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$L(2,1)$-Labeling Number of the Product and the Join Graph on Two Fans

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ABSTRACT

$L(2,1)$-labeling number of the product and the join graph on two fans are discussed in this paper, we proved that $L(2,1)$-labeling number of the product graph on two fans is $\lambda(G) \le \Delta + 3$, $L(2,1)$-labeling number of the join graph on two fans is $\lambda(G) \le 2\Delta + 3$.

Keywords: Labeling Number; Join Graph; Product Graph

1. Introduction

Throughout this paper, we consider connected graphs without loops or multiple edges. For a graph $G, V(G)$ and $E(G)$ are used to denote the vertex set and edge set of $G, \delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of a graph *G*, respectively. For a vertex $v \in V(G)$, the neighborhood of *v* in *G* is $N_G(v) = \{u \in V(G), u \text{ is adjacent to } v \text{ in } G\}$. Vertices in $N_G(V)$ are called neighbors of *v*, $|N_G(V)|$ denotes the number of vertices in $N_G(V)$. The other terminology and notations are referred to [1].

For a given graph *G*, an integer $k > 0$, an $L(2,1)$-labeling of *G* is defined as a function $f: V(G) \to \{0,1,2,\cdots,k\}$ such that $|f(u) - f(v)| \ge 2$ if $uv \in E(G)$; and $|f(u) - f(v)| \ge 1$ if $d_G(u,v) = 2$, where $d_G(u,v) = 2$, the distance of *u* and *v*, is the length (number of edges) of a shortest path between *u* and *v*. the $L(2,1)$-labeling number, denoted $\lambda(G)$, is the least integer k such that *G* has a $L(2,1)$-labeling.

The Motivated by the channel assignment problem introduced by Hale in [2], the $L(2,1)$ labeling have been studied extensively in the past decade. In 1992, in [3] Griggs and Yeh proposed the famous conjecture, for any graph $G, \lambda(G) \le \Delta^2$.

Griggs and Yeh in [3] proved that the conjecture true fop path, tree, circle, wheel and the graph with diameter 2, G. J. chang and David Kuo in [4] proved that $\lambda(G) \le \Delta^2 + \Delta$ for any graph. Recently Kral D and Skrekovski R in [5] proved the upper is $\lambda(G) \le \Delta^2 + \Delta - 1$. It is difficult to prove the conjecture. Now, the study of $L(2,1)$-labeling is focus on special graph. Georges [6,7] give some good results. Zhang and Ma studied the labeling of some special graph, giving some good results in [8-11].

In this paper, we studied the $L(2,1)$-labeling number of the product and the join graph on two fans.

2. $L(2,1)$-Labeling Number of the Join Graph on Two Fans

Definition 2.1 Let F_m be a fan with *m* + 1 vertices $u_0, u_1, u_2, \cdots, u_m$, in which $d(u_0) = m$.

Definition 2.2 Let *G* and *H* be two graphs, the join of *G* and *H* denoted $G \vee H$, is a graph obtained by starting with a disjoint union of *G* and *H*, and adding edges joining each vertex of *G* to each vertex of *H*.

Theorem 2.1 Let $G = F_m \vee F_n$, if $m \ge 4, n \ge 4$, then $\lambda(G) \le \Delta + 3$.

Proof. In $F_m \vee F_n$, for arbitrary vertex *u* and *v*, such that $d_G(u,v) \le 2$, clearly $\Delta(G) \le n + m + 1$.

Let *k* denote the maximum labeling number of F_n

First, we give a $L(2,1)$-labeling of F_n as follows, $f(v_0) = 0$.

