of the standard normal distribution alone. Given these results, the per-unit monetary payoff of a CDD call option is stated in Proposition 2.

Proposition 2

The per-unit monetary payoff of a European call option with strike D written on C_N, where the distribution of C_N is Gaussian with mean and variance established in Proposition 1, *is given by*

[3]The choice of European option is not limiting in the sense that many more complex derivative strategies are in fact combinations of simple European options.

$$\sqrt{\mathrm{Var}[C_N]}\left[\phi(\xi)+\xi\Phi(\xi)\right],\quad \xi=\frac{\mathbb{E}[C_N]-D}{\sqrt{\mathrm{Var}[C_N]}}.$$

The focus of subsequent subsections is to develop and prove the results stated in Proposition 1.

3.1. Mean of C_N

It follows directly from Equation (1.6) that

$$\mathbb{E}[C_N]=\mathbb{E}[\mathcal{T}_1]+\cdots+\mathbb{E}[\mathcal{T}_N]$$

where

$$\mathbb{E}[\mathcal{T}_k]=\frac{1}{\sqrt{2\pi S_k}}\int_T^\infty(\theta-T)\exp\left[-\frac{(\theta-\bar{T}_k)^2}{2S_k}\right]\mathrm{d}\theta. \quad (1.10)$$

Let $z_k=(\bar{T}_k-T)/\sqrt{S_k}$, then the change of variable $\theta=\bar{T}_k-\sqrt{S_k}z$ gives immediately

$$\begin{aligned}\mathbb{E}[\mathcal{T}_k]&=\frac{\sqrt{S_k}}{\sqrt{2\pi}}\int_{-\infty}^{z_k}(z_k-z)\mathrm{e}^{-z^2/2}\mathrm{d}z\\&=\sqrt{S_k}\left[z_k\Phi(z_k)+\phi(z_k)\right],\end{aligned} \quad (1.11)$$

where $\phi(z)$ and $\Phi(z)$ are respectively the probability density function and cumulative distribution function of the standard normal. The quoted expression for $\mathbb{E}[C_N]$ follows immediately from result (1.11). Moreover, it should be noted in passing that the proof of Proposition 2 is analogous to the derivation of Equation (1.11).

3.2. Variance of C_N

The computation of the variance of C_N is less straightforward. The key steps in this calculation are outlined here with the detail being relegated to **Appendices 1** and **2**. The analysis begins by noting that $\mathrm{Var}[C_N]$ can be expressed as the sum of variances and covariances in the usual form

$$\mathrm{Var}[C_N]=\sum_{k=1}^{N}\mathrm{Var}[\mathcal{T}_k]+2\sum_{k=1}^{N-1}\sum_{j=k+1}^{N}\mathrm{Cov}[\mathcal{T}_k,\mathcal{T}_j]. \quad (1.12)$$

Straightforward calculation indicates that

$$\begin{aligned}\mathrm{Var}[\mathcal{T}_k]&=\int_T^\infty\frac{(\theta-T)^2}{\sqrt{2\pi S_k}}\exp\left[-\frac{(\theta-\bar{T}_k)^2}{2S_k}\right]\mathrm{d}\theta\\&\quad-S_k\left[z_k\Phi(z_k)+\phi(z_k)\right]^2,\end{aligned} \quad (1.13)$$

which under the change of variable $\theta=\bar{T}_k-\sqrt{S_k}z$ becomes

$$\begin{aligned}\mathrm{Var}[\mathcal{T}_k]&=S_k\int_{-\infty}^{z_k}(z_k-z)^2\phi(z)\mathrm{d}z\\&\quad-S_k\left[z_k\Phi(z_k)+\phi(z_k)\right]^2.\end{aligned} \quad (1.14)$$

It is demonstrated in **Appendix 1** that

$$\mathrm{Var}[\mathcal{T}_k]=S_k\left[\Phi(z_k)-(\phi(z_k)+z_k\Phi(z_k))\times(\phi(-z_k)-z_k\Phi(-z_k))\right] \quad (1.15)$$

thereby completing the computation of the first item on the right hand side of Equation (1.12).

The second item on the right hand side of Equation (1.12) is a sum of covariances of generic form

$$\mathrm{Cov}[\mathcal{T}_t,\mathcal{T}_{t+s}]=\iint_T(T_t-T)(T_{t+s}-T)f(T_{t+s},T_t)\mathrm{d}T_t\mathrm{d}T_{t+s}-\mathbb{E}[\mathcal{T}]_t\mathbb{E}[\mathcal{T}]_{t+s} \quad (1.16)$$

in which t and s (> 0) are to be given appropriate values. First, the integral on the right hand side of Equation (1.16) is simplified using the change of variables $T_t=\bar{T}_t-\sqrt{S_t}z$ and $T_{t+s}=\bar{T}_{t+s}-\sqrt{S_{t+s}}w$ to get

$$\mathrm{Cov}[\mathcal{T}_t,\mathcal{T}_{t+s}]=\sqrt{S_tS_{t+s}}\times\int_{-\infty}^{z_t}\int_{-\infty}^{z_{t+s}}(z_t-z)(z_{t+s}-w)\tilde{f}(z_{t+s},z_t)\mathrm{d}z\mathrm{d}w-\mathbb{E}[\mathcal{T}]_t\mathbb{E}[\mathcal{T}]_{t+s}, \quad (1.17)$$

where $z_t=(\bar{T}_t-T)/\sqrt{S_t}$ and $z_{t+s}=(\bar{T}_{t+s}-T)/\sqrt{S_{t+s}}$ and $\tilde{f}(z_{t+s},z_t)$ is the joint probability density of z and w, namely

$$\frac{1}{2\pi}\sqrt{\frac{S_{t+s}}{S_{t+s}-\beta^2S_t}}\mathrm{e}^{-\psi(z,w)}, \quad (1.18)$$

where $\beta=\mathrm{e}^{-\alpha s}$ and

$$\psi(z,w)=\frac{S_{t+s}z^2-2zw\beta\sqrt{S_tS_{t+s}}+S_{t+s}w^2}{2(S_{t+s}-\beta^2S_t)}.$$

The integral in Equation (1.17) is expressed as a repeated integral in which integration is first performed with respect to w and then again with respect to z. The detailed calculations can be found in **Appendix 2**, but the outcome of these operations is that

$$\begin{aligned}\mathrm{Cov}[\mathcal{T}_t,\mathcal{T}_{t+s}]&=\sqrt{S_tS_{t+s}}\left(z_t\phi(z_{t+s})\Phi(\chi_{t+s})+z_{t+s}\phi(z_t)\Phi(\eta_{t+s})\right)-\mathbb{E}[\mathcal{T}]_t\mathbb{E}[\mathcal{T}]_{t+s}+\left(z_tz_{t+s}\sqrt{S_tS_{t+s}}+\beta S_t\right)\\&\quad\times\int_{-\infty}^{z_t}\Phi\left(\frac{z_{t+s}\sqrt{S_{t+s}}-\beta z\sqrt{S_t}}{\sqrt{S_{t+s}-\beta^2S_t}}\right)\phi(z)\mathrm{d}z+\sqrt{S_t(S_{t+s}-\beta^2S_t)}\phi(z_t)\phi(\eta_{t+s}),\end{aligned} \quad (1.19)$$

where η_{t+s} and χ_{t+s} are defined respectively by

$$\eta_{t+s} = \frac{z_{t+s}\sqrt{S_{t+s}} - \beta z_t\sqrt{S_t}}{\sqrt{S_{t+s} - \beta^2 S_t}},$$
$$\chi_{t+s} = \frac{z_t\sqrt{S_{t+s}} - \beta z_{t+s}\sqrt{S_t}}{\sqrt{S_{t+s} - \beta^2 S_t}}. \tag{1.20}$$

