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# Some Equivalent Forms of Bernoulli's Inequality: A Survey*

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## ABSTRACT

The main purpose of this paper is to link some known inequalities which are equivalent to Bernoulli's inequality.

**Keywords:** Bernoulli's Inequality; Young's Inequality; Jensen's Inequality; Hölder's Inequality; Cauchy's Inequality; Minkowski's Inequality; Schlömich's Inequality; AGM Inequality; Jacobsthal's Inequality; Equivalent

## 1. Introduction

Based on the theory of inequalities, many classical inequalities not only promote the development of the inequality theory, but also lead to many applications in pure mathematics and in applied mathematics. Bernoulli's inequality is one of the most distinguished inequalities. In this paper, a new proof of Bernoulli's inequality via the dense concept is given. Some strengthened forms of Bernoulli's inequality are established. Moreover, some equivalent relations between this inequality and other known inequalities are tentatively linked. The organization of this paper is as follows:

In Section 2, a new proof of Bernoulli's inequality by means of the concept of density is raised. In Section 3, some strengthened forms of Bernoulli's inequality are establised. In Section 4, we link some known inequalities which are equivalent to Bernoulli's inequality. In Section 5, we collect some variants of Young's inequality which are equivalent to Bernoulli's inequality. For related results, we refer to [1-35].

## 2. Preliminaries

In order to complete these tasks, we need the definition and some basic results of the convex function as follows:

**Definition 2.1**

Let $f: I \to R$ be a function, where *I* is an interval of *R*.

1) Suppose that *P* and *Q* are <u>any</u> two points on the graph of $y = f(x)$, if the chord $\overline{PQ}$ can not below the arc *PQ* of the graph of *f*, then we say that *f* is a convex function on *I*. That is, for any two point $x, y \in I$ and any $t \in [0,1]$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \tag{1}$$

then *f* is a convex function on *I*. We say that *f* is called *concave* on *I* if $-f$ is convex on *I*.

If, for any two points $x, y \in I$ with $x \ne y$ and any $t \in (0,1)$,

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y),$$

then we say that $f(x)$ is a strictly convex function on *I*.

2) *I* is said to be *midpoint convex* or *J-convex* on *I* if for any two points $x, y \in I$,

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{2}\left[f(x) + f(y)\right]. \tag{2}$$

It is well-known fact that every convex function on an interval $(a,b)$ is continuous; if *f* is mid-point convex and continuous on an interval *I*, then it is convex on *I*. The following Jensen's inequality can be shown by the mathematical induction directly.

**Lemma 2.2** (**Jensen's inequality**, [3], page 31) *Let* $f(x)$ *be a convex function on I. Then for any* $q_1, q_2, \cdots, q_n \in (0,1)$ *with* $\sum_{i=1}^{n} q_i = 1$ *and for any* $x_1, x_2, \cdots, x_n \in I$,

*Dedicated to the Respected Professor Haruo Murakami.

$$f\left(\sum_{i=1}^{n} q_i x_i\right) \le \sum_{i=1}^{n} q_i f\left(x_i\right). \quad (3)$$

If $f$ is strictly convex, then (3) is strictly unless the $x_i$ are all identically.

**Lemma 2.3** *Let* $f: I \to R$ *be a function. Then the following statements are equivalent*:

1) $f$ is strictly convex on $I$,

2) For any two distinct points $x, y \in I$ and any $\lambda > 1$ satisfying $\lambda x + (1-\lambda) y \in I$,

$$f\left(\lambda x + (1-\lambda) y\right) < \lambda f(x) + (1-\lambda) f(y);$$

3) For any two distinct points $x, y \in I$ and any $\lambda < 0$ satisfying $\lambda x + (1-\lambda) y \in I$,

$$f\left(\lambda x + (1-\lambda) y\right) < \lambda f(x) + (1-\lambda) f(y).$$

**Proof** 1) $\Rightarrow$ 2). Let $x, y \in I$ be distinct and let $\lambda > 1$ be arbitrary. If $z = \lambda x + (1-\lambda) y \in I$, then $x = \frac{1}{\lambda} z + \left(1 - \frac{1}{\lambda}\right) y$ and $x$ is between $y$ and $z$. It follows from the strict convexity of $f$ on $I$ that

$$f(x) < \frac{1}{\lambda} f(z) + \left(1 - \frac{1}{\lambda}\right) f(y).$$

Hence 2) holds.

2) $\Rightarrow$ 3). Let $x, y \in I$ be distinct and let $\lambda < 0$ be arbitrary. If $z = \lambda x + (1-\lambda) y \in I$, then $x = \left(1 - \frac{1}{\lambda}\right) y + \frac{1}{\lambda} z$ and both $y$ and $z$ are distinct. By the assumption of (b), we have

$$f(x) < \left(1 - \frac{1}{\lambda}\right) f(y) + \frac{1}{\lambda} f(z).$$

It follows from $\lambda < 0$ that 3) holds.

3) $\Rightarrow$ 1). Let $x, y \in I$ be distinct and let $0 < \lambda < 1$ be arbitrary. If $z = \lambda x + (1-\lambda) y$, then $y = \left(1 + \frac{1}{\lambda - 1}\right) x + \frac{-1}{\lambda - 1} z$ and $\left(1 + \frac{1}{\lambda - 1}\right) < 0$. It follows from the assumption of 3) that

$$f(y) < \left(1 + \frac{1}{\lambda - 1}\right) f(x) - \frac{1}{\lambda - 1} f(z).$$

This prove 1) holds. Thus the proof is complete.

