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Some Properties of a Kind of Singular Integral Operator with Weierstrass Function Kernel

Lixia Cao
Department of Information and Computing Sciences, Mathematics College, Northeast Petroleum University, Daqing, China
Email: caolixia98237@163.com

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ABSTRACT

We considered a kind of singular integral operator with Weierstrass function kernel on a simple closed smooth curve in a fundamental period parallelogram. Using the method of complex functions, we established the Bertrand Poincaré formula for changing order of the corresponding integration, and some important properties for this kind of singular integral operator.

Keywords: Weierstrass Function Kernel; Singular Integral Operator; Bertrand Poincaré Formula; Properties

1. Introduction

The properties of singular integral operator with Cauchy or Hilbert kernel on simple closed smooth curve or open arc have been elaborately discussed in [1-3]. Based on these, for the boundary curve is a closed curve or an open arc, the authors discussed the singular integral operators and corresponding equation with Cauchy kernel or Hilbert kernel in [1-3]. In recent years, many authors discussed the numerical solution of a class of systems of Cauchy singular integral equations with constant coefficients, Numerical methods for nonlinear singular Volterra integral equations in [4-6].

In this paper, we consider a kind of singular integral operator with Weierstrass function kernel on a simple closed smooth curve in a fundamental period parallelogram. Our goal is to develop the Bertrand poincaré formula for changing order of the corresponding integration, and some important properties of the above singular integral operator.

2. Preliminaries

Definition 1 Suppose that ω_1, ω_2 are complex constants with $\operatorname{Im}(\omega_1/\omega_2) \neq 0$, and **P** denotes the fundamental period parallelogram with vertices $\pm\omega_1 \pm \omega_2$. Then the function

$$\zeta(z) = 1/z + \sum_{m,n}{}' \left[1/(z - \Omega_{mn}) + 1/\Omega_{mn} + z/\Omega_{mn}^2 \right]$$

is called the Weierstrass ζ-function, where

$$\Omega_{mn} = 2m\omega_1 + 2n\omega_2$$

$\sum_{m,n}'$ denotes the sum of all $m, n = 0, \pm 1, \pm 2, \cdots$, except for $m = n = 0$.

Definition 2 Suppose that L_0 is a smooth closed curve in the counterclockwise direction, lying entirely in the fundamental period parallelogram **P**, with $z_0 (\neq 0)$ and the origin lying in the domain S_0^+ enclosed by L_0. The following operator

$$K\varphi \equiv a(t_0)\varphi(t_0) + \frac{1}{\pi i}\int_{L_0} \varphi(t) K(t_0, t)\left[\zeta(t - t_0) + \zeta(t_0 - z_0)\right] \mathrm{d}t, t_0 \in L_0 \quad (1)$$

is called the singular integral operator with ζ-function kernel on L_0, where $\varphi(t) \in H(L_0)$ is the unknown function, and

$$K(t_0, t) \in H(L_0 \times L_0),\ a(t) \in H(L_0)$$

are the given functions.

Letting $b(t) = K(t,t)$, then (1) becomes

$$K\varphi \equiv a(t_0)\varphi(t_0) + \frac{b(t_0)}{\pi i}\int_{L_0} \varphi(t)\left[\zeta(t - t_0) + \zeta(t_0 - z_0)\right] \mathrm{d}t + \frac{1}{\pi i}\int_{L_0} \left[K(t_0, t) - K(t_0, t_0)\right]\left[\zeta(t - t_0) + \zeta(t_0 - z_0)\right]\varphi(t)\, \mathrm{d}t \quad (2)$$

Since $\zeta(t)$ is uniformly convergent in any closed bounded region lying entirely in **P**,

$$\left|\zeta(t-t_0)+\zeta(t_0-z_0)\right|\le 1/\left|t-t_0\right|+M$$

for any $t_0,t\in L_0$, where M is some positive finite constant. By noting that $K(t_0,t)\in H^{\alpha}(0<\alpha\le 1)$, we obtain

$$\left|\left[K(t_0,t)-K(t_0,t_0)\right]\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\right|\le N/\left|t-t_0\right|^{\lambda}$$