In particular each component of $\mathrm{Cov}\left[\mathcal{T}_t, \mathcal{T}_{t+s}\right]$, with the exception of the integral, may be evaluated from the probability density function $\phi(z)$ and cumulative distribution function $\Phi(z)$ of the standard normal with appropriately chosen arguments. The usefulness of expression (1.19) for $\mathrm{Cov}\left[\mathcal{T}_t, \mathcal{T}_{t+s}\right]$ can be improved if the value of the integral appearing in this formula can be expressed, albeit approximately, in terms of $\phi(\ )$ and $\Phi(\ )$ with appropriately chosen arguments.

For positive values of the parameter q, this objective can be achieved by making the approximation

$$\Phi\left(\frac{z_{t+s}\sqrt{S_{t+s}} - \beta z\sqrt{S_t}}{\sqrt{S_{t+s} - \beta^2 S_t}}\right) \approx 1 - \Phi(-\eta_{t+s})\,\mathrm{e}^{-\left(p(z-z_t)+q(z-z_t)^2/2\right)}, \tag{1.21}$$

noting, in particular, that the approximation agrees with the interpolated function at $z = z_t$ and as $z \to -\infty$ independently of the values of the parameters p and q. The quality of the approximation is improved by choosing the values of p and q to ensure that the first and second derivatives of the interpolating function match those of the interpolated function when $z = z_t$. The outcome of this matching procedure is that

$$p = -\frac{\beta\sqrt{S_t}}{\sqrt{S_{t+s} - \beta^2 S_t}}\frac{\phi(\eta_{t+s})}{\Phi(-\eta_{t+s})},$$
$$q = p^2\left(1 - \frac{\eta_{t+s}\Phi(-\eta_{t+s})}{\phi(\eta_{t+s})}\right). \tag{1.22}$$

In particular, it is easy to show that $q > 0$, as required. The use of the interpolating formula (1.21) to evaluate the integral in expression (1.19) leads to the conclusion that

$$\int_{-\infty}^{z_t}\Phi(\xi_{t+s})\phi(z)\,\mathrm{d}z \approx \Phi(z_t) - \frac{1}{\sqrt{q+1}}\Phi\left(\frac{p+z_t}{\sqrt{1+q}}\right)\exp\left[\frac{1}{2}\left(\frac{(p+z_t)^2}{1+q} - z_t^2\right)\right]. \tag{1.23}$$

Expression (1.23) is now incorporated into expression (1.19) to give the final approximate form

$$\begin{aligned}
&\mathrm{Cov}\left[\mathcal{T}_t, \mathcal{T}_{t+s}\right] \\
&= \sqrt{S_t S_{t+s}}\left(z_t\phi(z_{t+s})\Phi(\chi_{t+s}) + z_{t+s}\phi(z_t)\Phi(\eta_{t+s})\right) \\
&\quad - \mathbb{E}[\mathcal{T}]_t\mathbb{E}[\mathcal{T}]_{t+s} + \left(z_t z_{t+s}\sqrt{S_t S_{t+s}} + \beta S_t\right)\Phi(z_t) \\
&\quad - \sqrt{S_t\left(S_{t+s} - \beta^2 S_t\right)}\phi(z_t)\phi(\eta_{t+s}) \\
&\quad - \frac{\left(z_t z_{t+s}\sqrt{S_t S_{t+s}} + \beta S_t\right)}{\sqrt{q+1}}\Phi\left(\frac{p+z_t}{\sqrt{1+q}}\right) \\
&\quad \times\exp\left[\frac{1}{2}\left(\frac{(p+z_t)^2}{1+q} - z_t^2\right)\right]
\end{aligned} \tag{1.24}$$

Expressions (1.15) and (1.24) (with t replaced by k and $t+s$ replaced by j) when substituted into expression (1.12) provide a closed-form approximation for the variance of the cumulative temperature index which is then treated as a Gaussian random variable with the computed variance and mean value given by expression (1.11).

4. Approximating the Variance

A closed-form expression for the variance of the cumulative temperature index was derived in the previous subsection. Curiously a heuristic argument based on interpolation can be used to generate a simpler expression for this variance, one that exhibits good accuracy despite the empirical nature of the derivation. The argument begins by noting that the k-th day in the lifetime of a CDD option will contribute to the cumulative temperature index driving the value of the option with probability

$$p_k = \Phi(z_k),\ \ z_k = \frac{\bar{T}_k - T}{\sqrt{S_k}}, \tag{1.25}$$

where $\Phi(z)$ is the cumulative distribution function of the standard normal and T is the temperature above which CDDs are accumulated. If the k-th day always contributes to the cumulative temperature index then the variance of that contribution would be S_k. On the other hand if the k-th day never contributes to the cumulative temperature index then the variance of that contribution would be zero. Since in reality the k-th day contributes fraction p_k of the time then linear interpolation suggests that the variance of this contribution may be reasonably approximated by $S_k p_k$. Based on this idea, the first summation on the right hand side of Equation (1.12) has approximate values

$$\sum_{k=1}^{m}\mathrm{Var}\left[\mathcal{T}_k\right] \approx \sum_{k=1}^{m} p_k S_k. \tag{1.26}$$

The second summation on the right hand side of Equation (1.12) is a correction to expression (1.26) to take account of the fact that contributions to the value of the temperature index from different days are not independent. The contribution made by the quantity $\text{Cov}\left[\mathcal{T}_t, \mathcal{T}_{t+s}\right]$ to the variance of the temperature index is argued in a similar way. In the absence of clipping, the variance of this product is equal to $\text{Cov}\left[\theta_k, \theta_j\right]$ with value $S_k e^{-\alpha(j-k)}$ assuming that $j > k$. However, the product $\mathcal{T}_k \mathcal{T}_j$ is nonzero with probability $p_k p_j$ and therefore the same linear interpolation argument suggests that $\text{Cov}\left[\mathcal{T}_k, \mathcal{T}_j\right]$ is reasonably approximated by $p_k p_j S_k e^{-\alpha(j-k)}$. Based on this idea, the second summation on the right hand side of Equation (1.12) has approximate value

$$2\sum_{k=1}^{N-1}\sum_{j=k+1}^{N} \text{Cov}\left[\mathcal{T}_k, \mathcal{T}_j\right] \approx 2\sum_{k=1}^{N-1} p_k S_k \sum_{j=k+1}^{N} p_j e^{-\alpha(j-k)}. \quad (1.27)$$

In conclusion, linear interpolation suggests that the variance of $\mathcal{T}$ is well approximated by the formula

$$\text{Var}\left[C_N\right] = \sum_{k=1}^{N} p_k S_k + 2\sum_{k=1}^{N-1} p_k S_k \sum_{j=k+1}^{N} p_j e^{-\alpha(j-k)}. \quad (1.28)$$

In fact Equation (1.28) is the first-order approximation to the closed-from expression of the variance in Proposition 1. Consequently, it is expected that this approximation will perform particularly well when the level of truncation is low and also when the persistence in temperature is low which means that deviations in temperature, $\theta(t)$, are restored to their mean value relatively quickly.