Next, we will prove Bernoulli's inequality by means of the concept of density without differentiation or integration.

**Lemma 2.4**

$$(1+x)^{\alpha} < 1 + \alpha x \text{ for all } \alpha \in (0,1) \text{ and } x > -1 \text{ with } x \ne 0. \quad (4)$$

The equality is obvious for case $x = 0$ or for case $\alpha = 0$ or 1.

**Proof** Let

$$E := \left\{\alpha \in (0,1) \middle| (1+x)^{\alpha} < 1 + \alpha x, x > -1, x \ne 0\right\}.$$

**Claim 1:** $E$ is dense in $(0,1)$.

It suffices to show that $E$ satisfies the following three properties.

1) $\frac{1}{2} \in E$.

2) If $\alpha \in E$, then $1 - \alpha \in E$.

3) If $\alpha, \beta \in E$, then $\alpha \cdot \beta \in E$ and $\frac{\alpha + \beta}{2} \in E$.

Let $x > -1$ be arbitrary with $x \ne 0$. Then $\frac{-x}{1+x} = -1 + \frac{1}{1+x} > -1$ and $\frac{-x}{1+x} \ne 0$. Thus

$$\left(1 + \frac{x}{2}\right)^2 = 1 + x + \frac{x^2}{4} > 1 + x.$$

So, $(1+x)^{1/2} < 1 + \frac{1}{2} x.$ This proves 1) and hence $E$ is nonempty.

If $\alpha \in E$, then

$$(1+x)^{1-\alpha} = (1+x)\left(1 + \frac{-x}{1+x}\right)^{\alpha} < (1+x)\left(1 + \frac{-\alpha x}{1+x}\right) = 1 + (1-\alpha) x.$$

This proves 2).

Next, if $0 \le \alpha < \beta \le 1$ are such that $\alpha, \beta \in E$, then for every $x > -1$ with $x \ne 0$,

$$(1+x)^{\alpha \cdot \beta} = \left[(1+x)^{\alpha}\right]^{\beta} < (1 + \alpha x)^{\beta} < 1 + \alpha \cdot \beta x.$$

This proves the first part of 3). On the other hand, it follows from $\alpha, \beta \in E$ that

$$(1+x)^{\alpha} < 1 + \alpha x$$

and

$$(1+x)^{\beta} < 1 + \beta x.$$

Therefore,

$$\begin{aligned}(1+x)^{\alpha+\beta} &= (1+x)^{\alpha}(1+x)^{\beta} < (1+\alpha x)(1+\beta x) \\ &= 1 + (\alpha+\beta) x + (\alpha \cdot \beta) x^2 \\ &= \left(1 + \frac{\alpha+\beta}{2} x\right)^2 + \left[\alpha \cdot \beta - \left(\frac{\alpha+\beta}{2}\right)^2\right] x^2 \quad (5) \\ &\le \left(1 + \frac{\alpha+\beta}{2} x\right)^2.\end{aligned}$$

Thus, we complete the proof of 3). Since 1)-3) imply that $\sum_{k=1}^{n} a_k 2^{-k} \in E$ for $a_1, a_2, \cdots, a_n \in \{0,1\}$ and

$n = 1,2,3,\cdots$. Therefore $E$ must be dense in $(0,1)$.

Finally, if $\alpha \in (0,1)$ is arbitrary and $\beta \in E$ with $\alpha < \beta < 1$, then for every $x > 0$,

$$(1+x)^\alpha < (1+x)^\beta \le 1 + \beta x \to 1 + \alpha x \text{ as } \beta \downarrow \alpha.$$

This proves

$$(1+x)^\alpha \le 1 + \alpha x \text{ for } x > 0.$$

Similarly, if $\alpha \in (0,1)$ is arbitrary and $\beta \in E$ with $0 < \beta < \alpha$, then, for every $-1 < x < 0$,

$$(1+x)^\alpha < (1+x)^\beta \le 1 + \beta x \to 1 + \alpha x \text{ as } \beta \uparrow \alpha.$$

This proves

$$(1+x)^\alpha \le 1 + \alpha x \text{ for } -1 < x < 0.$$

Therefore, for every $\alpha \in (0,1)$, we have

$$(1+x)^\alpha \le 1 + \alpha x \text{ for } x > -1 \text{ with } x \ne 0.$$

It follows from (5) again that (4) holds. This completes the proof.

**Corollary 2.5** *The following statements are equivalent*:

1) $-\ln x$ is strictly convex on $(0,\infty)$;

2) $x^\alpha y^{1-\alpha} < \alpha x + (1-\alpha) y$ for all $x, y > 0$ with $x \ne y$ and for all $0 < \alpha < 1$;

3) Young's inequality holds, that is, $XY < \frac{1}{p} X^p + \frac{1}{q} Y^q$, where $X$, $Y > 0$ with $X^p \ne Y^q$ and $1 < p,q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$;

4) (4) holds.