$(0<\lambda\le 1)$, where N is some positive finite constant. Write

$$K^0\varphi\equiv a(t_0)\varphi(t_0)+\frac{b(t_0)}{\pi i}\int_{L_0}\varphi(t)\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\mathrm{d}t,$$

$$k(t_0,t)=\frac{1}{\pi i}\left[K(t_0,t)-K(t_0,t_0)\right]\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]$$

$$k\varphi\equiv\int_{L_0}k(t_0,t)\varphi(t)\mathrm{d}t$$

then (1) can be rewritten in the form

$$\left(K^0+k\right)\varphi, \tag{3}$$

where k is a Fredholm operator and K^0 is called the characteristic operator of K. Now the index of K is defined as $\kappa=\frac{1}{2\pi}\left[\arg\frac{D(t)}{S(t)}\right]_{L_0}$, where

$$S(t)=a(t)+b(t),\ D(t)=a(t)-b(t)$$

and for definiteness we assume that $a^2(t)-b^2(t)\ne 0$, namely we assume that K is an operator of normal type.

Now the associated operator of (1) takes the form

$$K'\psi\equiv a(t_0)\psi(t_0)-\frac{1}{\pi i}\int_{L_0}K(t,t_0)\psi t\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\mathrm{d}t,\ t_0\in L, \tag{4}$$

or

$$K'\psi\equiv a(t_0)\psi(t_0)-\frac{1}{\pi i}\int_{L_0}b(t)\psi(t)\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\mathrm{d}t+\int_{L_0}k(t,t_0)\psi(t)\mathrm{d}t,\quad t_0\in L, \tag{4)'$$

and so that the associated operator of K^0 becomes

$$K^{0\prime}\psi\equiv a(t_0)\psi(t_0)-\frac{1}{\pi i}\int_{L_0}b(t)\psi(t)\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\mathrm{d}t,\ t_0\in L.$$

In addition, if we write

$$k_1\psi(t_0)=\int_{L_0}\left\{k(t,t_0)-\frac{1}{\pi i}\left[b(t)-b(t_0)\right]\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\right\}\psi(t)\mathrm{d}t,\ t_0\in L,$$

then (4) can be rewritten as

$$K'\psi\equiv a(t_0)\psi(t_0)-\frac{b(t_0)}{\pi i}\int_{L_0}\psi(t)\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\mathrm{d}t+k_1\psi(t),t_0\in L, \tag{5}$$

where

$$\left|\left[b(t)-b(t_0)\right]\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\right|\le D/\left|t-t_0\right|^{\lambda}\quad (0\le\lambda<1,\ D \text{ is some finite constant}).$$

So $k_1\psi$ is a Fredholm operator, and then the characteristic operator of K' operator becomes

$$K'^0\psi\equiv a(t_0)\psi(t_0)\ -\frac{b(t_0)}{\pi i}\int_{L_0}\psi(t)\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\mathrm{d}t \tag{6}$$

Therefore, we concluded that $K'^0\psi=K^{0\prime}\psi$ usually can not be established, that is $K'^0\psi\ne K^{0\prime}\psi$.

For convenience, we write

$$\mathrm{X}(z,\sigma)=\zeta(z-\sigma)+\zeta(\sigma-z_0)$$

where the fixed nonzero point z_0 and the origin lie in S_0^+. It is not difficult to get the following results.

Lemma 1 Suppose that $f(t,\tau)\in H(L_0\times L_0)$, and with the same L_0 as mentioned before, then

a) $\int_{L_0}\mathrm{d}t\int_{L_0}f(t,\tau)\mathrm{X}(\tau,t)\mathrm{d}\tau=\int_{L_0}\mathrm{d}\tau\int_{L_0}f(t,\tau)\mathrm{X}(\tau,t)\mathrm{d}t,$

$$\int_{L_0}\mathrm{X}(\tau,t)\mathrm{d}t\int_{L_0}f(t,\tau)\mathrm{d}\tau=\int_{L_0}\mathrm{d}\tau\int_{L_0}f(t,\tau)\mathrm{X}(\tau,t)\mathrm{d}t$$

b) (Poincare-Bertrand formula)

$$\int_{L_0} \mathrm{X}(t,t_0)\mathrm{d}t\int_{L_0} f(t,\tau)\mathrm{X}(\tau,t)\mathrm{d}\tau = -\pi^2 f(t_0,t_0)$$

3. Some Properties of Operator *K*

1) If $\varphi \in H$, then $K\varphi \in H$.