To test the accuracy of the approximate closed-form expression for $\text{Var}\left[C_N\right]$ stated in Proposition 1, tranches of one million realizations of Equation (1.1), each of duration 90 days, were constructed for fixed values of α and σ. Specifically, each realization $(\theta_0, \cdots, \theta_{90})$ was obtained by drawing θ_0 from the marginal density of θ expressed in the form $N(0, S^2)$, and subsequent values of θ were determined exactly using the iteration

$$\theta_k = e^{-\alpha}\theta_{k-1} + S e^{-\alpha/2}\sqrt{2\sinh(\alpha)}\,\xi_k, \quad k = 1, \cdots, N, \quad (1.29)$$

where $\xi_k \sim N(0,1)$. Realizations of $\theta(t)$ generated in this way had mean value zero and stationary standard deviation S which was set at 4C° for all simulation experiments. A threshold value of θ was chosen, say Θ, and a cumulative CDD for the 90 day period was constructed from a realization $(\theta_0, \cdots, \theta_{90})$ using the formula

$$C = \sum_{k=1}^{90} \max\left(\theta_k - \Theta, 0\right). \quad (1.30)$$

For a given value of α and a given value of Θ, each tranche of one million realization of Equation (1.1) generated one million independently and identically distributed realizations of CDDs. **Table 1** shows the result of seven experiments for the case $\alpha = 0.2$ and thresholds $\Theta \in (-3S, -2S, -S, 0, S, 2S, 3S)$. **Table 2** shows the equivalent result when $\alpha = 0.5$ and the thresholds are unchanged.

Table 1. For α = 0.2 the column headed "Θ" gives threshold temperature relative to zero for contributions to cumulative CDD. Columns headed "Mean" and "Std Dev" give the mean cumulative CDD and its standard deviation based on one million simulations. Estimates of this standard deviation based on Proposition 1 (Exact) and the heuristic argument of Section 1.4 (Approx) are shown.

Θ	Mean	Std Dev	Exact	Approx
−12	1080.1	116.63	116.67	116.69
−8	722.98	114.25	114.23	114.32
−4	389.92	99.608	99.545	99.272
0	143.57	63.325	63.269	61.422
4	29.975	24.680	24.465	23.148
8	3.0560	5.7022	5.3141	6.2514
12	0.1379	0.8556	0.5688	1.4022

Table 2. For α = 0.5 the column headed "Θ" gives threshold temperature relative to zero for contributions to cumulative CDD. Columns headed "Mean" and "Std Dev" give the mean cumulative CDD and its standard deviation based on one million simulations. Estimates of this standard deviation based on Proposition 1 (Exact) and the heuristic argument of Section 1.4 (Approx) are shown.

Θ	Mean	Std Dev	Exact	Approx
−12	1080.1	75.730	75.751	75.766
−8	723.01	74.189	74.181	74.346
−4	389.95	64.740	64.703	65.310
0	143.60	41.281	41.246	42.409
4	29.982	16.243	16.154	18.359
8	3.0537	3.8667	3.7333	5.9155
12	0.1379	0.6207	0.5527	1.3970

It is clear from these results that the variance of cumulative CDDs predicted by the closed-form approximation of Proposition 1 is achieved in practice. Minor differences between the approximate variance in Proposition 1 and that achieved by simulation become evident only when the threshold temperature lies two standard deviations or more above the mean temperature largely due to the fact that under these circumstances realizations of CDDs will be dominated by zero values. However this is not a scenario that will be occur in practice.

The most interesting observation in **Tables 1** and **2** lies in the unexpected accuracy of the heuristic estimate of variance. In the region of most interest, that is when the threshold temperature lies on or below the average daily temperature taken to be zero in this analysis, the heuristic approach delivers parsimonious estimates of variance that, although marginally inferior to the estimates of true variance provided by Proposition 1, are negligibly different from it for all practical purposes.

5. Parameter Estimation

To use this model for predicting the payoffs from temperature-based derivatives an estimate of the parameter α in Equation (1.1) is required. This parameter measures the rate at which deviations of temperature from the seasonal are restored to this mean. In order to do so, it is first necessary to obtain estimates of $\bar{T}(t)$ and $\sigma(t)$. Following Campbell and Diebold [6], $\bar{T}(t)$ and $\sigma(t)$ are approximated by the Fourier series

$$\begin{aligned}\bar{T}(s) &= a_0 + b_0 s + \sum_{k=1}^{n} a_k \cos(\omega_k s) + b_k \sin(\omega_k s), \\ \sigma^2(s) &= c_0 + \sum_{k=1}^{n} c_k \cos(\omega_k s) + d_k \sin(\omega_k s),\end{aligned} \quad (1.31)$$

where $\omega_k = 2k\pi/365$ and $s = 0$ is assumed to be the calender date of the first observation of average daily temperature. The contribution $b_0 s$ in the expression for $\bar{T}(s)$ is present to take account of any annual trend in daily average temperature. Otherwise expressions (1.31) assume that seasonal variations in daily average temperature follow an annual cycle which is independent of calendar year. Consequently, the expression for $S(t)$ corresponding to the expression (1.31) for $\sigma^2(s)$ is

$$S(s) = p_0 + \sum_{k=1}^{n} p_k \cos(\omega_k s) + q_k \sin(\omega_k s), \quad (1.32)$$

where the Fourier coefficients $c_0, c_1, \cdots, c_n, d_1, \cdots, d_n$ are related to the Fourier coefficients $p_0, p_1, \cdots, p_n, q_1, \cdots, q_n$ by the formulae

$$c_0 = 2\alpha p_0, \quad \left.\begin{aligned} c_k &= 2\alpha p_k + \omega_k q_k, \\ d_k &= -\omega_k p_k + 2\alpha q_k, \end{aligned}\right] \quad (1.33)$$

where k takes all integer values from $k = 1$ to $k = n$ inclusive. Two strategies to estimate the value of α and the coefficients in the Fourier series (1.31) are now described.

