

Riemann Boundary Value Problem of Non-Normal Type on the Infinite Straight Line

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ABSTRACT

We consider a Riemann boundary value problem of non-normal type on the infinite straight line. By using the method of complex functions, we investigate the method for solving this Riemann boundary value problem of non-normal type and give the general solutions and the solvable conditions for it.

Keywords: Non-Normal Type; Riemann Boundary Value Problem; The Infinite Straight Line

1. Introduction

Various kinds of Riemann boundary value problems (BVPs) for analytic functions on closed curves or on open arc, doubly periodic Riemann BVPs, doubly quasi-periodic Riemann BVPs, and BVPs for polyanalytic functions have been widely investigated in [1-8]. The main approach is to use the decomposition of polyanalytic functions and their generalization to transform the boundary value problems to their corresponding boundary value problems for analytic functions. Recently, inverse Riemann BVPs for generalized analytic functions or bianalytic functions have been investigated in [9-12].

In this paper, we consider a kind of Riemann BVP of non-normal type on the infinite straight line and discuss the solvable conditions and the general solution for it.

2. A Riemann Boundary Value Problem of Non-Normal Type on the Infinite Straight Line

Let X be the real axis oriented in the positive direction. And let Z^+ , Z^- denote the upper half-plane and the lower half-plane cut by X . Our objective is to find a sectionally holomorphic function $\Phi(z)$ satisfying the following boundary condition

$$\Phi^+(x) = \frac{\Pi_1(x)}{\Pi_2(x)} G(x) \Phi^-(x) + g(x), \quad x \in X \quad (1)$$

where the two given functions $G(x), g(x) \in \hat{H}(X)$ and $G(x) \neq 0$ on X with $G(\infty)$ existing and $G(\infty) \neq 0$ (clearly $g(\infty)$ exists), and

$$\Pi_1(x) = \prod_{j=1}^m (x - \alpha_j)^{\lambda_j}, \quad \Pi_2(x) = \prod_{l=1}^n (x - \beta_l)^{\mu_l},$$

with

$$\alpha_j, \beta_l (\alpha_j \neq \beta_l) (j = 1, \dots, m; l = 1, \dots, n)$$

being on X and $\lambda_j, \mu_l \in \mathbb{N}$. And the integer

$$\kappa = \text{Ind}_X G(x) = \frac{1}{2\pi} [\arg G(x)]_X$$

is the index of problem (1).

Write

$$\lambda = \sum_{j=1}^m \lambda_j, \quad \mu = \sum_{l=1}^n \mu_l.$$

Without loss of generality, we can consider problem (1) in class R_0 , that is, the two limits $\Phi^+(\infty)$ and $\Phi^-(\infty)$ exist as $z (\in \bar{Z}^+ \text{ or } \bar{Z}^-) \rightarrow \infty$. Clearly, here we have $\Phi^+(\infty) = \Phi^-(\infty)$.

3. Homogeneous Problem

The homogeneous problem of (1) is as follows

$$\Phi^+(x) = \frac{\Pi_1(x)}{\Pi_2(x)} G(x) \Phi^-(x), \quad x \in X. \quad (2)$$

It is found that $\lambda = \mu$ is required for solving problem (2) in class R_0 . Here we suppose that $\lambda = \mu$. Let

$$G_0(x) = \left(\frac{x+i}{x-i} \right)^\kappa G(x), \quad (3)$$

then $G_0(x) \in \hat{H}$ and $G_0(x) \neq 0$ with $G_0(\infty) = G(\infty) \neq 0$,

and

$$\text{Ind}_x G_0(x) = \frac{1}{2\pi} [\arg G_0(x)]_x = 0. \tag{4}$$

Write

$$\Gamma(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log G_0(x)}{x-z} dx. \tag{5}$$

Since $\text{Ind}_x G_0(x) = 0$, by taking logarithm of $G_0(x)$ for some branch we obtain a single-valued function $\log G_0(x)$ with $\log G_0(x) \in \hat{H}$, hence $\Gamma^\pm(x)$ exists with $\Gamma^\pm(x) \in \hat{H}$ and $\Gamma(\pm\infty) = 0$. And by simple calculation we see that $e^{\Gamma(z)}$ is sectionally holomorphic.