**Proof**

$$\begin{aligned}
&(d) \text{ holds, that is,}\\
&(1+x)^\alpha < 1 + \alpha x \text{ for all } x > -1 \text{ with } x \ne 0 \text{ and for all } 0 < \alpha < 1\\
&\Leftrightarrow x^\alpha < 1 + \alpha(x-1) = (1-\alpha) + \alpha x \text{ for all } x > 0 \text{ with } x \ne 1 \text{ and for all } 0 < \alpha < 1\\
&\Leftrightarrow (xy)^\alpha < (1-\alpha) + \alpha \cdot xy \text{ for all } x, y > 0 \text{ with } x \ne y \text{ and for all } 0 < \alpha < 1\\
&\Leftrightarrow (b) \text{ holds, that is, } x^\alpha y^{1-\alpha} < \alpha x + (1-\alpha) y \text{ for all } x, y > 0 \text{ with } x \ne y \text{ and for all } 0 < \alpha < 1\\
&\Leftrightarrow -\ln x \text{ is strictly convex on } (0,\infty), \text{ that is, } (a) \text{ holds}\\
&\Leftrightarrow XY < \frac{1}{p} X^p + \frac{1}{q} Y^q \text{ for } X, Y > 0 \text{ with } X^p \ne Y^q \text{ and for all } 1 < p,q < \infty \text{ with } \frac{1}{p} + \frac{1}{q} = 1;\\
&\Leftrightarrow (c) \text{ holds.}
\end{aligned}$$

The equality of Young's inequality is clear for case $X, Y \ge 0$ with $X^p = Y^q$. This completes the proof.

Next, we prove some equivalent results which are related to $-\ln x$:

**Lemma 2.6** *For any* $c \in (0,\infty)$, *the following statements are equivalent:*

1) $-\ln x$ is strictly convex on $(0,\infty)$;

2) $-\ln x$ is strictly convex on $(c,\infty)$;

3) $-\ln x$ is strictly convex on $(0,c)$.

**Proof** Clearly, 1) $\Rightarrow$ 2) and 3).

Now, we prove 3) $\Rightarrow$ 1) and 2) $\Rightarrow$ 1). Let $x, y, t > 0$ be with $x \ne y$ and let $\alpha \in (0,1)$ be arbitrary. Since

$$\ln(\alpha(tx) + (1-\alpha)(ty)) = \ln t + \ln(\alpha x + (1-\alpha) y),$$

we have

$$\begin{aligned}
&-\ln(\alpha(tx) + (1-\alpha)(ty))\\
&< \alpha(-\ln(tx)) + (1-\alpha)(-\ln(ty))\\
&\Leftrightarrow -\ln(\alpha x + (1-\alpha) y)\\
&< \alpha(-\ln x) + (1-\alpha)(-\ln y).
\end{aligned}$$

Thus, if $t$ is small such that $tx, ty \in (0,c)$, we obtain that 3) implies 1). Similarly, if $t$ is enough large so that $tx, ty \in (c,\infty)$, we obtain that 2) implies 1). This completes the proof.

**Lemma 2.7** *Let* $x_i \in (-1,0) \cup (0,\infty)$, $c_i \in (-\infty,0) \cup (1,\infty)$, $i = 1,2,\cdots$ *satisfying* $c_i x_i$, $i = 1, 2, \cdots$ *be all positive or all negative. If, for all* $i = 1,2,\cdots$ *with* $(1+x_i)^{c_i} > 1 + c_i x_i$, *then*

$$\prod_{i=1}^n (1+x_i)^{c_i} > 1 + \sum_{i=1}^n c_i x_i, \quad n = 1,2,\cdots.$$

**Proof** This lemma is true for $n = 1$ by assumption. Suppose that this lemma holds for $n = 1,2,\cdots,k$. Let $n = k+1$. If $1 + \sum_{i=1}^n c_i x_i \le 0$, then, clearly, the conclusion holds. Now, we assume $1 + \sum_{i=1}^n c_i x_i > 0$. Since $c_i x_i$, $i = 1,2,\cdots$ are all positive or all negative, we see that $1 + c_i x_i > 0$, $i = 1,2,\cdots,n$. Therefore,

$$\begin{aligned}
\prod_{i=1}^n (1+x_i)^{c_i} &> (1+x_n)^{c_n} \left[1 + \sum_{i=1}^k c_i x_i\right]\\
&> (1 + c_n x_n)\left[1 + \sum_{i=1}^k c_i x_i\right] > 1 + \sum_{i=1}^n c_i x_i.
\end{aligned}$$

 

This completes our proof.

## 3. Variants of Bernoulli's Inequality

In this section, we establish some variants of Bernoulli's Inequality.