5.1. Two-Step Estimator

Suppose that the data consists of observations of daily average temperatures $T_1, T_2, \cdots, T_N$ at times $t_1, t_2, \cdots, t_N$. The Fourier coefficients of $\bar{T}(s)$ can be estimated in a straightforward way by minimizing the objective function

$$\Psi(a_0, b_0, a_1, \cdots, a_n, b_1, \cdots, b_n) = \sum_{j=1}^{N} \left(T_j - \bar{T}(t_j)\right)^2.$$

Once these coefficients are known, then the deviations from the seasonal means $\theta_1, \theta_2, \cdots, \theta_n$ can be computed directly from the formula $\theta_j = T_j - \bar{T}(t_j)$. The problem is now to find the values of α and the coefficients $c_0, c_1, \cdots, c_n, d_1, \cdots, d_n$ which best fit the residuals $\theta_1, \theta_2, \cdots, \theta_n$.

Using a result established by Bibby and Sorensen [12], an unbiased estimate $\hat{\alpha}$ of α is given by the expression

$$-\log\left[\frac{\displaystyle\sum_{k=1}^{n} \frac{\theta_{k-1}}{\sigma_{k-1}^2} \sum_{j=1}^{n} \frac{\theta_j}{\sigma_j^2} - \sum_{k=1}^{n} \frac{\theta_{k-1}\theta_k}{\sigma_{k-1}^2} \sum_{j=1}^{n} \frac{1}{\sigma_{j-1}^2}}{\displaystyle\left(\sum_{k=1}^{n} \frac{\theta_{k-1}}{\sigma_{k-1}^2}\right)^2 - \sum_{k=1}^{n} \frac{\theta_{k-1}^2}{\sigma_{k-1}^2} \sum_{j=1}^{n} \frac{1}{\sigma_{j-1}^2}}\right]. \quad (1.34)$$

The difficulty, however, in using this expression is that σ_k^2 is unknown whereas what is known is the seasonal variance of the residuals. The strategy for finding the values of α and the coefficients $c_0, c_1, \cdots, c_n, d_1, \cdots, d_n$ is therefore the following.

Step 1: Compute the Fourier coefficients $p_0, p_1, \cdots p_n$ and $q_1, \cdots, q_n$ of $S(t)$ directly from the deviations $\theta_1, \theta_2, \cdots, \theta_N$.

Step 2: Choose an arbitrary value for α, say α_0, and compute the Fourier coefficients $c_0, c_1, \cdots, c_n, d_1, \cdots d_n$ from expression (1.33) with $\alpha = \alpha_0$. Knowing the Fourier coefficients of $\sigma^2(s)$ enables $\sigma_0^2, \cdots, \sigma_n^2$ to be computed from Equation (1.31). Expression (1.34) is now used to update the estimate of α_0. This procedure may then be iterated by recomputing in turn $c_0, c_1, \cdots, c_n$, $d_1, \cdots d_n$ and $\sigma_0^2, \cdots, \sigma_n^2$. This procedure is repeated until consecutive estimates of α are not deemed to be significantly different.

The estimate of α and the Fourier coefficients a_0, b_0, $a_1, \cdots, a_n, b_1, \cdots, b_n$ and $c_0, c_1 \cdots, c_n, d_1, \cdots d_n$ can either be used as they stand or can be used as an initial guess for the parameters of the maximum likelihood estimation procedure outlined in the next subsection.

5.2. Maximum-Likelihood Estimation

The feasibility of parameter estimation by maximum likelihood (ML) in this instance relies on the fact that the transitional probability density function of average daily temperature can be computed under the assumption that the deviations of average daily temperature from its mean value satisfies the stochastic differential Equation (1.1). Ito's lemma applied to the stochastic differential Equation (1.1) may be shown to lead to the formal solution

$$\theta(t) = \theta_j \mathrm{e}^{-\alpha(t-t_j)} + \int_{t_j}^{t} \mathrm{e}^{-\alpha(t-s)} \sigma(s) \mathrm{d}W_s, \ t > t_j. \quad (1.35)$$

with $\theta_j = \theta(t_j)$. The important observation from this solution is that $\theta(t)$ is a Gaussian random variable with mean value $E[\theta(t)] = \theta_j \mathrm{e}^{-\alpha(t-t_j)}$ and variance

$$\chi(t,t_j) = \int_{t_j}^{t} \mathrm{e}^{-2\alpha(t-s)}\sigma^2(s)\,\mathrm{d}s = S(t) - \mathrm{e}^{-2\alpha(t-t_j)}S(t_j), \tag{1.36}$$

where the latter expression for $\chi(t,t_j)t$ is derived directly from the definition of $S(t)$ given in Equation (1.3). Because $T = \bar{T}(t) + \theta(t)$, then the average daily temperature T is itself Gaussian distributed with mean value $\bar{T}(t) + (T_j - \bar{T}_j)\mathrm{e}^{-\alpha(t-t_j)}$ and variance $\chi(t,t_j) = S(t) - \mathrm{e}^{-2\alpha(t-t_j)}S(t_j)$ in which

$$\bar{T}(t) = a_0 + b_0 t + \sum_{k=1}^{n} a_k \cos(\omega_k t) + b_k \sin(\omega_k t). \tag{1.37}$$

Thus the average daily temperature $T(t)$ has transitional probability density function

$$f(T,t|T_j,t_j) = \frac{\mathrm{e}^{-\psi(T,t)}}{\sqrt{2\pi\chi(t,t_j)}}, \tag{1.38}$$

where

$$\psi(T,t) = \frac{\left(T - \bar{T}(t) - (T_j - \bar{T}_j)\mathrm{e}^{-\alpha(t-t_j)}\right)^2}{2\chi(t,t_j)}.$$

The likelihood of observing the sequence $T_1, T_2, \cdots, T_N$ of average daily temperatures at calendar times $t_1, t_2, \cdots, t_N$ is therefore

$$\mathcal{L}(\alpha; a_0, a_1, \cdots, a_n, b_1, \cdots b_n; c_0, c_1, \cdots, c_n, d_1, \cdots, d_n) = \prod_{j=1}^{N-1} f(T_{j+1}, t_{j+1}|T_j, t_j). \tag{1.39}$$

In practice, the parameters are estimated by minimizing the negative log-likelihood function

$$-\log\mathcal{L} = \frac{N-1}{2}\log 2\pi + \frac{1}{2}\sum_{j=1}^{N-1}\log\left(S_{j+1} - \mathrm{e}^{-2\alpha(t_{j+1}-t_j)}S_j\right) + \frac{1}{2}\sum_{j=1}^{N-1}\frac{\left(T_{j+1} - \bar{T}_{j+1} - (T_j - \bar{T}_j)\mathrm{e}^{-\alpha(t_{j+1}-t_j)}\right)^2}{S_{j+1} - \mathrm{e}^{-2\alpha(t_{j+1}-t_j)}S_j}, \tag{1.40}$$

where the notation $S_j = S(t_j)$ has been used. The optimal values for the parameters of this model are taken to be those which minimize expression (1.39). Although model (1.1) is specified in terms of the intrinsic function $\sigma(t)$, from a purely technical point of view it is easier to treat the Fourier coefficients of $S(t)$ as the parameters to be determined by the ML procedure.

