***Applied Mathematics*, 2013, 4, 1340-1346**
http://dx.doi.org/10.4236/am.2013.49181 Published Online September 2013 (http://www.scirp.org/journal/am)

Randomly Weighted Averages on Order Statistics

Homei Hajir, Hasanzadeh Leila, Mina Ghasemi
Department of Statistics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran
Email: homei@tabrizu.ac.ir.

Received July 23, 2012; revised January 4, 2013; accepted January 11, 2013

ABSTRACT

We study a well-known problem concerning a random variable uniformly distributed between two independent random variables. Two different extensions, randomly weighted average on independent random variables and randomly weighted average on order statistics, have been introduced for this problem. For the second method, two-sided power random variables have been defined. By using classic method and power technical method, we study some properties for these random variables.

Keywords: Two-Sided Power; Moment; Weighted Averages; Power Distribution

1. Introduction

Van Asch [1] introduced the notion of a random variable Z uniformly distributed between two independent random variables X_1 and X_2 which arose in studying the distribution of products of random 2 × 2 matrices for stochastic search of global maxima. By letting X_1 and X_2 to have identical distribution, he derived that: 1) for X_1 and X_2 on [−1, 1], Z is uniform on [−1, 1] if and only if X_1 and X_2 have an Arcsine distribution; and 2) Z possesses the same distribution as X_1 and X_2 if and only if X_1 and X_2 are degenerated or have a Cauchy distribution. Soltani and Homei [2] following Johnson and Kotz [3] extended Van Asch's results. They put $X_1,\cdots,X_n$ to be independent, and considered

$$S_n = R_1X_1 + R_2X_2 + \cdots + R_{n-1}X_{n-1} + R_nX_n,\ n \ge 2 .$$

where random proportions

$$R_i = U_{(i)} - U_{(i-1)},\ i = 1,\cdots,n-1,$$

$$R_n = 1 - \sum_{i=0}^{n-1} R_i,\ U_{(1)},\cdots,U_{(n)}$$

are order statistics from a uniform distribution on [0, 1], and $U_{(0)} = 0$. These random proportions are uniformly distributed over the unit simplex. They employed Stieltjes transform and that: 1) S_n possesses the same distribution as $X_1,\cdots,X_n$ if and only if $X_1,\cdots,X_n$ are degenerated or have a Cauchy distribution; and 2) Van Asch's result for Arcsine holds for Z only.

In this paper, we introduce two families of distributions, suggested by an anonymous referee of the article, to whom the author expresses his deepest gratitude. We say that Z_1 is a random variable between two independent random variables with power distribution, if the conditionally distribution of Z_1 given at $X_1 = x_1,\ X_2 = x_2$ is

$$F_{Z_1|x_1,x_2(z)} = \begin{cases} 1 & z \ge \max(x_1,x_2), \\ \left(\dfrac{z-x_1}{x_2-x_1}\right)^n & x_1 < z < x_2, \\ 1-\left(\dfrac{z-x_1}{x_2-x_1}\right)^n & x_2 < z < x_1, \\ 1 & z \ge \max(x_1,x_2). \end{cases} \quad (1.1)$$

The distribution $F_{Z_1|x_1,x_2}(z)$ will be said to follow a conditionally directed power distribution, When n is an integer. For n = 1, the distribution given by (1.1) simplifies to the distribution Z that was introduced before. Also we used Stieltjes methods, for more on the Stieltjes transform, see Zayed [4].

For n = 2, we call Z_1 directed triangular random variable. For further generalizing Van Asch results, we introduce a seemingly more natural conditionally power distribution. We call Z_2 two-sided power (TSP) random variable if the conditionally distribution of Z_2 given at $X_1 = x_1, X_2 = x_2$ is

$$F_{Z_2|x_1,x_2(z)} = \begin{cases} 1 & z \ge y_2, \\ \left(\dfrac{z-y_1}{y_2-y_1}\right)^n & y_1 < z < y_2. \\ 0 & z \le y_1. \end{cases} \quad (1.2)$$

The distribution $F_{Z_2|x_1,x_2}$ will be said to follow a conditionally undirected power distribution, when y_1 = min $y_1 = \min(x_1, x_2)$, $y_2 = \max(x_1, x_2)$ and *n* is an integer.