Since $\ln x$ is strictly concave and strictly increasing on $(0,\infty)$, its inverse function $e^x$ is strictly convex and strictly increasing. Using Lemma 2.7, we have the following

**Theorem 3.1** *The following inequalities are equivalent*:

$(r_0')$ $-\ln x$ is strictly convex on $(1,\infty)$;

$(r_1')$ $(1+y)^\alpha < 1+\alpha y$, where $0<\alpha<1$ and $0<y$, that is, $x^\alpha < \alpha x + (1-\alpha)$, where $0<\alpha<1$ and $x>1$;

$(r_{1,n}')$ $\prod_{i=1}^{n}(1+y_i)^{c_i} < 1+\sum_{i=1}^{n} c_i y_i$, where $0<c_i<1$, $0<y_i$, $i=1,2,\cdots,n$ satisfy $\sum_{i=1}^{n} c_i = 1$;

$(r_1'')$ $(1+y)^\alpha < 1+\alpha y$, where $0<\alpha<1$ and $-1<y<0$, that is, $x^\alpha < \alpha x + (1-\alpha)$, where $0<\alpha<1$ and $0<x<1$;

$(r_{1,n}'')$ $\prod_{i=1}^{n}(1+y_i)^{c_i} < 1+\sum_{i=1}^{n} c_i y_i$, where $0<c_i<1$, $-1<y_i<0$, $i=1,2,\cdots,n$ satisfy $\sum_{i=1}^{n} c_i = 1$;

$(r_2')$ $(1+y)^\alpha > 1+\alpha y$, where $\alpha>1$ and $0<y$, that is, $x^\alpha > \alpha x + (1-\alpha)$, where $\alpha>1$ and $x>1$;

$(r_{2,n}')$ $\prod_{i=1}^{n}(1+y_i)^{c_i} > 1+\sum_{i=1}^{n} c_i y_i$, where $c_i>1$, $0<y_i$, $i=1,2,\cdots,n$;

$(r_2'')$ $(1+y)^\alpha > 1+\alpha y$, where $\alpha>1$ and $-1<y<0$, that is, $x^\alpha > \alpha x + (1-\alpha)$, where $\alpha>1$ and $0<x<1$;

$(r_{2,n}'')$ $\prod_{i=1}^{n}(1+y_i)^{c_i} > 1+\sum_{i=1}^{n} c_i y_i$, where $c_i>1$, $-1<y_i<0$, $i=1,2,\cdots,n$;

$(r_3')$ $(1+y)^\alpha > 1+\alpha y$, where $\alpha<0$ and $0<y$, that is, $x^\alpha > \alpha x + (1-\alpha)$, where $\alpha<0$ and $x>1$;

$(r_{3,n}')$ $\prod_{i=1}^{n}(1+y_i)^{c_i} > 1+\sum_{i=1}^{n} c_i y_i$, where $c_i<0$, $0<y_i$, $i=1,2,\cdots,n$;

$(r_3'')$ $(1+y)^\alpha > 1+\alpha y$, where $\alpha<0$ and $-1<y<0$, that is, $x^\alpha > \alpha x + (1-\alpha)$, where $\alpha<0$ and $0<x<1$;

$(r_{3,n}'')$ $\prod_{i=1}^{n}(1+y_i)^{c_i} > 1+\sum_{i=1}^{n} c_i y_i$, where $c_i<0$, $-1<y_i<0$, $i=1,2,\cdots,n$;

$(r_4')$ $(1+y)^\alpha > \dfrac{1}{1-\dfrac{\alpha y}{1+y}}$, where $0<\alpha<1$ and $y>0$;

$(r_4'')$ $(1+y)^\alpha > \dfrac{1}{1-\dfrac{\alpha y}{1+y}}$, where $0<\alpha<1$ and $-1<y<0$;

$(r_5')$ $(1+y)^\alpha < \dfrac{1}{1-\dfrac{\alpha y}{1+y}}$, where $\alpha>1$ and $0<y<\dfrac{1}{\alpha-1}$;

$(r_5')$ $(1+y)^\alpha < \dfrac{1}{1-\dfrac{\alpha y}{1+y}}$, where $\alpha>1$ and $-1<y<0$;

$(r_6')$ $(1+y)^\alpha < \dfrac{1}{1-\dfrac{\alpha y}{1+y}}$, where $\alpha<0$ and $y>0$;

$(r_6'')$ $(1+y)^\alpha < \dfrac{1}{1-\dfrac{\alpha y}{1+y}}$, where $\alpha<0$ and $\dfrac{1}{\alpha-1}<y<0$;

 

$(r_7')$ $\left(1+\frac{y}{q}\right)^q < \left(1+\frac{y}{p}\right)^p$, where $p > q > 0$ and $y > 0$;

$(r_7'')$ $\left(1+\frac{y}{q}\right)^q < \left(1+\frac{y}{p}\right)^p$, where $p > q > 0$ and $-q < y < 0$;

$(r_8')$ $\left(1+\frac{y}{q}\right)^q < \left(1+\frac{y}{p}\right)^p$, where $q < p < 0$ and $0 < y < -p$;

$(r_8'')$ $\left(1+\frac{y}{q}\right)^q < \left(1+\frac{y}{p}\right)^p$, where $q < p < 0$ and $y < 0$;

$(r_9')$ $\left(1+\frac{y}{q}\right)^q > \left(1+\frac{y}{p}\right)^p$, where $q < 0 < p$ and $0 < y < -q$;

$(r_9'')$ $\left(1+\frac{y}{q}\right)^q > \left(1+\frac{y}{p}\right)^p$, where $q < 0 < p$ and $-p < y < 0$.

**Proof** Let $\varphi(x) := \frac{-x}{1+x}$, where $x \neq -1$, then $\varphi(\varphi(x)) = x$, $1+\varphi(x) = \frac{1}{1+x}$, $\varphi : (-1,\infty) \to (-1,\infty)$ is a strictly convex function, $\varphi(0,\infty) = (-1,0)$ and $\varphi(-1,0) = (0,\infty)$.