Write

$$X(z) = \begin{cases} (z+i)^{-\kappa} e^{\Gamma(z)}, & z \in Z^+; \\ (z-i)^{-\kappa} e^{\Gamma(z)}, & z \in Z^-, \end{cases} \tag{6}$$

then $X^+(x) = G_0(x)X^-(x)$. Substituting this into (2) gives

$$\frac{\Pi_2(x)\Phi^+(x)}{X^+(x)} = \frac{\Pi_1(x)\Phi^-(x)}{X^-(x)}.$$

If we write

$$\Omega(z) = \begin{cases} \frac{\Pi_2(z)\Phi(z)}{X(z)}, & z \in Z^+; \\ \frac{\Pi_1(z)\Phi(z)}{X(z)}, & z \in Z^-, \end{cases}$$

then we get $\Omega^+(x) = \Omega^-(x)$. Thus $\Omega(z)$ is analytic on the whole complex plane and has at most $\kappa + \lambda$ order at ∞ . From [5], we know that $\Omega(z)$ must be an arbitrary polynomial $P_{\kappa+\lambda}(z)$ of degree $\kappa + \lambda$ with $P_{\kappa+\lambda}(z) \equiv 0$ if $\kappa + \lambda < 0$. Therefore, the homogeneous problem (2) has general solution in class R_0 as follows

$$\Phi(z) = \begin{cases} \frac{X(z)P_{\kappa+\lambda}(z)}{\Pi_2(z)}, & z \in Z^+; \\ \frac{X(z)P_{\kappa+\lambda}(z)}{\Pi_1(z)}, & z \in Z^-. \end{cases} \tag{7}$$

Considering the requirements that $\Phi^+(x)$ and $\Phi^-(x)$ are bounded at

$$\alpha_j, \beta_l \quad (j = 1, \dots, m; l = 1, \dots, n),$$

we can let

$$P_{\kappa+\lambda}(z) = \Pi_1(z)\Pi_2(z)P_{\kappa-\lambda}(z),$$

where $P_{\kappa-\lambda}(z)$ is an arbitrary polynomial of degree $\kappa - \lambda$ with $P_{\kappa-\lambda}(z) \equiv 0$ if $\kappa - \lambda < 0$. Now we get the general solution in class R_0 for the homogeneous problem (2) as follows

$$\Phi(z) = \begin{cases} \Pi_1(z)X(z)P_{\kappa-\lambda}(z), & z \in Z^+; \\ \Pi_2(z)X(z)P_{\kappa-\lambda}(z), & z \in Z^-. \end{cases} \tag{8}$$

Thus we get the following results.

Theorem 3.1. For the homogeneous problem (2) in class R_0 , the following two cases arise.

1) When $\kappa \geq \lambda$, it is always solvable and its general solution is given by (8), where $P_{\kappa-\lambda}(z)$ is an arbitrary polynomial of degree $\kappa - \lambda$.

2) When $\kappa < \lambda$, it only has zero-solution.

4. Nonhomogeneous Problem

For nonhomogeneous problem (1), the key is to find out the special solution.