Again for *n* = 1, the distribution given by (1.1) simplifies to the distribution *Z* that was introduced by Van Asch. The main aim of this article is providing a generalization of notion to the results of Van Asch for some other values of *n* (other than *n* = 1). This article is organized as follows. We introduce preliminaries and previous works in Section 2. In Section 3, we give some Characterizations for distribution of Z_1 given in (1.1), when *n* = 2. In Section 4, we find distribution of Z_2 given in (1.2) by direct and power method, and give some examples and Characterizations of such distributions by use of Soltani and Homei' results [5].

2. Preliminaries and Previous Works

In this section, we first review some results of Van Asch [1] and then modify them a little Bit to fit in our framework, to be introduced in the forthcoming sections. Using the Heaviside function $\left(U(x)=0, x<0, x=1, x\geq 0\right)$, we conclude that for any given distinct values X_1 and X_2 the conditional distribution $F_{Z_1|x_1,x_2}(z)$ in (1.1) is

$$F_{Z_1|x_1,x_2}(z)=\left(\frac{z-x_1}{x_2-x_1}\right)^n U(z-x_1) -\sum_{i=1}^{n}\binom{n}{i}\left(\frac{z-x_1}{x_2-x_1}\right)^i U(z-x_2). \tag{2.1}$$

Lemma 2.1. For distinct real's x_1, x_2, z and integer n, we have

$$\frac{-1}{(z-x_1)(x_2-x_1)}+\frac{(-1)^n}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x_2^{n-1}}\left(\frac{1}{(z-x_2)}\cdot\frac{1}{(x_2-x_1)}\right) =\frac{1}{(x_1-z)(x_2-z)^n}.$$

Proof. It follows from the Leibniz formula.

Let $h\in D^{\alpha}(I)$, where $I\subseteq\mathbb{R}$ is an interval, and $D^{\alpha}(I)$ is the set of all real functions f that are α-Times differentiable on *I*. If $g(z)=\frac{1}{(z-x)}$. for some constants c and $k\in\{1,\cdots,n\}$. Then

$$P=M-g(z)\frac{\mathrm{d}^k}{\mathrm{d}z^k}h(z)+kg(z)\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\{h(z)g(z)\}-P\Big] =\frac{\mathrm{d}^k}{\mathrm{d}z^k}\{h(z)g(z)\}.$$

We use the Leibnitz formula for the $(k-1)$ th derivative of a product, namely

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\{h(z)g(z)\} =N=\sum_{i=0}^{k-1}\binom{k-1}{i}\left\{\frac{\mathrm{d}^i}{\mathrm{d}z^i}h(z)\right\}\left\{\frac{\mathrm{d}^{k-i}}{\mathrm{d}z^{k-i}}g(z)\right\}.$$

Let

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\{h(z)g(z)\}=\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\{h(z)g(z)\}=M+N.$$

where

$$M=\sum_{i=0}^{k-1}\binom{k-1}{i}\left\{\frac{\mathrm{d}^{i+1}}{\mathrm{d}z^{i+1}}h(z)\right\}\left\{\frac{\mathrm{d}^{k-1-i}}{\mathrm{d}z^{k-1-i}}g(z)\right\},$$

$$N=\sum_{i=0}^{k-1}\binom{k-1}{i}\left\{\frac{\mathrm{d}^{i}}{\mathrm{d}z^{i}}h(z)\right\}\left\{\frac{\mathrm{d}^{k-i}}{\mathrm{d}z^{k-i}}g(z)\right\}.$$

Since $\frac{\mathrm{d}^r}{\mathrm{d}z^r}g(z)=\frac{r!}{(z-x)^{r+1}}$. It follows that $\frac{\mathrm{d}^r}{\mathrm{d}z^r}g(z)=rg(z)\left\{\frac{\mathrm{d}^{r-1}}{\mathrm{d}z^{r-1}}g(z)\right\}$. Consequently, $N=kg(z)\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\{h(z)g(z)\}-P$ where after some algebraic work $P=M-g(z)\frac{\mathrm{d}^k}{\mathrm{d}z^k}h(z)$.

Therefore,

$$M+N=g(z)\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}h(z)+k\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\{h(z)g(z)\}\right].$$

This completes the proof.

Another tool for proving our main theorem is the following formula taken from the Schwartz Distribution theory, namely,

$$\int_{-\infty}^{\infty}\varphi(x)\Lambda^{[n]}(\mathrm{d}x)=\frac{-1^n}{n!}\int_{-\infty}^{\infty}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\varphi(x)\Lambda(x) \tag{2.2}$$

where Λ is a distribution Function and $\Lambda^{[n]}$ is the *n*-th distributional derivative of Λ.