$(r_1') \Leftrightarrow (r_0')$:

$(r_0')$ holds

$\Leftrightarrow -\ln(\alpha x + (1-\alpha)y) < -\alpha \ln(1+x) - (1-\alpha)\ln(1+y)$ for all $x, y > 1$ with $x \neq y$ and for all $0 < \alpha < 1$

$\Leftrightarrow x^{\alpha} y^{1-\alpha} < \alpha x + (1-\alpha) y$ for all $x > y > 1$ and for all $0 < \alpha < 1$

$\Leftrightarrow \left(\frac{x}{y}\right)^{\alpha} \left(\frac{x}{y}\right) + (1-\alpha)$ for all $x > y > 1$ and for all $0 < \alpha < 1$

$\Leftrightarrow z^{\alpha} < 1 - \alpha + \alpha z$ for all $z > 1$ and for all $0 < \alpha < 1$

$\Leftrightarrow (r_1')$ holds.

$(r_1') \Rightarrow (r_{1,n}'')$: Let $n = 2$. WLOG, we assume $y_1 > y_2 > 0$ and $0 < c_1 < 1$. Then

$$(1+y_1)^{c_1}(1+y_2)^{1-c_1} = (1+y_2)\left(1+\frac{y_1}{1+y_2}\right)^{c_1} = (1+y_2)\left(1+\frac{y_1-y_2}{1+y_2}\right)^{c_1}$$

$$< (1+y_2)\left(1+c_1 \cdot \frac{y_1-y_2}{1+y_2}\right) \text{ by } (r_1'') = 1 + c_1 y_1 + (1-c_1) y_2.$$

Now, we assume $(r_{1,m}')$ holds for $m = 1,2,\cdots,n$ $(n \geq 2)$. Set $m = n+1$. We have for $y_1, y_2, \cdots, y_m > 0$ and $c_1, c_2, \cdots, c_m > 0$ with $\sum_{i=1}^{m} c_i = 1$. Let $r = 1 - c_1$. It follows from above argument and the induction assumption that

$$\prod_{i=1}^{m}(1+y_i)^{c_i} = (1+y_1)^{c_1}\left(\prod_{i=2}^{m}(1+y_i)^{\frac{c_i}{r}}\right)^r < (1+y_1)^{c_1}\left(1+\sum_{i=2}^{m}\frac{c_i y_i}{r}\right)^r < 1 + c_1 y_1 + \sum_{i=2}^{m} c_i y_i.$$

This proves $(r_{1,n}'')$. $(r_{1,n}') \Leftrightarrow (r_1')$ is obvious.

Moreover, it follows from Lemma 2.7 that $(r_2') \Leftrightarrow (r_{2,n}')$, $(r_2'') \Leftrightarrow (r_{2,n}'')$, $(r_3') \Leftrightarrow (r_{3,n}')$ and $(r_3'') \Leftrightarrow (r_{3,n}'')$.

$(r_1') \Leftrightarrow (r_1'')$: By $\varphi(\varphi(x)) = x$, $x \neq -1$ and $\varphi(0,\infty) = (-1,0)$,

$(r_1')$ holds

$\Leftrightarrow (1+x)^{1-\alpha} < 1 + (1-\alpha)x$, $0 < \alpha < 1$ and $x > 0$

$\Leftrightarrow \left(1+\varphi(x)\right)^{\alpha-1} < 1+(1-\alpha)x = 1+(1-\alpha)\dfrac{-\varphi(x)}{1+\varphi(x)}, \quad 0<\alpha<1$ and $x>0$

$\Leftrightarrow \left(1+\varphi(x)\right)^{\alpha} < 1+\varphi(x)+(1-\alpha)\left(-\varphi(x)\right) = 1+\alpha\varphi(x), 0<\alpha<1$ and $x>0$

$\Leftrightarrow (r_1'')$ holds.

$(r_2'') \Leftrightarrow (r_1'')$:

$(r_2'')$ holds

$\Leftrightarrow (1+x)^{\alpha} > 1+\alpha x$, where $\alpha>1$ and $-1<x<0$

$\Leftrightarrow (1+x)^{\alpha} > 1+\alpha x$, where $\alpha>1$ and $-\dfrac{1}{\alpha}<x<0$

$\Leftrightarrow \left(1+\dfrac{y}{\alpha}\right)^{\alpha} > 1+y$, where $\alpha>1$ and $y:=\alpha x\in(-1,0)$

$\Leftrightarrow 1+\dfrac{y}{\alpha} > (1+y)^{\frac{1}{\alpha}}$, where $\alpha>1$ and $-1<y<0$

$\Leftrightarrow (r_1'')$ holds.

$(r_2') \Leftrightarrow (r_1')$: $(r_2')$ holds

$\Leftrightarrow (1+x)^{\alpha} > 1+\alpha x$, where $\alpha>1$ and $x>0$

$\Leftrightarrow \left(1+\dfrac{y}{\alpha}\right)^{\alpha} > 1+y$, where $\alpha>1$ and $y:=\alpha x\in(0,\infty)$

$\Leftrightarrow 1+\dfrac{y}{\alpha} > (1+y)^{\frac{1}{\alpha}}$, where $\alpha>1$ and $y>0$

$\Leftrightarrow (r_1')$ holds.