Similar to the case in homogeneous problem (2), the canonical function $X(z)$ is given by (6) but with

$$\Gamma(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log G_0(x)}{x-z} dx \quad (z \notin X)$$

satisfying

$$X^+(x) = G(x)X^-(x).$$

By this, problem (1) can be rewritten as

$$\Pi_2(x) \frac{\Phi^+(x)}{X^+(x)} = \Pi_1(x) \frac{\Phi^-(x)}{X^-(x)} + \Pi_2(x) \frac{g(x)}{X^+(x)}, \quad x \in X. \tag{9}$$

We note that Plemelj formula can not be used directly here, because that when $\kappa > 0$

$$\lim_{x \rightarrow \infty} \frac{g(x)}{X^+(x)} = \lim_{x \rightarrow \infty} (x+i)^\kappa e^{-\Gamma^+(x)} g(x) = \infty$$

is not a finite constant, and so $\frac{g(x)}{X^+(x)} \notin \hat{H}$ (unless

$\kappa \leq 0$). For a unified treatment, regardless of the value of κ , we always let

$$Y(z) = (z+i)^\kappa X(z) = \begin{cases} e^{\Gamma(z)}, & z \in Z^+; \\ \left(\frac{z+i}{z-i}\right)^\kappa e^{\Gamma(z)}, & z \in Z^-. \end{cases}$$

Multiplying $(x+i)^{-\kappa}$ to the two sides of (9) gives

$$\frac{\Pi_2(x)\Phi^+(x)}{Y^+(x)} = \frac{\Pi_1(x)\Phi^-(x)}{Y^-(x)} + \frac{\Pi_2(x)g(x)}{Y^+(x)}.$$

We know that $\frac{g(x)}{Y^+(x)} \in \hat{H}$ and so that

$$\frac{\Pi_2(x)g(x)}{Y^+(x)} \in \hat{H}.$$

If we let

$$\Psi(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Pi_2(\tau) \frac{g(\tau)}{Y^+(\tau)} \frac{d\tau}{\tau - z}, z \notin X, \quad (10)$$

then we get

$$\Psi^+(x) - \Psi^-(x) = \Pi_2(x) \frac{g(x)}{Y^+(x)}, \quad x \in X,$$

and

$$\begin{aligned} & \frac{\Pi_2(x)\Phi^+(x)}{Y^+(x)} - \Psi^+(x) \\ &= \frac{\Pi_1(x)\Phi^-(x)}{Y^-(x)} - \Psi^-(x), x \in X, \end{aligned} \quad (11)$$

with $\Psi(\infty) = 0$. Similar to the reasoning for (7) for problem (2), we know that if problem (1) has solution in class R_0 , then can easily write out the form. But for problem (11), the function

$$F(z) = \begin{cases} \frac{\Pi_2(z)\Phi(z)}{Y(z)} - \Psi(z), & z \in Z^+, \\ \frac{\Pi_1(z)\Phi(z)}{Y(z)} - \Psi(z), & z \in Z^- \end{cases}$$

is analytic everywhere except at the possible unique pole $z = -i$, therefore the following two cases arise.

Case 1. $\kappa \geq 0$.

When $\kappa \geq 0$, $F(z)$ has a pole of order κ at $z = -i$. To eliminate the singularity, we multiply $(z+i)^\kappa$ to $F(z)$ and get a polynomial of degree $\kappa + \lambda$:

$$(z+i)^\kappa F(z) = P_{\kappa+\lambda}(z).$$

Therefore

$$\Phi(z) = \begin{cases} \frac{X(z)P_{\kappa+\lambda}(z) + Y(z)\Psi(z)}{\Pi_2(z)}, & z \in Z^+; \\ \frac{X(z)P_{\kappa+\lambda}(z) + Y(z)\Psi(z)}{\Pi_1(z)}, & z \in Z^- \end{cases} \quad (12)$$

is actually the general solution for problem (1) in class R_0 , where $(z+i)^{-\kappa} Y(z) = X(z)$. For convenience, we deform the function $\Phi(z)$ given by (12) into

$$\Phi(z) = \begin{cases} \frac{(z+i)^{-\kappa} e^{\Gamma(z)} [P_{\kappa+\lambda}(z) + (z+i)^\kappa \Psi(z)]}{\Pi_2(z)}, & z \in Z^+; \\ \frac{(z-i)^{-\kappa} e^{\Gamma(z)} [P_{\kappa+\lambda}(z) + (z+i)^\kappa \Psi(z)]}{\Pi_1(z)}, & z \in Z^-. \end{cases} \quad (13)$$