The conditional distribution $F_{Z_1|x_1,x_2}(z)$ given by (1.1) leads us to a linear functional on complex Valued functions *f*: $\mathbb{R}\to\mathbb{C}$, defined on the set of real numbers $\mathbb{R}$:

$$F_{Z_1|x_1,x_2}(f) =\frac{f(x_1)}{(x_2-x_1)^n}-\sum_{i=1}^{n}\frac{1}{(n-i)!(x_2-x_1)^i}\frac{\mathrm{d}^{n-i}}{\mathrm{d}z^{n-i}}f(x_2).$$

It easily follows that

$$F_{Z_1|x_1,x_2}(af+bg)=aF_{Z_1|x_1,x_2}(f)+bF_{Z_1|x_1,x_2}(g). \tag{2.3}$$

For any complex-valued functions *f*, *g* and complex constants *a*, *b*. We note that $F_{Z_1|x_1,x_2}(z)=F_{Z_1|x_1,x_2}(f_z)$,

Whenever $f_z(x)=(z-x)^nU(z-x)$ and

$$F_{Z_1|x_1,x_2}(f) =\frac{f_z(x_1)}{(x_2-x_1)^n}-\sum_{i=1}^{n}\frac{1}{(n-i)!(x_2-x_1)^i}\frac{\mathrm{d}^{n-i}}{\mathrm{d}z^{n-i}}f_z(x_2).$$

Also we note that $U(z-x)=\frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}f_z(x)$. Thus

$$P(Z_1\le z)=\int_{\mathbb{R}}U(z-x)\mathrm{d}F_{Z_1}(x)=\int_{\mathbb{R}^2}F_{Z_1|x_1,x_2}(z)\prod_{i=1}^{2}F_{X_i}(\mathrm{d}x_i),$$

can be viewed as:

$$\int_{\mathbb{R}}\frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}f_z(x)\mathrm{d}F_{Z_1}(x)=\int_{\mathbb{R}^2}F_{Z_1|x_1,x_2}(f_z)\prod_{i=1}^{2}F_{X_i}(\mathrm{d}x_i). \tag{2.4}$$

Therefore by using (2.3) along with (2.4) and a standard argument in the integration theory, we obtain that

$$\int_{\mathbb{R}}\frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)\mathrm{d}F_{Z_1}(x)=\int_{\mathbb{R}^2}F_{Z_1|x_1,x_2}(f)\prod_{i=1}^{2}F_{X_i}(\mathrm{d}x_i). \tag{2.5}$$

For any infinitely differentiable functions f for which the corresponding integrals are finite. Now (2.5) together with (2.2) lead us to

$$\int_{\mathbb{R}}f(x)\mathrm{d}F_{Z_1}^{(n)}(x)=\int_{\mathbb{R}^2}F_{Z_1|x_1,x_2}(f)\prod_{i=1}^{2}F_{X_i}(\mathrm{d}x_i). \tag{2.6}$$

For the above mentioned functions *f*, where $F_{Z_1}^{(n)}$ is the (n)-th distributional derivative of the distribution of Z_1.

Let us denote the Stieltjes transform of a distribution *H* by

$$S(H,z)=\int_{\mathbb{R}}\frac{1}{(z-x)}H(\mathrm{d}x).$$

For every *z* in the set of complex numbers $\mathbb{C}$ which does not belong to the support of *H*, *i.e.*, $z\in\mathbb{C}(\mathrm{suppH})^c$.

The following lemma indicates how the Stieljes transform of Z_1 and X_1, X_2 are related.

Lemma 2.2. Let Z_1 be a random variables that satisfies (1.1). Suppose that the random variables X_1 and X_2 are independent and continuous with distribution functions F_{X_1} and F_{X_2} respectively. Then

$$\frac{1}{n}S^{(n)}(F_{Z_1},z)=-S(F_{X_1},z)S^{(n-1)}(F_{X_2},z),\quad z\in\mathbb{C}(\mathrm{suppH})^c.$$

Proof. It follows from (2.6) that

$$S\left(F_{Z_1}^{(n)},z\right)=\int_{\mathbb{R}^2}F_{Z_1|x_1,x_2}(g_z)\prod_{i=1}^{2}F_{X_i}(\mathrm{d}x_i).$$