If $j = 1, 2, \cdots, n-5, n-4$,

$$f(v_j) = j + 3 \quad \text{when} \quad j(\bmod 4) = 1,$$

$$f(v_j) = j \quad \text{when} \quad j(\bmod 4) = 2,$$

$$f(v_j) = j + 2 \quad \text{when} \quad j(\bmod 4) = 3,$$

$$f(v_j) = j-1 \text{ when } j(\bmod 4) = 0.$$

If $n(\bmod 4) = 0$, let

$$f(v_{n-3}) = n, \quad f(v_{n-2}) = n-2,$$

$$f(v_{n-1}) = n+1, f(v_n) = n-1.$$

If $n(\bmod 4) = 1$, let

$$f(v_{n-3}) = n-3, \quad f(v_{n-2}) = n,$$

$$f(v_{n-1}) = n-2, f(v_n) = n+1.$$

If $n(\bmod 4) = 2$, let

$$f(v_{n-3}) = n-1, \quad f(v_{n-2}) = n+1,$$

$$f(v_{n-1}) = n-3, f(v_n) = n.$$

If $n(\bmod 4) = 3$, let

$$f(v_{n-3}) = n, f(v_{n-2}) = n-4,$$

$$f(v_{n-1}) = n-1, f(v_n) = n+1.$$

Clearly, $k = n+1$.
Then we label the vertex of F_m as follows,
If $i = 1, 2\cdots, m-5, m-4$,

$$f(u_0) = \max\{f(u_i) | i = 1, 2, \cdots, m\} + 2,$$

$$f(u_i) = k+i+3 \text{ when } i(\bmod 4) = 1,$$

$$f(u_i) = k+i \text{ when } i(\bmod 4) = 2,$$

$$f(u_i) = k+i+2 \text{ when } i(\bmod 4) = 3,$$

$$f(u_i) = k+i-1 \text{ when } i(\bmod 4) = 0.$$

If $m(\bmod 4) = 0$, let

$$f(u_{m-3}) = k+m, f(u_{m-2}) = k+m-2,$$

$$f(u_{m-1}) = k+m+1, f(u_m) = k+m-1;$$

If $m(\bmod 4) = 1$, let

$$f(u_{m-3}) = k+m-3, f(u_{m-2}) = k+m,$$

$$f(u_{m-1}) = k+m-2, f(u_m) = k+m+1;$$

If $m(\bmod 4) = 2$, let

$$f(u_{m-3}) = k+m-1, f(u_{m-2}) = k+m+1,$$

$$f(u_{m-1}) = k+m-3, f(u_m) = k+m;$$

If $m(\bmod 4) = 3$, let

$$f(u_{m-3}) = k+m, f(u_{m-2}) = k+m-4,$$

$$f(u_{m-1}) = k+m-1, f(u_m) = k+m+1.$$

From above,
If $m(\bmod 4) = 0$, $f(u_{m-1})$ is the maximum number in F_m, and $f(u_{m-1}) = k+m+1$, then

$$f(u_0) = k+m+1+2 = k+m+3$$
$$= n+1+m+3 = n+m+4.$$

If $m(\bmod 4) = 1$, $f(u_m)$ is the maximum number in F_m, and $f(u_m) = k+m+1$, then

$$f(u_0) = k+m+1+2 = k+m+3$$
$$= n+1+m+3 = n+m+4.$$

If $m(\bmod 4) = 2$, $f(u_{m-2})$ is the maximum number in F_m, and $f(u_{m-2}) = k+m+1$, then

$$f(u_0) = k+m+1+2 = k+m+3$$
$$= n+1+m+3 = n+m+4.$$

If $m(\bmod 4) = 3$, $f(u_m)$ is the maximum number in F_m, and $f(u_m) = k+m+1$, then

$$f(u_0) = k+m+1+2 = k+m+3$$
$$= n+1+m+3 = n+m+4.$$

So $f(u_0)$ is the maximum number in $G = F_m \vee F_n$, and $f(u_0) = n+m+4$, and $\Delta(G) = n+m+1$.
Obviously, *f* is a $(\Delta+3)$-$L(2,1)$-labeling of *G*,
Then $\lambda(G) \le \Delta+3$.

3. $L(2,1)$-Labeling Number of the Product Graph on Two Fans

Definition 3.1 The Cartesian product of graph *G* and *H*, denoted $G \times H$, which vertex set and edge set are the follows:

$$V(G\times H) = V(G)\times V(H)$$
$$= \{(u,v) | u \in V(G), v \in V(H)\}$$

$$E(G\times H) = \{(u,v)(u',v') | v = v' \text{ and }$$
$$uu' \in E(G) \text{ or } u = u' \text{ and } vv' \in E(H)\}.$$

Theorem 3.1 Let $G = F_m \times F_n$, if $3 \le n \le m < 2n$, then $\lambda(G) \le 2\Delta+3$.