$(r_3') \Leftrightarrow (r_2'')$: $(r_3')$ holds

$\Leftrightarrow (1+x)^{\alpha} > 1+\alpha x$, where $\alpha<0$ and $x>0$

$\Leftrightarrow (1+x)^{\alpha-1} > \dfrac{1+x+(\alpha-1)x}{1+x}$, where $\alpha<0$ and $x>0$

$\Leftrightarrow \left(1+\varphi(x)\right)^{1-\alpha} > 1+(1-\alpha)\varphi(x)$, where $\alpha<0$ and $x>0$

$\Leftrightarrow (r_2'')$ holds.

$(r_3'') \Leftrightarrow (r_2')$: It follows from $1+x=\left(1+\varphi(x)\right)^{-1}$ that

$(r_3'')$ holds

$\Leftrightarrow (1+x)^{\alpha} > 1+\alpha x$, where $\alpha<0$ and $-1<x<0$

$\Leftrightarrow (1+x)^{\alpha-1} > \dfrac{1+x+(\alpha-1)x}{1+x} = 1+(1-\alpha)\varphi(x)$, where $\alpha<0$ and $-1<x<0$

$\Leftrightarrow \left(1+\varphi(x)\right)^{1-\alpha} > 1+(1-\alpha)\varphi(x)$, where $\alpha<0$ and $-1<x<0$

$\Leftrightarrow (r_2')$ holds.

$(r_4') \Leftrightarrow (r_4'')$: $(r_4')$ holds

$\Leftrightarrow (1+x)^{\alpha} > \dfrac{1}{1-\alpha\dfrac{x}{1+x}}$, where $0<\alpha<1$ and $x>0$

$\Leftrightarrow \left(1+\varphi(x)\right)^{-\alpha} > 11+\alpha\varphi(x)$, where $0<\alpha<1$ and $x>0$

$\Leftrightarrow (1+\varphi(x))^{1-\alpha} > \frac{1+\varphi(x)}{1+\alpha\varphi(x)} = \frac{1}{1-(1-\alpha)\frac{\varphi(x)}{1+\varphi(x)}}$, where $0<\alpha<1$ and $x>0$

$\Leftrightarrow (r_4'')$ holds.

$(r_4') \Leftrightarrow (r_1')$: $(r_4')$ holds

$\Leftrightarrow (1+x)^{\alpha} > \frac{1}{1-\alpha\frac{x}{1+x}} = \frac{1+x}{1+(1-\alpha)x}$, where $0<\alpha<1$ and $x>0$

$\Leftrightarrow (1+x)^{1-\alpha} < 1+(1-\alpha)x$, where $0<\alpha<1$ and $x>0$

$\Leftrightarrow (r_1')$ holds.

$(r_5') \Leftrightarrow (r_3')$: $(r_5')$ holds

$\Leftrightarrow (1+x)^{\alpha} < \frac{1}{1+\alpha\varphi(x)}$, where $\alpha>1$ and $0<x<\frac{1}{\alpha-1}$

$\Leftrightarrow (1+x)^{\alpha-1} < \frac{1}{1+(1-\alpha)x}$, where $\alpha>1$ and $0<x<\frac{1}{\alpha-1}$

$\Leftrightarrow (1+x)^{\beta} > 1+\beta x$, where $\beta := 1-\alpha \in (-\infty,0)$ and $x>0$

$\Leftrightarrow (r_3')$ holds.

$(r_5'') \Leftrightarrow (r_2')$: It follows from $\varphi(-1,0)=(0,\infty)$ and $1+\varphi(x)=(1+x)^{-1}$, where $x\neq -1$, that $(r_5'')$ holds

$\Leftrightarrow (1+y)^{\alpha} < \frac{1}{1-\alpha\frac{y}{1+y}}$, where $\alpha>1$ and $-1<y<0$

$\Leftrightarrow (1+x)^{-\alpha} = (1+y)^{\alpha} < \frac{1}{1+\alpha\varphi(x)} = \frac{1}{1+\alpha x}$, where $y=\frac{-x}{1+x}$, $\alpha>1$ and $x>0$

$\Leftrightarrow (r_2')$ holds.

$(r_6') \Leftrightarrow (r_5'')$: $(r_6')$ holds

$\Leftrightarrow (1+y)^{\alpha} < \frac{1}{1+\alpha\varphi(y)}$, where $\alpha<0$ and $y>0$

$\Leftrightarrow (1+\varphi(y))^{-1}(1+\varphi(y))^{1-\alpha} = (1+y)^{\alpha} < \frac{1}{1+\alpha\varphi(y)}$, where $\alpha<0$ and $y>0$

$\Leftrightarrow (1+x)^{1-\alpha} < (1+x)\frac{1}{1+x-(1-\alpha)x} = \frac{1}{1-\frac{(1-\alpha)x}{1+x}}$, where $\alpha<0$ and $-1<x:=\varphi(y)<0$

$\Leftrightarrow (r_5'')$ holds.