Proof Through calculation and estimation, we have

$$\left|\int_{L_0} K(t_1,t)\varphi(t)\zeta(t_1)\mathrm{d}t - \int_{L_0} K(t_2,t)\varphi(t)\zeta(t_2)\mathrm{d}t\right| \le M\left|t_1-t_2\right|^{\alpha} + N\left|\zeta(t_1)-\zeta(t_2)\right| \tag{7}$$

for any $t_1,t_2 \in L_0$, where M and N are all finite constant. While for any $t_1,t_2 \in L_0$, we have

$$\left|\zeta(t_1)-\zeta(t_2)\right| \le \left|\frac{t_1-t_2}{t_1t_2}\right| + \sum_{m,n}{}'\left|\frac{t_1-t_2}{(t_1-\Omega_{mn})(t_2-\Omega_{mn})}\right| + \sum_{m,n}{}'\left|\frac{t_1-t_2}{\Omega_{mn}^2}\right| \le Q\left|t_1-t_2\right| \tag{8}$$

where Q is some finite constant. Substituting (8) into (7), we obtain

$$\int_{L_0} K(t_0,t)\varphi(t)\zeta(t_0)\mathrm{d}t \in H \tag{9}$$

Similarly we know that

$$\int_{L_0} K(t_0,t)\varphi(t)\zeta(t-t_0)\mathrm{d}t,\ \int_{L_0} K(t_0,t)\varphi(t)\zeta(t_0-z_0)\mathrm{d}t,\ a(t_0)\varphi(t_0) \in H$$

Consequently, we have $K\varphi \in H$.

2) If K_1,K_2 are singular integral operator, then K_1K_2 is also a singular integral operator. That is, if

$$K_j\varphi \equiv a_j(t_0)\varphi(t_0) + \frac{1}{\pi i}\int_{L_0} \varphi(t)K_j(t_0,t)\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\mathrm{d}t,\ j=1,2$$

then

$$\begin{aligned} K_1K_2\varphi &= \left[a_1(t_0)a_2(t_0)+b_1(t_0)b_2(t_0)\right]\varphi(t_0) \\ &+\int_{L_0} \frac{\left[a_1(t_0)K_2(t_0,t)+a_2(t)K_1(t_0,t)\right]}{\pi i}\varphi(t)\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\mathrm{d}t \\ &+\frac{1}{(\pi i)^2}\int_{L_0}\left\{\int_{L_0} K_1(t_0,t_1)K_2(t_1,t)\left[\zeta(t_1-t_0)+\zeta(t_0-z_0)\right]\left[\zeta(t-t_1)+\zeta(t_1-z_0)\right]\mathrm{d}t_1\right\}\varphi(t)\mathrm{d}t \end{aligned}, \tag{10}$$

where the sum of the former two terms in the right hand of Equation (10) are the characteristic operator, and the remainder in that is a Fredholm operator.

Proof By definition, we deduce that

$$\begin{aligned} K_1K_2\varphi &= a_1(t_0)a_2(t_0)\varphi(t_0) + \frac{1}{\pi i}\int_{L_0} a_1(t_0)K_2(t_0,t)\varphi(t)\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\mathrm{d}t \\ &+\frac{1}{\pi i}\int_{L_0} a_2(t)K_1(t_0,t)\varphi(t)\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\mathrm{d}t + C(t_0) \end{aligned}$$

where

$$C(t_0) = \frac{1}{(\pi i)^2}\int_{L_0} K_1(t_0,t)\left[\zeta(t-t_0)+\zeta(t_0-z_0)\right]\left\{\int_{L_0} K_2(t,t_1)\left[\zeta(t_1-t)+\zeta(t-z_0)\right]\varphi(t_1)\mathrm{d}t_1\right\}\mathrm{d}t$$