And

$$\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}S(F_{Z_1},z)=\int_{\mathbb{R}^2}F_{Z_1|x_1,x_2}(g_z)\prod_{i=1}^{2}F_{X_i}(\mathrm{d}x_i)$$

for $g_z(x)=\frac{1}{z-x}$. Now, it follows that

$$F_{Z_1|x_1,x_2}(g_z)=\frac{\frac{1}{z-x_1}}{(x_2-x_1)^n}-\sum_{i=1}^{n}\frac{1}{(n-i)!(x_2-x_1)^i}\frac{\mathrm{d}^{n-i}}{\mathrm{d}z^{n-i}}\frac{1}{z-x_2}.$$

And by using Lemma 2.1, we have

$$F_{Z_1|x_1,x_2}(g_z)=\frac{(-1)^n}{(z-x_1)(z-x_2)^n}.$$

Therefore,

$$\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}S(F_{Z_1},z)=\int_{\mathbb{R}^2}\frac{(-1)^n}{(z-x_1)(z-x_2)^n}\prod_{i=1}^{2}F_{X_i}(\mathrm{d}x_i),$$

and

$$\frac{1}{n}S^{(n)}(F_{Z_1},z)=-S(F_{X_1},z)S^{(n-1)}(F_{X_2},z),\quad z\in\mathbb{C}(\mathrm{suppH})^c. \tag{2.7}$$

This finishes the proof.

Note that Van Asch's lemma is the case of $n = 1$:

$$-S'(F_{Z_1},z)=S(F_{X_1},z)S(F_{X_2},z).$$

We also note that the Stieltjes transform of Cauchy distribution, *i.e.*, $S(F,z)=\frac{1}{z+c}$ satisfies (2.7).

3. Characterizations

Now, we apply Lemma 2.2 for some characterizations, when X_1 and X_2 are not identically distributed.

Theorem 3.1. Let X_1 and X_2 be independent random variables and *Z* be a randomly weighted average given in (1.1).

For $n = 2$ we have,

a) if X_1 has uniform distribution on [−1, 1], then Z_1 has semicircle distribution on [−1, 1] if and only if X_2 has Arcsin distribution on [−1, 1];

b) if X_1 has uniform distribution on [−1, 1], then Z_1 has power semicircle distribution on [−1, 1] if and only if X_2 has power semicircle distribution *i.e.*,

$$f(z)=\frac{3(1-z^2)}{4},\quad -1\le z\le 1.$$

c) if X_1 has Beta (1,1) distribution on [0, 1], then Z_1 has Beta $\left(\frac{3}{2},\frac{3}{2}\right)$ distribution if and only if X_2 has Beta

$\left(\frac{1}{2},\frac{1}{2}\right)$ distribution;

d) if X_1 has uniform distribution on [0, 1], then Z_1 has Beta (2, 2) distribution if and only if X_2 has Beta (2, 2) distribution.

Proof. 1) For the "if" part we note that the random variable X_1 has uniform distribution and X_2 has Arcsin distribution on [−1, 1]; then

$$S\left(F_{X_1},z\right)=\frac{1}{2}\left(\ln|z+1|-\ln|z-1|\right).$$

And $S\left(F_{X_2},z\right)=\frac{1}{\sqrt{z^2-1}}$.

From Lemma 2.2 and substituting the corresponding Stieltjes transforms of distributions, we get

$$S''\left(F_{Z_1},z\right)=\frac{2}{\left(z^2-1\right)^{\frac{3}{2}}}.$$

The solution $S\left(F_{Z_1},z\right)=2\left(z-\sqrt{z^2-1}\right)$.

Which is the Stieltjes transform of the semicircle distribution on [−1, 1].

For the "only if" part we assume that the random variable Z_1 has semicircle distribution. Then it follows from Lemma 2.2 that

$$S\left(F_{X_2},z\right)\frac{1}{1-z^2}=\frac{-1}{\left(z^2-1\right)^{\frac{3}{2}}}.$$

The proof is completed.

2) By an argument similar to that given in 1) and solving the following differential equations,

$$S''\left(F_{Z_1},z\right)=\frac{-2}{z(z-1)}\left(6\left(z^2-z\right)\left(\ln|z|-\ln|z-1|\right)-6z+3\right).$$

(for the "if" part), and

$$\mathrm{S}\left(F_{X_2},z\right)=6\left(z^2-z\right)\left(\ln|z|-\ln|z-1|\right)+6z-3.$$

(for the "only if" part).