6. Empirical Illustration

The task is now to provide a means of gauging the efficacy of the analytical expressions for the mean and variance of C_N given derived previously in terms of the the expected payoffs to options contracts. Payoffs based on the analytical results of the paper are compared to historical pricing as outlined in [4,5]. The metric for comparison is taken to be the mean "profit" of a 90-day call option contract. Profit is defined from the point of view of the buyer of the call option as the difference between the actual tick value of the contract and the expected tick value or "price" of the option. Of course, this is not meant to represent a true price for the option, as this notional pricing strategy takes no account of discounting or overhead expenses. But of course, any pricing scheme will stand or fall by its ability to estimate the expected tick value accurately.

6.1. Data

The data set comprises daily maximum and minimum temperature records in degrees Celsius for Brisbane (1/1/1887-31/8/2007), Melbourne (1/1/1856-31/8/2007), Perth (1/1/1897-31/8/2007) and Sydney (1/1/1859-31/8/2007). These locations were chosen primarily because they had accurate temperature records of over 100 years duration measured at comparable weather stations[4].

Figure 1 shows the long-term expected values (upper panel) and standard deviations (lower panel) of daily temperatures for each day of the year. The figure shows that the behaviour of the mean and standard deviation is amenable to modelling by a low-order Fourier series approximation. In this exercise the order of the series is taken to be 3. The Fourier approximation is applied only over the period over which the option is to be written, namely, 1 January to 31 March, inclusive.

Descriptive statistics for cumulative CDDs are reported in **Table 3**. There are two observations of note arising from **Table 3**. First, the distribution of cumulative CDDs for Melbourne is skewed to the right as evidenced by a mean which is significantly larger than the median. Second, Perth is notable for the diffuse nature of the distribution of cumulative CDDs, recording a standard deviation significantly larger than those of the other cities.

The distributions of cumulative CDDs for each city is illustrated in **Figure 2** which plots both the distribution of historical cumulative CDDs (shaded region) and the predicted distributions for 1950 (dashed line) and 2007 (solid line) generated by closed-form approximations to the distributions of CDDs derived in the paper. To the uniformed eye, the distribution of historical cumulative CDDs may appear well behaved and taken as reasonable evidence in favour of using historical records to price temperature-based derivatives. When compared to the

[4]All the raw data were supplied by Climate Information Services, National Climate Centre, Australian Bureau of Meteorology. The construction of the temperature record for each city is discussed in **Appendix 3**.

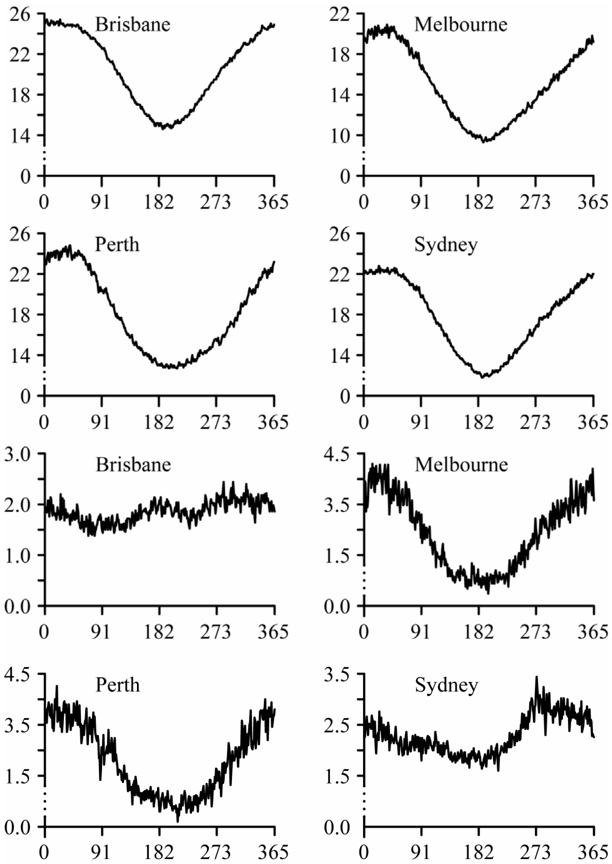

Figure 1. The expected value of the average daily temperatures (upper panel) and the expected value of the volatility of average daily temperatures (lower panel) are shown for Brisbane, Melbourne, Perth and Sydney.

Table 3. Mean, median, standard deviation, minimum and maximum cumulative CDDs in four Australian cities.

	Summary Statistics				
	N	Mean (SD)	Med.	Min.	Max.
BNE	121	584.2 (54.5)	584.6	463.3	705.9
MEL	152	207.9 (64.1)	195.6	93.5	391.4
PER	111	489.6 (83.3)	492.2	298.3	688.3
SYD	149	350.0 (60.1)	350.2	225.5	533.3

distributions for 1950 and 2007 generated by the analytical approach, however, the potential for error inherent in the historical approach becomes evident. Not only does the mean of the predicted distribution change noticeably over time, but the distribution also has lower volatility.

6.2. Payoffs

The profits generated by two call-option contracts with different strike prices, written on the period 1 January to