Considering the requirements that $\Phi^\pm(x)$ are bounded on X , and the fact that

$$\begin{aligned} \Phi^+(z) &= \frac{(z+i)^{-\kappa} e^{\Gamma(z)} [P_{\kappa+\lambda}(z) + (z+i)^\kappa \Psi(z)]}{\Pi_2(z)} \\ &= \frac{(z+i)^{-\kappa} e^{\Gamma(z)} F^+(z)}{\Pi_2(z)} \end{aligned}$$

with $F^+(z) = P_{\kappa+\lambda}(z) + (z+i)^\kappa \Psi(z)$, the smallest degree of $(t - \beta_l)$ in the numerator of the above formula should be μ_l , we suppose that

$$\begin{aligned} F(t) &= a_{-n} (t - \beta_l)^{\mu_l + n} + \dots + a_0 (t - \beta_l)^{\mu_l} \\ &\quad + a_1 (t - \beta_l)^{\mu_l - 1} + \dots + a_l (t - \beta_l) + a_{l+1}, \end{aligned}$$

then when $t \rightarrow \beta_l$, we have

$$\begin{aligned} F(t) &\rightarrow a_{l+1}, F'(t) \rightarrow a_l, F''(t) \rightarrow 2a_{l-1}, \dots, \\ F^{(\mu_l - 1)}(t) &\rightarrow a_2 (\mu_l - 1)! \end{aligned}$$

and the smallest degree of $(t - \beta_l)$ in numerator is μ_l (i.e. $\Phi^+(x)$ is bounded on X) only when

$$F(t) = F'(t) = F''(t) = \dots = F^{(\mu_l - 1)}(t) = 0,$$

that is,

$$\begin{aligned} P_{\kappa+\lambda}^{(s)}(\beta_l) + [(z+i)^\kappa \Psi(z)]^{+(s)} \Big|_{z=\beta_l} &= 0, \\ s &= 0, 1, \dots, \mu_l - 1; \quad l = 1, 2, \dots, n. \end{aligned} \quad (14)$$

In a similar way, we know that $\Phi^-(x)$ is bounded on X only when

$$\begin{aligned} P_{\kappa+\lambda}^{(r)}(\alpha_j) + [(z+i)^\kappa \Psi(z)]^{- (r)} \Big|_{z=\alpha_j} &= 0, \\ r &= 0, 1, \dots, \lambda_j - 1; \quad j = 1, 2, \dots, m \end{aligned} \quad (15)$$

are satisfied.

Hence, we get the results that $\Phi^\pm(x)$ are bounded on X only when the conditions (14) and (15) are all satisfied. While it is troublesome to solve the system composed by (14) and (15) for the coefficients of $P_{\kappa+\lambda}(z)$.

Here we aim to determine the coefficients by using Hermite interpolation polynomial.

Firstly, we make the polynomial $Q_\rho(z)$ of degree $\rho = \lambda + \mu - 1 = 2\lambda - 1$ ($\lambda = \mu$) such that

$$Q_\rho^{(r)}(\alpha_j) = \left[(z+i)^\kappa \Psi(z) \right]^{- (r)} \Big|_{z=\alpha_j}, \quad r = 0, 1, \dots, \lambda_j - 1; \quad j = 1, 2, \dots, m,$$

$$Q_\rho^{(s)}(\beta_l) = \left[(z+i)^\kappa \Psi(z) \right]^{+ (s)} \Big|_{z=\beta_l}, \quad s = 0, 1, \dots, \mu_l - 1; \quad l = 1, 2, \dots, n.$$