The proof can be completed.

3) By Lemma (2.2), we have

$$-\frac{1}{2}S''\left(F_Z,z\right)=\frac{-1}{z(z-1)\sqrt{z(z-1)}},$$

(for the"if" part), and

$$-\frac{1}{z(z-1)}S\left(F_{X_2},z\right)=\frac{-1}{z(z-1)\sqrt{z(z-1)}},$$

(for the "only if " part).

The proof can be completed by solving the above differential equations.

4) By Lemma (2.2), we have

$$S''\left(F_{Z_1},z\right)=\frac{-2}{z(z-1)}\left(6\left(z^2-z\right)\left(\ln|z|-\ln|z-1|\right)-6z+3\right)$$

(for the "if" part), and

$$\mathrm{S}\left(F_{X_2},z\right)=6\left(z^2-z\right)\left(\ln|z|-\ln|z-1|\right)+6z-3$$

(for the "only if " part).

Solving the differential equations, can complete the proof.

4. TSP Random Variables

In Section 3, we used a powerful method, based on the use of Stieltjes transforms, to obtain the distribution of z_1 given in (1.1). It seems that one can not use that method to find distribution of z_2 given in (1.2). So we employ a direct method to find the distribution of z_2. Let us follow Lemma 4.1 to find a simple method to get the distribution of z_2 following [2] and the work of them leads us to the following lemma.

Lemma 4.1. Suppose W has a power distribution with parameter n, $n\geq 1$, n is an integer, and let $y_1=\min\left(X_1,X_2\right)$, $y_2=\max\left(X_1,X_2\right)$ where $X_1,\cdots,X_n$ independent random variables are. Let

$$X=Y_1+W\left(Y_2-Y_1\right).$$

Then

1) X is a TSP random variable.

2) X can be equivalently defined by

$$X=\frac{1}{2}\left(X_1+X_2\right)+\left(W-\frac{1}{2}\right)\left|X_1-X_2\right|.$$

Proof. 1)

$$F_{X|x_1,x_2}(z)=\left(PY_1+W\left(Y_2-Y_1\right)\leq z\middle|X_1=x_1,X_2=x_2\right)$$
$$=P\left(y_1+W\left(y_2-y_1\right)\leq z\right)=\left(\frac{z-y_1}{y_2-y_1}\right)^n.$$

Proof. 2)

$$X\middle|X_1=x_1,\ X_2=x_2\sim U\left[\min\left(x_1,x_2\right),\max\left(x_1,x_2\right)\right],$$

$$W=\frac{X-\min\left(x_1,x_2\right)}{\max\left(x_1,x_2\right)-\min\left(x_1,x_2\right)}\sim U[0,1],$$

and also

$$\min\left(x_1,x_2\right)=\frac{x_1+x_2-\left|x_1-x_2\right|}{2},$$

$$\max\left(x_1,x_2\right)=\frac{x_1+x_2+\left|x_1-x_2\right|}{2}.$$

then

$$W = \frac{X - \frac{x_1 + x_2}{2} - \frac{|x_1 - x_2|}{2}}{|x_1 - x_2|},$$

so

$$X = \frac{1}{2}(x_1 + x_2) + \left(W - \frac{1}{2}\right)|x_1 - x_2|.$$

4.1. Moments of TSP Random Variables

The following theorem provides equivalent conditions For $\mu'_X = Ez_2^k$.

Theorem 4.1.1. Suppose that z_2 is a TSP random variable satisfying (1.2). If X_1 and X_2 are random variables and $E|x_i|^k = \infty$, $i = 1,2$ for all integers k then

1) $Ez_2^k = n\frac{\Gamma(k+1)}{\Gamma(n+k+1)}\sum_{i=0}^{k}\frac{\Gamma(k+i-n)}{\Gamma(k-i+1)}E\left(y_1^i y_2^{k-i}\right)$.

2) $Ez_2^k = \binom{k}{i}\left(\frac{1}{2}\right)^{k-i}E\left(W - \frac{1}{2}\right)^i E\left(X_1 + X_2\right)^{k-i}\left|X_1 + X_2\right|^i$.

3) $Ez_2^k = \binom{k}{i}\left(\frac{n}{n+i}\right)E\left(y_1^{k-i}\left(y_2 - y_1\right)^i\right)$.