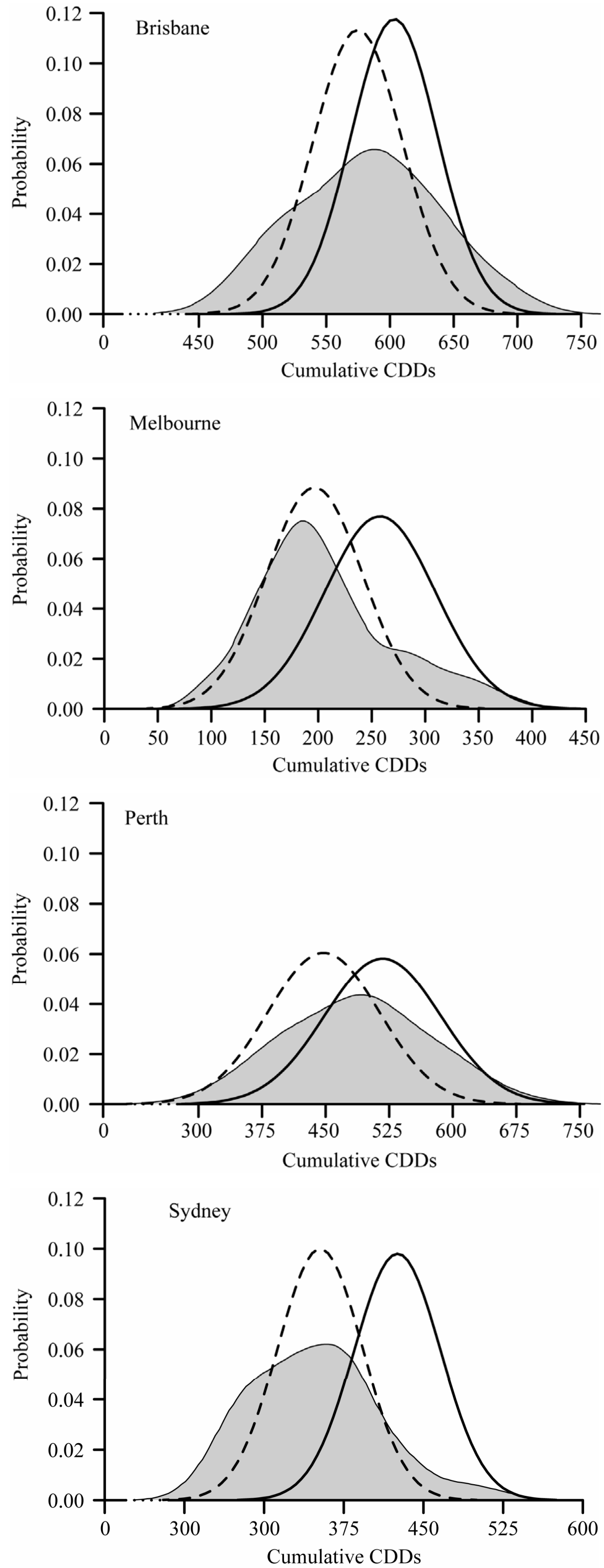

Figure 2. Density of historical cumulative CDDs based on data up to and including 1949 (shaded area), predicted density of cumulative CDD for 1950 (dashed line) and predicted density of cumulative CDD for 2007 (solid line).

31 March are now reported in **Tables 4** and **5** respectively. The call options used in the experiment have respective strike prices set to be approximately $D = \mu + 0.5\sigma$ and $D = \mu + 0.75\sigma$ where μ is the unconditional mean and σ is the unconditional standard deviation of CDDs up to the current year under consideration.

The experiments begin by pricing these options for the year 1950 using data up to and including 1949. The actual payoff for 1950 is recorded, the profit or loss stored and the data set updated to include the latest observation on cumulative CDDs. These steps are repeated up to and including 2007 giving a total of 58 separate profits for each option. The means and standard deviations of the profits are regarded as measures of the performance of each of the methods used to determine expected tick values.

Table 4. Means and standard deviations of profits to a 90-day call option defined on CDDs with strike price *D* approximately equal to μ + 0.5σ, where μ and σ are the unconditional mean and standard deviation of available historical CDDs. The option is priced for each year from 1950 to 2007 inclusive.

	BNE	MEL	PER	SYD
Strike D	600	240	530	380
Historical				
Mean Payoff	−8.1	−14.3	−23.8	7.8
SDev Payoff	33.1	45.8	43.2	48.9
Quarterly Model				
Mean Payoff	7.2	13.2	2.2	11.7
SDev Payoff	29.6	41.5	41.8	35.5
Annual Model				
Mean Payoff	5.8	15.4	18.3	4.0
SDev Payoff	29.1	41.4	40.0	34.6

Table 5. Means and standard deviations of profits to a 90-day call option defined on CDDs with strike price *D* approximately equal to μ + 0.75σ, where μ and σ are the unconditional mean and standard deviation of available historical CDDs. The option is priced for each year from 1950 to 2007 inclusive.

	BNE	MEL	PER	SYD
Strike D	620	260	550	400
Historical				
Mean Payoff	−17.7	−24.7	−35.1	−4.2
SDev Payoff	25.3	38.3	36.1	42.7
Quarterly Model				
Mean Payoff	6.2	11.9	1.3	9.8
SDev Payoff	22.7	34.2	34.2	30.1
Annual Model				
Mean Payoff	5.5	13.3	13.4	4.6
SDev Payoff	22.4	34.2	36.6	29.2

The historical pricing reported in **Tables 4** and **5** is self-explanatory, but the implementation of the closed-form approximations needs further elucidation. Two variations of this method are implemented, namely an annual version and a quarterly version. The annual approach fits the mean and seasonal variance of average daily temperature using data for the entire year and the best estimates of the parameters are used in computing the closed-from approximations of the distribution of cumulative CDDs. By contrast, the quarterly version focusses on the period from 1 January to the 31 March in each year and fits the mean and seasonal variance of average daily temperature for this 90-day period alone. In other words, the fitting procedure is implemented only on the period over which the contract is written. The main reason for adopting this approach is that the behaviour of temperature in parts of the year unrelated to the period of the option are not being allowed to influence parameter estimates for the mean and variance processes. Another benefit of this approach is that better resolution of the mean and variance processes with the same number of parameters.

The first striking conclusion to be drawn from these results is just how bad historical pricing performs for the Australian temperature data. Interestingly enough, it appears that historical pricing in three of the cities has substantially over-priced the call options. This result is counter-intuitive as the conventional view is that there is an upward trend in temperature which would result in the under-pricing of call options priced on the history of cumulative CDDs.

The resolution of this conundrum is to be found in the behaviour of temperature between the years 1890 and 1920. During this period, Brisbane, Melbourne and Perth recorded substantial outliers in cumulative CDDs, the likes of which were not seen again until late in the sample period. These outliers will have had a disproportionate affect on the pricing of temperature derivatives in the 1960s, 1970s and 1980s. Their existence also explains the deterioration of profits based on historical pricing when moving from lower to higher exercise prices. The weather station in Sydney where the temperature data were recorded did not show these extreme temperature events and consequently historical pricing for Sydney performs significantly better.

Taken as a whole, the closed-approximations used to price the call options generate mean profits closer to zero and with lower standard deviations than historical pricing. Nevertheless, this method appears to underprice somewhat, even though these pricing errors are smaller in magnitude than those generated by the historical method. This underpricing is again a manifestation of the outliers in cumulative CDDs but in this case, not enough weight is given to them. There is little difference in terms of performance of quarterly and annual models, with the exception of Perth where the quarterly model performs better. It is conjectured that this is due to the ability of the quarterly model to better resolve the extreme temperature variations that are prone to take place in Perth. Unlike the case documented for historical pricing, there seems little difference in performance when moving from the lower to the higher exercise price for the the closed-form approach.

7. Conclusions

This paper has derived closed-form expressions for approximating the distribution of temperature indices. The major practical use for these approximations is in estimating the payoffs to temperature-based weather derivatives. Although the cumulative cooling degree day index is the focus of this research, the methods used are equally applicable to derivatives based on cumulative heating degree days. Common practice when modelling average daily temperature is to regard the deviations of temperature from its expected value as an Ornstein-Uhlenbeck process. The key result derived in this paper, is that if this model of temperature is adopted, then the distribution of cumulative cooling degree days may be constructed as the sum of truncated, correlated Gaussian deviates. The mean and variance of the resultant Gaussian distribution depend on the parameters of the underlying temperature process and its autocorrelation structure.

The efficacy of these approximate distributions is tested by estimating the payoffs to temperature-based derivatives. Time series data spanning over a hundred years of average daily temperatures in four major Australian cities are used to estimate the payoffs to European call options written on cooling degree days. The robust conclusion to emerge from this line of research is that the closed-form distributions perform more reliably than the historical pricing method that is commonly advocated in the literature.