Proof. In $F_m, d(u_0) = m$, the other vertices $u_i(1,2,\cdots,m)$, In $F_n, d(v_0) = n$, the other vertices

$$v_j(j = 1,2,\cdots,n),$$

$$V = \{w_{ij} | w_{ij} = (u_i, v_j), 1 \le i \le m, 1 \le j \le n\}$$

denote the vertex of $G = F_m \times F_n$, Obviously, $\Delta(G) = m+n$, for $n \ge 3$.
We give a $L(2,1)$-labeling of *G* as follows, First, let

$$f(w_{00}) = 0$$

$$f(w_{1j}) = 2j, j = 1,2,\cdots,n$$

$$f(w_{2j}) = 2j+3, j = 1,2,\cdots,n,$$

We have the maximum labeling number is 2*n* + 3.

Then let

$$f(w_{i,j+1}) = f(w_{i-2,j}), (i = 3,4,\cdots,m, j = 1,2,\cdots,n)$$

$$f(w_{i,1}) = f(w_{i-2,n}), (i = 3,4,\cdots,m, j = 1,2,\cdots,n-1);$$

$$f(w_{i,0}) = 2n + 2i + 2, (i = 1,2,\cdots,m),$$

From above, $2n + 2m + 2$ is the maximum labeling number.

Finally, let $f(w_{0,0}) = 2n + 2i + 2, (i = 1,2,\cdots,m)$, Obviously, $2n + 2m + 3$ is the maximum labeling number in these $f(w_{0,0}) = 2n + 2i + 2, (i = 1,2,\cdots,m)$, since $n \le m < 2n$, then the maximum labeling number no more than $2n + 2m + 3$, and $\Delta(G) = m + n$, so $\lambda(G) \le 2\Delta + 3$.

REFERENCES

[1] J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications," Macmillan, New York, 1976.

[2] W. K. Hale, "Frequency Assignment: Theory and Applications," *IEEE Proceedings*, Vol. 68, No. 12, 1980, pp. 1497-1514. doi:10.1109/PROC.1980.11899

[3] J. R. Griggs and R. K. Yeh, "Labeling Graphs with a Condition at Distance 2," *SIAM Journal on Discrete Mathematics*, Vol. 5, No. 4, 1992, pp. 586-595. doi:10.1137/0405048

[4] G. J. Chang and D. Kuo, "The *L*(2,1)-Labeling Problem on Graphs," *SIAM Journal on Discrete Mathematics*, Vol. 9, No. 2, 1996, pp. 309-316. doi:10.1137/S0895480193245339

[5] D. Král and R. A. Škrekovski, "Theorem about the Channel Asscgnment Problem," *SIAM Journal on Discrete Mathematics*, Vol. 16, No. 3, 2003, pp. 426-437. doi:10.1137/S0895480101399449

[6] J. P. Georges, D. W. Mauro and M. I. Stein, "Labeling Products of Complete Graphs with a Condition at Distance Two," *SIAM Journal on Discrete Mathematics*, Vol. 14, No. 1, 2000, pp. 28-35. doi:10.1137/S0895480199351859

[7] J. P. Georges, D. W. Mauro and M. A. Whittlesey, "Relating Path Covering to Vertex Labelling with a Condition at Distance Two," *Discrete Mathematics*, Vol. 135, 1994, pp. 103-111. doi:10.1016/0012-365X(93)E0098-O

[8] S. M. Zhang and Q. L. Ma, "On List (2,1)-Labelling of Some Planar Graphs," *Ars Combinatoria*, Vol. 84, 2007, pp. 231-241.

[9] S. M. Zhang and Q. L. Ma, "Labelling Some Planar Graphs with a Condition at Distance Two," *Journal of Applied Mathematics and Computing*, Vol. 24, No. 1-2, 2007, pp. 421-426.

[10] S. M. Zhang and J. H. Wang, "*L*(p,q)-Labeling of Planar Graph with High Maximum Degree," *Journal of Shandong University*, Vol. 42, No. 4, 2007, pp. 39-43.

[11] S. M. Zhang and Q. L. Ma, "*L*(d,1)-Total Labeling of Outerplannar Graphs," *Journal of Jinnan University*, Vol. 20, No. 3, 2006, pp. 258-260.