By virtue of Lemma 1 (b), $C(t_0)$ can be rewritten in the form

$$C(t_0) = b_1(t_0)b_2(t_0)\varphi(t_0) + \frac{1}{(\pi i)^2}\int_{L_0}\left\{\int_{L_0} K_1(t_0,t_1)K_2(t_1,t)\left[\zeta(t_1-t_0)+\zeta(t_0-z_0)\right]\left[\zeta(t-t_1)+\zeta(t_1-z_0)\right]\mathrm{d}t_1\right\}\varphi(t)\mathrm{d}t.$$

Consequently, (10) is established.

Now we write

$$\int_{L_0} K_1(t_0,t_1)K_2(t_1,t)\zeta(t_1-t_0)\zeta(t-t_1)\mathrm{d}t_1 = \langle 1\rangle + \langle 2\rangle + \langle 3\rangle + \langle 4\rangle$$

where

$$\langle 1\rangle = \int_{L_0} \frac{K_1(t_0,t_1)K_2(t_1,t)}{(t_1-t_0)(t-t_1)}\mathrm{d}t_1,\ \langle 2\rangle = \int_{L_0} \frac{K_1(t_0,t_1)K_2(t_1,t)}{t_1-t_0}\sum_{m,n}{}'\left[\frac{1}{t-t_1-\Omega_{mn}}+\frac{1}{\Omega_{mn}}+\frac{t-t_1}{\Omega_{mn}^2}\right]\mathrm{d}t_1,$$

$$\langle 3\rangle = \int_{L_0} \frac{K_1(t_0,t_1)K_2(t_1,t)}{t-t_1}\sum_{m,n}{}'\left[\frac{1}{t_1-t_0-\Omega_{mn}}+\frac{1}{\Omega_{mn}}+\frac{t_1-z_0}{\Omega_{mn}^2}\right]\mathrm{d}t_1,$$

$$\langle 4\rangle = \int_{L_0} K_1(t_0,t_1)K_2(t_1,t)\sum_{m,n}{}'\left[1/(t-t_1-\Omega_{mn})+1/\Omega_{mn}+(t-t_1)/\Omega_{mn}^2\right]\cdot\sum_{m,n}{}'\left[1/(t_1-t_0-\Omega_{mn})+1/\Omega_{mn}+(t_1-t_0)/\Omega_{mn}^2\right]\mathrm{d}t_1.$$

By [1], we know that $\langle 1\rangle$ is a Fredholm integral. For $\langle 4\rangle$, we know from

$$K_1(t_0,t_1),K_2(t_1,t)) \in H(L_0\times L_0)$$

that $\langle 4\rangle$ is continuous about the variable $t\in L_0$, and so that $\int_{L_0}\langle 4\rangle\varphi(t)\mathrm{d}t$ is also a Fredholm integral. By nothing that $\langle 2\rangle$ and $\langle 3\rangle$ have the same form, we only need to discuss either one of them. Here we consider the integral $\langle 2\rangle$. Write

$$h(z) = \sum_{m,n}{}'\left[1/(t-z-\Omega_{mn})+1/\Omega_{mn}+(t-z)/\Omega_{mn}^2\right]$$

then $h(t_1)\in H(L_0)$ is analytic in **P** and so that $h(t_1)\in H(L_0)$. Consequently, we read from

$$K_1(t_0,t_1),K_2(t_1,t)\in H(L_0\times L_0)$$

that $\langle 2\rangle\in H(L_0)$ and so that $\langle 2\rangle$ is continuous on L_0, therefore $\int_{L_0}\langle 2\rangle\varphi(t)\mathrm{d}t$ is also a Fredholm integral.

So far, we conclude that K_1K_2 is a singular integral operator.