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Appendix 1

Proof of Result (1.15)

It has been shown in Equation (1.13) that

$$\begin{aligned}\mathrm{Var}\left[\mathcal{T}_k\right] = {} & S_k\int_{-\infty}^{z_k}\left(z_k - z\right)^2\phi(z)\mathrm{d}z \\ & - S_k\left[z_k\Phi(z_k)+\phi(z_k)\right]^2.\end{aligned}$$

The manipulation of this integral uses the fact that the Gaussian probability density function enjoys the property $z\phi(z) = -\phi'(z)$. Thus

$$\begin{aligned}& S_k\int_{-\infty}^{z_k}\left(z_k - z\right)^2\phi(z)\mathrm{d}z \\ & = S_k\int_{-\infty}^{z_k}\left(z_k^2 - 2zz_k + z^2\right)\phi(z)\mathrm{d}z \\ & = S_k z_k^2\Phi(z_k) + S_k\int_{-\infty}^{z_k}\left(2z_k - z\right)\phi'(z)\mathrm{d}z \\ & = S_k z_k^2\Phi(z_k) + S_k\left[\left(2z_k - z\right)\phi(z)\right]_{-\infty}^{z_k} + S_k\int_{-\infty}^{z_k}\phi(z)\mathrm{d}z \\ & = S_k\left(z_k^2+1\right)\Phi(z_k) + S_k z_k\phi(z_k).\end{aligned}$$

It is now straightforward algebra to verify the assertion in Equation (1.14), namely that $\mathrm{Var}\left[\mathcal{T}_k\right]$ has value

$$S_k\left[\Phi(z_k) - \left(\phi(z_k) + z_k\Phi(z_k)\right)\left(\phi(-z_k) - z_k\Phi(-z_k)\right)\right],$$

where the calculation has noted that $\phi(z)$ is an even-valued function of *z* and that $1-\Phi(z) = \Phi(-z)$.

Appendix 2

Proof of Result (1.19)

The calculation of $\mathrm{Cov}\left[\mathcal{T}_t, \mathcal{T}_{t+s}\right]$ requires *I*, the value of the integral

$$\sqrt{S_t S_{t+s}}\int_{-\infty}^{z_t}\int_{-\infty}^{z_{t+s}}\left(z_t - z\right)\left(z_{t+s} - w\right)\tilde{f}\left(z_{t+s}, z_t\right)\mathrm{d}z\mathrm{d}w \quad (1.41)$$

in which $\tilde{f}\left(z_{t+s}, z_t\right)$ is the probability density function

$$\frac{1}{2\pi}\sqrt{\frac{S_{t+s}}{S_{t+s}-\beta^2 S_t}}\mathrm{e}^{-\psi},$$

with

$$\psi = \frac{S_{t+s}z^2 - 2zw\beta\sqrt{S_t S_{t+s}} + S_{t+s}w^2}{2\left(S_{t+s}-\beta^2 S_t\right)}$$

and $\beta = \mathrm{e}^{-\alpha s}$. By re-expressing ψ in the form

$$\frac{S_{t+s}}{2\left(S_{t+s}-\beta^2 S_t\right)}\left(w - \beta z\sqrt{\frac{S_t}{S_{t+s}}}\right)^2 - \frac{z^2}{2},$$

expression (1.41) is re-expressed as the repeated integral

$$I = S_{t+s}\sqrt{\frac{S_t}{S_{t+s}-\beta^2 S_t}}\int_{-\infty}^{z_t}\left(z_t - z\right)\phi(z)g(z)\mathrm{d}z \quad (1.42)$$

where $\phi(z)$ is the standard normal probability density function and $g(z)$ is the integral

$$\begin{aligned}& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z_{t+s}} -\left(z_{t+s} - w\right) \\ & \cdot\exp\left[-\frac{S_{t+s}\left(w - \beta z\sqrt{\frac{S_t}{S_{t+s}}}\right)^2}{2\left(S_{t+s}-\beta^2 S_t\right)}\right]\mathrm{d}w.\end{aligned} \quad (1.43)$$

Phase I

The evaluation of this integral is achieved by changing the variable of integration from *w* to ξ using the substitution

$$\xi = \sqrt{\frac{S_{t+s}}{S_{t+s}-\beta^2 S_t}}\left(w - \beta z\sqrt{\frac{S_t}{S_{t+s}}}\right).$$

The outcome of this operation is that $g(z)$ takes the simplified form

$$g(z) = \frac{S_{t+s}-\beta^2 S_t}{S_{t+s}}\int_{-\infty}^{\xi_{t+s}(z)}\left(\xi_{t+s}(z) - \xi\right)\phi(\xi)\mathrm{d}\xi \quad (1.44)$$

where

$$\xi_{t+s}(z) = \sqrt{\frac{S_{t+s}}{S_{t+s}-\beta^2 S_t}}\left(z_{t+s} - \beta z\sqrt{\frac{S_t}{S_{t+s}}}\right).$$

It now follows immediately from the definition of $\Phi(z)$, the cumulative function of the standard normal distribution, and the basic properties of $\phi(z)$ that

$$g(z) = \frac{S_{t+s}-\beta^2 S_t}{S_{t+s}}\left(\phi(\xi_{t+s}) + \xi_{t+s}\Phi(\xi_{t+s})\right) \quad (1.45)$$

in which the dependence of ξ_{t+s} on *z* has been suppressed for representational convenience. Consequently

$$\begin{aligned}I = {} & \sqrt{S_t\left(S_{t+s}-\beta^2 S_t\right)}\int_{-\infty}^{z_t}\left(z_t - z\right)\phi(z) \\ & \times\left[\phi(\xi_{t+s}) + \xi_{t+s}\Phi(\xi_{t+s})\right]\mathrm{d}z.\end{aligned} \quad (1.46)$$

This completes the first phase in the computation of the value of *I* using repeated integration.

Phase II

The second phase of calculation continues by dividing the right hand side of Equation (1.46) into the two integrals

$$\sqrt{S_t\left(S_{t+s}-\beta^2 S_t\right)}\int_{-\infty}^{z_t} z_t\phi(z)\left(\phi(\xi_{t+s})+\xi_{t+s}\Phi(\xi_{t+s})\right)\mathrm{d}z$$
$$-\sqrt{S_t\left(S_{t+s}-\beta^2 S_t\right)}\int_{-\infty}^{z_t} z\phi(z)\left(\phi(\xi_{t+s})+\xi_{t+s}\Phi(\xi_{t+s})\right)\mathrm{d}z.$$

The function $\xi_{t+s}(z)$ is now replaced by its definition in the first of these integrals, and after some rearrangement, I is expressed as the sum of four integrals, namely

$$\begin{aligned}I &= z_t\sqrt{S_t\left(S_{t+s}-\beta^2 S_t\right)}\int_{-\infty}^{z_t}\phi(\xi_{t+s})\phi(z)\mathrm{d}z + z_t z_{t+s}\sqrt{S_t S_{t+s}}\int_{-\infty}^{z_t}\Phi(\xi_{t+s})\phi(z)\mathrm{d}z \\ &\quad -\beta z_t S_t\int_{-\infty}^{z_t}\Phi(\xi_{t+s})z\phi(z)\mathrm{d}z - \sqrt{S_t\left(S_{t+s}-\beta^2 S_t\right)}\int_{-\infty}^{z_t} z\phi(z)\left[\varphi(\xi_{t+s})+\xi_{t+s}\Phi(\xi_{t+s})\right]\mathrm{d}z.\end{aligned} \quad (1.47)$$