The polynomial $Q_\rho(z)$ exists uniguelly from [5]. Let

$$\Phi_0(z) = \begin{cases} \frac{(z+i)^{-\kappa} e^{\Gamma(z)} [(z+i)^\kappa \Psi(z) - Q_\rho(z)]}{\Pi_2(z)}, & z \in Z^+; \\ \frac{(z-i)^{-\kappa} e^{\Gamma(z)} [(z+i)^\kappa \Psi(z) - Q_\rho(z)]}{\Pi_1(z)}, & z \in Z^-. \end{cases} \tag{16}$$

We can see from (16) that $\Phi_0^\pm(x)$ are continuous on X , and through simple verification that $\Phi_0^\pm(x)$ satisfy

$$\Phi(z) = \begin{cases} \frac{(z+i)^{-\kappa} e^{\Gamma(z)} [(z+i)^\kappa \Psi(z) - Q_\rho(z)]}{\Pi_2(z)} + \Pi_1(z) X(z) P_{\kappa-\lambda}(z), & z \in Z^+; \\ \frac{(z-i)^{-\kappa} e^{\Gamma(z)} [(z+i)^\kappa \Psi(z) - Q_\rho(z)]}{\Pi_1(z)} + \Pi_2(z) X(z) P_{\kappa-\lambda}(z), & z \in Z^-, \end{cases} \tag{17}$$

where $P_{\kappa-\lambda}(z) = 0$ when $\kappa = \lambda - 1$.

b) If $\lambda - \kappa - 1 > 0$, that is, $\kappa < \lambda - 1$, since the homogeneous problem (2) corresponding to (1) only has zero-solution in R_0 , the existence of a solution $\Phi_0(z)$ for nonhomogeneous problem (1) in R_0 implies the

the condition in problem (1), and that the order of $\Phi_0(z)$ at ∞ is $-\kappa + \rho - \lambda = \lambda - \kappa - 1$.

Now we aim to make $\Phi_0(z)$ belong to R_0 by adding restricted conditions.

a) If $\lambda - \kappa - 1 \leq 0$, that is, $\kappa \geq \lambda - 1$, $\Phi_0(z)$ exactly belongs to R_0 and is justly the solution for homogeneous problem of (2), and also a particular solution for nonhomogeneous problem (1) in R_0 . Combining with the general solution (8) of homogeneous problem (2), we known that when $\kappa \geq \lambda - 1$, the general solution of problem (1) in R_0 is

uniqueness. Because $\Phi_0(z)$ has generally singularity of order $\lambda - \kappa - 1$ at ∞ , $\Phi_0(z)$ belongs to R_0 if and only if $\lambda - \kappa - 1$ conditions on $g(x)$ or on $Q_\rho(z)$ are satisfied.

By rewriting $\Phi_0(z)$ as

$$\Phi_0(z) = \begin{cases} \frac{(z+i)^{-\kappa} e^{\Gamma(z)} [(z+i)^\kappa \Psi(z) - Q_\rho(z)]}{\Pi_2(z)} = \frac{e^{\Gamma(z)} \Psi(z)}{\Pi_2(z)} - \frac{e^{\Gamma(z)} Q_\rho(z)}{(z+i)^\kappa \Pi_2(z)}, & z \in Z^+; \\ \frac{(z-i)^{-\kappa} e^{\Gamma(z)} [(z+i)^\kappa \Psi(z) - Q_\rho(z)]}{\Pi_1(z)} = \frac{e^{\Gamma(z)} (z+i)^\kappa \Psi(z)}{(z-i)^\kappa \Pi_1(z)} - \frac{e^{\Gamma(z)} Q_\rho(z)}{(z-i)^\kappa \Pi_1(z)}, & z \in Z^-. \end{cases}$$

Here we write

$$Q_\rho(z) = A_0 z^\rho + A_1 z^{\rho-1} + \dots + A_\rho (\rho = \lambda + \mu - 1 = 2\lambda - 1). \tag{18}$$

Since the order of the denominator in $\Phi_0(z)$ is $\kappa + \lambda (\geq 1) \geq -1$ (is always true), it is enough that $Q_\rho(z)$ is at most a polynomial of degree $\kappa + \lambda$, that is,

$$A_0 = A_1 = \dots = A_{\lambda-\kappa-2} = 0. \tag{19}$$

Therefore, only when condition (19) is satisfied, (16) is actually the general solution for nonhomogenous problem (1), now the homogeneous problem (2) only has

zero-solution.