Proof. 1) By using Lemma 2.1, we obtain that

$$\begin{aligned}
&= \sum_{i=0}^{k}\binom{k}{i}E\left(W^{k-i}(1-W)^i\right)E\left(Y_2^{k-i}Y_1^i\right)\\
&= \sum_{i=0}^{k}\binom{k}{i}E\left(W^{k-i}(1-W)^i\right)E\left(Y_2^{k-i}Y_1^i\right)\\
&= n\frac{\Gamma(k+1)}{\Gamma(n+k+1)}\sum_{i=0}^{k}\frac{\Gamma(k-i+n)}{\Gamma(k-i+1)}E\left(y_1^i y_2^{k-i}\right).
\end{aligned}$$

Proof. 2) This can be easily proved by Lemma 4.1 2).

Proof. 3)

$$\begin{aligned}
Ez_2^k &= \left(Y_1 + W\left(Y_2 - Y_1\right)\right)^k\\
&= E\sum_{i=0}^{k}\binom{k}{i}\left(W^i y_1^{k-i}\left(y_2 - y_1\right)^i\right)\\
&= \sum_{i=0}^{k}\binom{k}{i}E(W)^i E\left(y_1^{k-i}\left(y_2 - y_1\right)^i\right)
\end{aligned}$$

$$Ez_2^k = \binom{k}{i}\left(\frac{n}{n+i}\right)E\left(y_1^{k-i}\left(y_2 - y_1\right)^i\right).$$

Let us consider expectation and variance of z_2. First, we suppose that $EY_1 = \mu_1$, $EY_2 = \mu_2$, $\mathrm{Var}Y_1 = \sigma_1^2$, $\mathrm{Var}Y_2 = \sigma_2^2$, and $\mathrm{Cov}(Y_1, Y_2) = \sigma_{12}$. Then

$$EZ_2 = \frac{\mu_1 + n\mu_2}{n+1},$$

and also, if $EX_1 = EX_2 = 0$ then

$$E(Z_2) = EY_1 + \frac{n}{n+1}\left(EY_2 - EY_1\right).$$

By $X_1 + X_2 = Y_1 + Y_2$. We have

$$E(Z_2) = E(Y_1) + \frac{n}{n+1}(-2EY_1) = \frac{1-n}{1+n}EY_1. \quad (4.2)$$

It can easily follow from (4.2) that the Arcsin result of Van Asch [1] is only true for $n = 1$, about the variance, we have

$$\mathrm{Var}Z_2 = \frac{n(\mu_1 - \mu_2)^2 + n(n+1)^2\sigma_2^2 + 2(n+1)\left(\sigma_1^2 + n\sigma_{12}\right)}{(n+1)^2(n+2)}.$$

Following the computation of expectation and variance, we evaluate them for some well-known distributions. If X_1 and X_2 have standard normal distributions, then from Theorem 3.1.1 2) and the fact that $X_1 - X_2$ and $X_1 + X_2$ are independent, it follows that their first, second and third order moments are equal, respectively, to

$$EZ_2 = \frac{1}{\sqrt{\pi}}\left(\frac{n-1}{n+1}\right),$$

$$EZ_2^2 = \left(\frac{n^2 + n + 2}{(n+1)(n+2)}\right),$$

and

$$EZ_2^3 = \frac{1}{2\sqrt{\pi}}\left(\frac{5n^3 + 12n^2 + 13n - 30}{(n+3)(n+1)(n+2)}\right).$$

Also, in case X_1 and X_2 have uniform distributions, Theorem 4.1.1 2) implies that,

$$Ez_2^k = n\frac{\Gamma(k+1)}{\Gamma(n+k+1)}\sum_{i=0}^{k}\frac{\Gamma(k-i+n)}{\Gamma(k-i+1)}\frac{2}{(k+2)(i+1)}.$$

$$EZ_2 = \frac{2n+1}{3(n+1)},$$

and

$$\mathrm{Var}Z_2 = \frac{1}{18}\frac{n^3 + 3n^2 + 6n + 2}{(n+1)^2(n+1)}.$$

Theorem 4.1.2. Suppose that z_2 is a TSP random variable satisfying (4.1), then

1) z_2 is location invariant;

2) If X_1 and X_2 have symmetric distribution around μ, then z_2 has symmetric distribution around μ, only when $n = 1$.

Proof.

1) Is immediate.