$(r_6'') \Leftrightarrow (r_3'')$: $(r_6'')$ holds

$\Leftrightarrow (1+y)^{\alpha} < \frac{1}{1-\frac{\alpha y}{1+y}} = \frac{1}{1+\alpha\varphi(y)}$, where $\alpha<0$ and $\frac{1}{\alpha-1}<y<0$

$\Leftrightarrow (1+y)^{\alpha} = (1+\varphi(y))^{-\alpha} < \frac{1}{1+\alpha\varphi(y)}$, where $\alpha<0$ and $\frac{1}{\alpha-1}<y<0$

$\Leftrightarrow (1+x)^{-\alpha} < \frac{1}{1+\alpha x}$, where $\alpha<0$ and $\varphi\left(\frac{1}{\alpha-1}\right) = \frac{-1}{\alpha} < x := \varphi(y) < 0$

$\Leftrightarrow (1+x)^{\alpha} > 1+\alpha x$, where $\alpha<0$ and $\frac{-1}{\alpha}<x<0$

$\Leftrightarrow (r_3'')$ holds.

$(r_7') \Leftrightarrow (r_1') : (r_7')$ holds

$$\Leftrightarrow \left(1+\frac{y}{q}\right)^q < \left(1+\frac{y}{p}\right)^p, \text{ where } p > q > 0 \text{ and } y > 0$$

$$\Leftrightarrow (1+x)^q < \left(1+\frac{q}{p}x\right)^p, \text{ where } p > q > 0 \text{ and } x := \frac{y}{q} \in (0,\infty)$$

$$\Leftrightarrow (1+x)^\alpha < 1+\alpha x, \text{ where } \alpha := \frac{q}{p} \in (0,\infty) \text{ and } x > 0$$

$\Leftrightarrow (r_1')$ holds.

$(r_7'') \Leftrightarrow (r_1') : (r_7'')$ holds

$$\Leftrightarrow \left(1+\frac{y}{q}\right)^q < \left(1+\frac{y}{p}\right)^p, \text{ where } p > q > 0 \text{ and } -q < y < 0$$

$$\Leftrightarrow (1+x)^q < \left(1+\frac{q}{p}x\right)^p, \text{ where } p > q > 0 \text{ and } x := \frac{y}{q} \in (-1,0)$$

$$\Leftrightarrow (1+x)^\alpha < 1+\alpha x, \text{ where } \alpha := \frac{q}{p} \in (0,1), -1 < x < 0$$

$\Leftrightarrow (r_1'')$ holds.

$(r_8') \Leftrightarrow (r_1'') : (r_8')$ holds

$$\Leftrightarrow \left(1+\frac{y}{q}\right)^q < \left(1+\frac{y}{p}\right)^p, \text{ where } q < p < 0 \text{ and } 0 < y < -p$$

$$\Leftrightarrow \left(1+\frac{p}{q}x\right)^q < (1+x)^p, \text{ where } q < p < 0 \text{ and } x := \frac{y}{q} \in (-1,0)$$

$$\Leftrightarrow (1+x)^\alpha < 1+\alpha x, \text{ where } \alpha := \frac{q}{p} \in (0,1) \text{ and } -1 < x < 0$$

$\Leftrightarrow (r_1'')$ holds.

$(r_8'') \Leftrightarrow (r_2') : (r_8'')$ holds

$$\Leftrightarrow \left(1+\frac{y}{q}\right)^q < \left(1+\frac{y}{p}\right)^p, \text{ where } q < p < 0 \text{ and } y < 0$$

$$\Leftrightarrow (1+x)^q < \left(1+\frac{q}{p}x\right)^p, \text{ where } q < p < 0 \text{ and } x := \frac{y}{q} \in (0,\infty)$$

$$\Leftrightarrow (1+x)^\alpha > 1+\alpha x, \text{ where } \alpha := \frac{q}{p} \in (1,\infty) \text{ and } x > 0$$

$\Leftrightarrow (r_2')$ holds.

$(r_9') \Leftrightarrow (r_3'') : (r_9')$ holds

$$\Leftrightarrow \left(1+\frac{y}{q}\right)^q < \left(1+\frac{y}{p}\right)^p, \text{ where } q < 0 < p \text{ and } 0 < y < -q$$

$$\Leftrightarrow \left(1+\frac{y}{q}\right)^{q/p} > 1+\frac{y}{p} = 1+\frac{q}{p}\frac{y}{q}, \text{ where } q < 0 < p \text{ and } 0 < y < -q$$

$$\Leftrightarrow (1+x)^\alpha > 1+\alpha x, \text{ where } \alpha := \frac{q}{p} < 0 \text{ and } -1 < x := \frac{y}{q} < 0$$

 

$\Leftrightarrow (r_3'')$ holds.

$(r_9'') \Leftrightarrow (r_3''):(r_9'')$ holds

$\Leftrightarrow \left(1+\frac{y}{p}\right)^p < \left(1+\frac{p}{q}\frac{y}{p}\right)^q$, where $q<0<p$ and $-p<y<0$

$\Leftrightarrow (1+x)^{p/q} > 1+\frac{p}{q}x$, where $q<0<p$ and $-1<x:=\frac{y}{q}<0$

$\Leftrightarrow (1+x)^\alpha > 1+\alpha x$, where $\alpha<0$ and $-1<x<0$

$\Leftrightarrow (r_3'')$ holds.

This prove our Theorem.