Case 2. $\kappa < 0$.

When $\kappa < 0$, $F(z)$ is analytic on the whole complex plane and has λ order at ∞ , that is,

$F(z) = P_\lambda(z)$, and now the homogeneous problem (2) only has zero-solution. Therefore the general solution for nonhomogenous problem (1) in R_0 is given by

$$\Phi(z) = \begin{cases} \frac{[P_\lambda(z) + \Psi(z)] Y(z)}{\Pi_2(z)} = \frac{[P_\lambda(z) + \Psi(z)] e^{\Gamma(z)}}{\Pi_2(z)}, & z \in Z^+; \\ \frac{[P_\lambda(z) + \Psi(z)] Y(z)}{\Pi_1(z)} = \frac{[P_\lambda(z) + \Psi(z)] (z+i)^\kappa e^{\Gamma(z)}}{\Pi_1(z)}, & z \in Z^-. \end{cases} \tag{20}$$

If we make the Hermite interpolation polynomial of degree $\lambda - 1$, the fact that $\lambda - 1 < \lambda$ (the order of the denominator) implies that $\Phi(z)$ is unbounded on X , which contradicts with the hypothesis that $\Phi(z)$ is bounded on X . So it is infeasible to make the Hermite interpolation polynomial for this.

However, we have the following effective treatment for this.

a) Under this situation, $\Phi(z)$ may have the unique pole at $z = -i$. In order to eliminate the pole, we should put the following restrictions for it:

if $\kappa + \lambda \geq 0$, we only need

$$P_{\lambda}^{(j)}(-i) + \Psi^{(j)}(-i) = 0, j = 0, 1, 2, \dots, -\kappa - 1; \quad (21)$$

if $\kappa + \lambda = -1$, we only need to put

$$P_{\lambda}(-i) = -\Psi(-i); \quad (22)$$

if $\kappa + \lambda < -1$, apart from (22), the restrictions

$$\Psi^{(j)}(-i) = 0, j = 1, 2, \dots, -\kappa - 1,$$

or

$$\int_{-\infty}^{+\infty} \frac{\prod_2(x)g(x)}{Y^+(x)(x+i)^{j+1}} dx = 0, j = 1, 2, \dots, -\kappa - 1 \quad (23)$$

are necessary.

b) Considering the boundedness of $\Phi(z)$ on X , the following restrictions are also necessary:

$$P_{\lambda}^{(s)}(\alpha_j) + \Psi^{-(s)}(\alpha_j) = 0, r = 0, 1, \dots, \lambda_j - 1; j = 1, 2, \dots, m \quad (24)$$

$$P_{\lambda}^{(s)}(\beta_l) + \Psi^{+(s)}(\beta_l) = 0, s = 0, 1, \dots, \mu_l - 1; l = 1, 2, \dots, n. \quad (25)$$

Thus we get the following results.

Theorem 4.1. For the nonhomogeneous problem (1) in class R_0 , the following two cases arise.

1) $\kappa \geq 0$. If $\kappa \geq \lambda - 1$, problem (1) is always solvable and its general solution is given by (17); and if $\kappa < \lambda - 1$, if and only if $Q_{\rho}(z)$ satisfies condition (19), problem (1) has unique solution, given by (16).

2) When $\kappa < 0$, it has unique solution in form (20) when the restrictions (21) or (22) or (22)-(23), and (24)-(25) are satisfied.

Anyway, the degree of freedom of solution for nonhomogeneous problem (1) is $\kappa - \lambda + 1$.

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