2) We can assume without loss of generality that $\mu = 0$ If Z_2 has a symmetric distribution around zero, then

$$Y_1 + W(Y_2 - Y_1) \overset{d}{=} -[Y_1 + W(Y_2 - Y_1)].$$

We note that

$$Y_1 + W(Y_2 - Y_1) \overset{d}{=} [-Y_1 + W(-Y_2 - (-Y_1))].$$

Since $-\min(X_1, X_2) = \max(-X_1, -X_2)$, $X_1 \overset{d}{=} -X_1$, And $X_2 \overset{d}{=} -X_2$, we have

$$Y_1 + W(Y_2 - Y_1) \overset{d}{=} Y_2 + W(Y_1 - Y_2). \tag{4.3}$$

By equating the conditional distributions given at $X_1 = x_1$ and $X_2 = x_2$ in (3.3), we conclude that $n = 1$ It can also easily follow from Theorem (4.1.1) that the Cauchy result of Van Asch [1] is true only for $n = 1$.

4.2. Distributions of TSP Random Variables

In this subsection, we investigate computing distributions by the direct method. We will give two examples of derivation based on (4.1). This method may be complicated in some cases, but we have chosen some easy to find examples. We use randomly weighted average on order statistics to find the distribution of z_2. Gauss hyper geometric function $F(a,b,c;z)$ which is a well-known special function that we used in this way.

Example 4.2.1. Let X_1, X_2 and W be independent random variables such that X_1 and X_2 are uniformly distributed over [0, 1], and W has a power function distribution with parameter. We find the value $f_{Z_2}(z;n)$ by means of $f_{Z_2|W(z|w)}$; therefore

$$f_{Z_2|W(z|w)} = \begin{cases} \dfrac{2z}{w} & 0 < z < w, \\ \dfrac{2(1-z)}{1-w} & w < z < 1. \end{cases} \tag{4.4}$$

By using the distribution of W, the density function $f_{Z_2}(z;n)$, can be expressed in terms of the Gauss hyper geometric function $F(a,b,c;z)$ which is a well-known special function. Indeed according to Euler's formula, the Gauss hyper geometric function assumes the integral representation

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t,$$

where a, b, c are parameters subject to $-\infty < a < \infty$, $c > b > 0$, whenever they are real and z is the variable.

$$f_{Z_2}(z;n) = \frac{2nz}{n-1}\left(1-z^{n-1}\right) + 2(1-z)z^n F(1,n,n+1,z), \tag{4.5}$$

where $n > 0$ and $n \neq 1$. There are some important functions as a Gauss hyper geometric function.

$$\log(1+z) = zF(1,1;2;-Z).$$

$$\mathrm{e}^z = \lim_{b\to\infty} F\left(a,b;b;\frac{z}{a}\right).$$

$$(1-z)^{-a} = F(a,1;1;z).$$

When $n = 1$ similar calculations lead to the following distribution

$$f_{Z_2}(z) = -2(1-z)\log(1-z) - 2z\log(z),\ 0 < z < 1.$$

When n is an integer, we obtain the following distribution.

$$f^*_{Z_2}(z,n) = 2\frac{n}{n-1}z\left(1-z^{n-1}\right) - 2nz(1-z)\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{(-1)^i}{i}\left((1-z)^i - 1\right),$$

$0 < z < 1$.

The probability density function $f_{Z_2}(z)$ was introduced by Johnson and Kotz [3], for the first time, under the title "uniformly randomly modified tine". So $f_{Z_2}(z;n)$ can be seen as an extension of the above mentioned distribution. We note that, from (4.1) and a simple Monte Carlo procedure using only simulated uniform variables, one can to simulate the distribution (4.5).

Theorem 4.2.2. Let z_2 be a undirected triangular random variable that satisfies (1.2). Suppose that the random variables X_1 and X_2 are independent and continuous with the distribution Functions F_{X_1} and F_{X_2}, respectively. Then

$$-\frac{1}{2}S'''(F_Z, z) = S'(F_{X_1}, z)S'(F_{X_2}, z) + 2S(F_{X_1}, F_{X_2}, z),$$

where

$$S(F_{X_1}, F_{X_2}, z) = \int_{\mathbb{R}^2} \frac{1}{(z-x_1)(z-x_2)(x_2-x_1)^2}\prod_{i=1}^{2} F_{X_i}(\mathrm{d}x_i).$$