3) Let $K_3 = K_1K_2$, where κ_j denotes the indices of $K_j\ (j=1,2,3)$, then $\kappa_3=\kappa_1+\kappa_2$.

Proof From 2), we know

$$a_3 = a_1a_2+b_1b_2,\ b_3 = a_1b_2+a_2b_1$$

and

$$S_3 = S_1S_2,\ D_3 = D_1D_2$$

so $\kappa_3=\kappa_1+\kappa_2$.

In addition, we can see from

$$a_3 = a_1a_2+b_1b_2 \text{ and } b_3 = a_1b_2+a_2b_1,$$

that when K_1,K_2 are normal, K_3 is also normal.

4) $(K_1K_2)K_2 = K_1(K_2K_3)$.

5) If K is a singular integral operator, and k is a Fredholm integral operator of the first kind, then kK and Kk are also Fredholm integral operators of the first kind.

6) If the indies of K and K' are κ and κ' respectively , then $\kappa'=-\kappa$.

7) $(K_1K_2)' = K_2'K_1'$.

Through careful calculation, we may obtain 4) - 7).

8) Generally speaking,

$$\int_{L_0}\psi K\varphi\mathrm{d}t = \int_{L_0}\varphi K'\psi\mathrm{d}t$$

can not be established for $\varphi,\psi\in H$.

Proof By definition and calculation, we have

$$\begin{aligned}\int_{L_0}\psi K\varphi\mathrm{d}t &= \int_{L_0}\psi t\left\{a(t)\varphi(t)+\frac{1}{\pi i}\int_{L_0}K(t,t_1)\varphi(t_1)\left[\zeta(t_1-t)+(t-z_0)\right]\mathrm{d}t_1\right\}\mathrm{d}t \\ &= \int_{L_0}a(t)\varphi(t)\psi(t)\mathrm{d}t+\frac{1}{\pi i}\int_{L_0}\psi(t)\int_{L_0}K(t,t_1)\varphi(t_1)\left[\zeta(t_1-t)+\zeta(t-z_0)\right]\mathrm{d}t_1\mathrm{d}t\end{aligned}. \tag{11}$$

Whereas

$$\int_{L_0}\varphi K'\psi\mathrm{d}t = \int_{L_0}a(t)\varphi(t)\psi(t)\mathrm{d}t-\frac{1}{\pi i}\int_{L_0}\varphi(t)\int_{L_0}K(t_1,t)\psi(t_1)\left[\zeta(t_1-t)+\zeta(t-z_0)\right]\mathrm{d}t_1\mathrm{d}t. \tag{12}$$

Let

$$W = \int_{L_0}\varphi(t)\int_{L_0}K(t_1,t)\psi(t_1)\left[\zeta(t_1-t)+\zeta(t-z_0)\right]\mathrm{d}t_1\mathrm{d}t$$

then by Lemma 1(a), we have

$$\begin{aligned}W &= \int_{L_0}\left\{\int_{L_0}K(t_1,t)\varphi(t)\psi(t_1)\left[\zeta(t_1-t)+\zeta(t-z_0)\right]\mathrm{d}t\right\}\mathrm{d}t_1 \\ &= -\int_{L_0}\psi(t)\left\{\int_{L_0}K(t,t_1)\varphi(t_1)\left[\zeta(t_1-t)-\zeta(t_1-z_0)\right]\mathrm{d}t_1\right\}\mathrm{d}t\end{aligned}, \tag{13}$$

Substituting (13) into (12), we see that

$$\int_{L_0} \varphi K' \psi \mathrm{d}t = \int_{L_0} a(t)\varphi(t)\psi t \mathrm{d}t + \frac{1}{\pi i}\int_{L_0} \psi(t)\left\{\int_{L_0} K(t,t_1))\varphi(t_1)\left[\zeta(t_1 - t) - \zeta(t_1 - z_0)\right]\mathrm{d}t_1\right\}\mathrm{d}t. \quad (14)$$

Therefore, $\int_{L_0} \psi K \varphi \mathrm{d}t = \int_{L_0} \varphi K' \psi \mathrm{d}t$ cannot be established.

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