The third and fourth integrals on the right hand side of this equation are now manipulated using integration by parts. Manipulation of the third integral gives

$$\begin{aligned}\int_{-\infty}^{z_t}\Phi(\xi_{t+s})z\phi(z)\mathrm{d}z &= \left[-\phi(z)\Phi(\xi_{t+s})\right]_{-\infty}^{z_t} - \beta\sqrt{\frac{S_t}{S_{t+s}-\beta^2 S_t}}\int_{-\infty}^{z_t}\phi(z)\phi(\xi_{t+s})\mathrm{d}z \\ &= -\phi(z_t)\Phi(\eta_{t+s}) - \beta\sqrt{\frac{S_t}{S_{t+s}-\beta^2 S_t}}\int_{-\infty}^{z_t}\phi(z)\phi(\xi_{t+s})\mathrm{d}z,\end{aligned} \quad (1.48)$$

where

$$\eta_{t+s} = \frac{z_{t+s}\sqrt{S_{t+s}}-\beta z_t\sqrt{S_t}}{\sqrt{S_{t+s}-\beta^2 S_t}}. \quad (1.49)$$

Manipulation of the fourth integral gives

$$\begin{aligned}&\int_{-\infty}^{z_t} z\phi(z)\left(\phi(\xi_{t+s})+\xi_{t+s}\Phi(\xi_{t+s})\right)\mathrm{d}z \\ &= \left[-\phi(z)\left(\varphi(\xi_{t+s})+\xi_{t+s}\Phi(\xi_{t+s})\right)\right]_{-\infty}^{z_t} - \beta\sqrt{\frac{S_t}{S_{t+s}-\beta^2 S_t}}\int_{-\infty}^{z_t}\phi(z)\Phi(\xi_{t+s})\mathrm{d}z \\ &= -\phi(z_t)\left(\phi(\eta_{t+s})+\eta_{t+s}\Phi(\eta_{t+s})\right) - \beta\sqrt{\frac{S_t}{S_{t+s}-\beta^2 S_t}}\int_{-\infty}^{z_t}\phi(z)\Phi(\xi_{t+s})\mathrm{d}z.\end{aligned} \quad (1.50)$$

Results (1.48) and (1.50) are now incorporated into Equation (1.47) to get

$$\begin{aligned}I &= z_t S_{t+s}\sqrt{\frac{S_t}{S_{t+s}-\beta^2 S_t}}\int_{-\infty}^{z_t}\phi(\xi_{t+s})\phi(z)\mathrm{d}z + z_{t+s}\sqrt{S_t S_{t+s}}\phi(z_t)\Phi(\eta_{t+s}) \\ &\quad +\left(z_t z_{t+s}\sqrt{S_t S_{t+s}}+\beta S_t\right)\int_{-\infty}^{z_t}\Phi(\xi_{t+s})\phi(z)\mathrm{d}z + \sqrt{S_t\left(S_{t+s}-\beta^2 S_t\right)}\phi(z_t)\phi(\eta_{t+s}).\end{aligned} \quad (1.51)$$

The final stage of this calculation is to note that

$$\begin{aligned}\int_{-\infty}^{z_t}\phi(\xi_{t+s})\phi(z)\mathrm{d}z &= \frac{\phi(z_{t+s})}{\sqrt{2\pi}}\times\int_{-\infty}^{z_t}\exp\left[-\frac{S_{t+s}}{2\left(S_{t+s}-\beta^2 S_t\right)}\left(z-\beta z_{t+s}\sqrt{\frac{S_t}{S_{t+s}}}\right)^2\right]\mathrm{d}z \\ &= \sqrt{\frac{S_{t+s}-\beta^2 S_t}{S_{t+s}}}\phi(z_{t+s})\Phi\left(\frac{z_t\sqrt{S_{t+s}}-\beta z_{t+s}\sqrt{S_t}}{\sqrt{S_{t+s}-\beta^2 S_t}}\right).\end{aligned}$$

To summarize, the repeated integral (1.41) has final value

$$\begin{aligned}I &= \sqrt{S_t S_{t+s}}\left(z_t\phi(z_{t+s})\Phi(\chi_{t+s})+z_{t+s}\phi(z_t)\Phi(\eta_{t+s})\right) \\ &\quad +\left(z_t z_{t+s}\sqrt{S_t S_{t+s}}+\beta S_t\right)\int_{-\infty}^{z_t}\Phi(\xi_{t+s})\phi(z)\mathrm{d}z + \sqrt{S_t\left(S_{t+s}-\beta^2 S_t\right)}\phi(z_t)\phi(\eta_{t+s}),\end{aligned} \quad (1.52)$$

where the constants η_{t+s} and χ_{t+s} and the function $\xi_{t+s}(z)$ are defined respectively by

$$\eta_{t+s} = \frac{z_{t+s}\sqrt{S_{t+s}} - \beta z_t\sqrt{S_t}}{\sqrt{S_{t+s} - \beta^2 S_t}},$$
$$\chi_{t+s} = \frac{z_t\sqrt{S_{t+s}} - \beta z_{t+s}\sqrt{S_t}}{\sqrt{S_{t+s} - \beta^2 S_t}}, \tag{1.53}$$
$$\xi_{t+s}(z) = \frac{z_{t+s}\sqrt{S_{t+s}} - \beta z\sqrt{S_t}}{\sqrt{S_{t+s} - \beta^2 S_t}}.$$

Appendix 3

The construction of the temperature data for the four Australian cities used in the empirical illustration is now outlined in detail.

Brisbane: The temperature record contains 44043 observations starting on the 1/1/1887 and ending on 31/8/2007. The time series is constructed from data collected from three weather stations: Brisbane Regional Office (Station Number 40214) 1/1/1887-31/3/1986; Brisbane Airport (Station Number 40223) 1/4/1986-14/2/2000); and again from Brisbane Airport (Station Number 40842) 15/2/2000-31/8/2007.

Melbourne: The temperature record contains 55358 observations starting on 1/1/1856 and ending on 31/8/2007. The time series is a continuous set of observations made at the Melbourne Regional Office (Station Number 86071) weather station. The location of the office changed in the early 1980s although the name of station did not.

Perth: The temperature record contains 40393 observations starting on 1/1/1897 and ending on 31/8/2007. The time series is constructed from data collected at two weather stations: Perth Regional Office (Station Number 9034) 1/1/1897-2/6/1944; and Perth Airport (Station Number 9021) 3/6/1944-31/8/2007.

Sydney: The temperature record contains 54263 observations starting on 1/1/1859 and ending on 31/8/2007. The time series is a continuous set of observations made at the Sydney Observatory Hill (Station Number 66062) weather station.

Instances of single missing values were treated by averaging adjacent records. In a few rare cases where several days were missing, the long term average for those days was inserted. Finally, following Campbell and Diebold [6], all occurrences of the 29 February were removed.