Proof. By using an argument similar to that given in section 3, we can conclude that

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}F^{(2)}_{Z_2}(x) = \int_{\mathbb{R}^2} F_{Z_2|x_1,x_2}(f)\prod_{i=1}^{2} F_{X_i}(\mathrm{d}x_i).$$

So,

$$-\frac{1}{2}S'''(F_{Z_2}, z) = \int_{\mathbb{R}^2} F_{Z_2|x_1,x_2}(g_z)\prod_{i=1}^{2} F_{X_i}(\mathrm{d}x_i),$$

For $g_z(x) = \dfrac{1}{(z-x)^2}$. From

$$F_{Z_2|x_1,x_2}(g_z)=\frac{\frac{1}{(z-x_1)^2}}{(x_2-x_1)^2}+\frac{\frac{1}{(z-x_2)^2}}{(x_1-x_2)^2}.$$

And by using partial fractional rule, we have

$$F_{Z_2|x_1,x_2}(g_z)$$
$$=\frac{1}{(z-x_1)^2(z-x_2)^2}+\frac{2}{(x_2-x_1)^2}\frac{1}{(z-x_1)(z-x_2)}.$$

Therefore,

$$-\frac{1}{2}S'''(F_{Z_2},z)=\int_{\mathbb{R}^2}\frac{1}{(z-x_1)^2(z-x_2)^2}$$
$$+\frac{2}{(x_2-x_1)^2}\frac{1}{(z-x_1)(z-x_2)}\prod_{i=1}^{2}F_{X_i}(\mathrm{d}x_i).$$

And

$$-\frac{1}{2}S'''(F_Z,z)=S'(F_{X_1},z)S'(F_{X_2},z)+2S(F_{X_1},F_{X_2},z).$$

This finishes the proof.

It is worth mentioning that the present method yields other extensions too; the following is such an example.

Example 4.2.3. Suppose that X_1, X_2 W are independent random variables. If X_1 and X_2 have Uniform distributions on [0, 1] and W has Beta (2, 2) distribution, then z_2 has the same distribution as W.

If the product moments of order statistics are known, those of W can be derived from that of z_2. By using Theorem 4.1.1 1). Then the distribution of W is characterized by that of z_2.

By an argument similar to the one given in example 4.2.1, when W has a Beta distribution with Parameters n and m, we find the distribution $f_{Z_2}(z;n,m)$ as

$$\frac{B(n-1,m)}{B(n,m)}2z\left(1-I_z(n-1,m)\right)$$
$$+\frac{B(n,m-1)}{B(n,m)}2(1-z)I_z(n,m-1),$$
$$0<z<1$$

where $I_z(a,b)$ is incomplete Beta function:

$$I_x(a,b)=\frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t,\ (a,b>0).$$

5. Conclusion

We have described how directed methods could be used for obtaining the distributions, Characterizations and properties of the random mixture of variables defined in (1.1). The TSP random variable when X_1 and X_2 have uniform distributions, led us to a new family of distribution which can be regarded as some generalization of "uniformly randomly modified tine". The proposed model in the direct method can easily lead to distribution generalizations, though this is not possible for the Stieltjes method, but here the characteristics can be easily computed.

6. Acknowledgements

The author is deeply grateful to the anonymous referee for reading the original manuscript very carefully and for making valuable suggestions.

REFERENCES

[1] W. Van Asch, "A Random Variable Uniformly Distributed between Two Independent Random Variables," *Sankhaya*, Vol. 49, No. 2, 1987, pp. 207-211. doi:10.1080/00031305.1990.10475730

[2] A. R. Soltani and H. Homei, "Weighted Averages with Random Proportions That Are Jointly Uniformly Distributed over the Unit Simplex," *Statistics & Probability Letters*, Vol. 79, No. 9, 2009, pp. 1215-1218. doi:10.1016/j.spl.2009.01.009

[3] N. L. Johnson and S. Kotz, "Randomly Weighted Averages," *The American Statistician*, Vol. 44, No. 3, 1990, pp. 245-249. doi:10.2307/2685351

[4] A. I. Zayed, "Handbook of Function and Generalized Function Transformations," CRC Press, London, 1996.

[5] A. R. Soltani and H. Homei, "A Generalization for Two-Sided Power Distributions and Adjusted Method of Moments," *Statistics*, Vol. 43, No. 6, 2009, pp. 611-620. doi:10.1080/02331880802689506