By Theorem 3.1, we have the following

**Corollary 3.2** Let $\alpha$ be a constant. If $x>0$ and $x \neq 1$, then the following three inequalities are equivalent:

1) $x^\alpha - 1 > \alpha(x-1), \alpha > 1$,

2) $x^\alpha - 1 < \alpha(x-1), 0<\alpha<1$,

3) $x^\alpha - 1 > \alpha(x-1), \alpha < 0.$

**Proof** Clearly, it follows from Theorem 3.1 that

1) holds $\Leftrightarrow$ $(r_2')$ and $(r_2'')$ hold;

2) holds $\Leftrightarrow$ $(r_1')$ and $(r_1'')$ hold;

3) holds $\Leftrightarrow$ $(r_3')$ and $(r_3'')$ hold.

## 4. Main Results

Now, we can state and prove some inequalities which are equivalent to each other in the following

**Theorem 4.1** *Let* $a_i, p_i, q_i, t_i, a, b \in (0,\infty)$, $i=1,2,\cdots,n$, *and* $\sum_{i=1}^n q_i = 1$, *where n is a positive integer. Then the following some statements are equivalent*:

$(R_1)$ $(1+y)^\alpha \le 1+\alpha y$, where $0<\alpha<1$ and $y>-1$;

$(R_1^a)$ $(1+x)^\alpha \le 1+\frac{\alpha x}{1+x}$, where $-1<\alpha<0$ and $x>-1$;

$(R_1^b)$ $x^\alpha \le \alpha x + 1 - \alpha$, where $0<\alpha<1$ and $x>0$;

$(R_{1,n})$ $\prod_{i=1}^n (1+y_i)^{c_i} \le 1+\sum_{k=1}^n c_k y_k$, where $y_i>-1, c_i \ge 0$, $i=1,2,\cdots,n$ and $\sum_{i=1}^n c_i \le 1$;

$(R_{1,n}^a)$ $\prod_{i=1}^n (1+y_i)^{c_i} \le 1+\sum_{i=1}^n \frac{c_i y_i}{1+y_i}$, where $y_i>-1$ and $-1 \le c_i \le 0$, $i=1,2,\cdots,n$, $\sum_{i=1}^n c_i \ge -1$;

$(R_2)$ $(1+y)^\alpha \ge 1+\alpha y$, where $\alpha>1$ and $y>-1$;

$(R_{2,n})$ $\prod_{i=1}^n (1+x_i)^{c_i} \ge 1+\sum_{i=1}^n c_i x_i$, where $c_i>1$ and $x_1,x_2,\cdots,x_n>0$ or $-1<x_1,x_2,\cdots,x_n<0$;

$(R_2^a)$ $(1+x)^\alpha \ge 1+\frac{\alpha x}{1+x}$, where $\alpha<-1$ and $x>-1$;

$(R_2^b)$ $x^\alpha \ge \alpha x + 1 - \alpha$, where $\alpha>1$ and $x>0$;

$(R_{2,n}^a)$ $\prod_{i=1}^n (1+y_i)^{c_i} \ge 1+\sum_{i=1}^n \frac{c_i y_i}{1+y_i}$, where $y_i>-1$ and $c_i \le -1$, $i=1,2,\cdots,n$;

$(R_3)$ $(1+y)^\alpha \ge 1+\alpha y$, where $\alpha<0$ and $y>-1$;

$(R_{3,n})$ $\prod_{i=1}^n (1+x_i)^{c_i} \ge 1+\sum_{i=1}^n c_i x_i$, where $c_i<0$ and $x_1,x_2,\cdots,x_n>0$ or $-1<x_1,x_2,\cdots,x_n<0$;

$(R_3^a)$ $(1+x)^\alpha \ge 1+\frac{\alpha x}{1+x}$, where $\alpha>0$ and $x>-1$;

$(R_3^b)$ $x^\alpha \ge \alpha x + 1 - \alpha$, where $\alpha<0$ and $x>0$;

$(R_{3,n}^a)$ $\prod_{i=1}^{n}(1+y_i)^{c_i} \ge 1+\sum_{i=1}^{n}\frac{c_i y_i}{1+y_i}$, where $y_i > -1$ and $c_i \ge 0$, $i = 1,2,\cdots,n$;

$(R_4)$ $(1+y)^\alpha \ge \frac{1}{1-\frac{\alpha y}{1+y}}$, where $0<\alpha<1$ and $y>-1$;

$(R_{4,n})$ $\prod_{i=1}^{n}(1+y_i)^{c_i} \ge \frac{1}{1-\sum_{k=1}^{n}\frac{c_k y_k}{1+y_k}}$, where $y_i > -1, c_i \ge 0$, $i = 1,2,\cdots,n$, and $\sum_{i=1}^{n} c_i \le 1$;

$(R_5)$ $(1+y)^\alpha \le \frac{1}{1-\frac{\alpha y}{1+y}}$, where $\alpha > 1$ and $-1 < y < \frac{1}{\alpha-1}$;

$(R_6)$ $(1+y)^\alpha \le \frac{1}{1-\frac{\alpha y}{1+y}}$, where $\alpha < 0$ and $y > \frac{1}{\alpha-1}$;

$(R_7)$ $a^\alpha b^{1-\alpha} \le \alpha a + (1-\alpha)b$, where $0<\alpha<1$;

$(R_7^a)$ $-\ln(x)$ is convex on $(0,\infty)$;

$(R_8)$ $a^\alpha b^{1-\alpha} \ge \alpha a + (1-\alpha)b$, where $\alpha > 1$;

$(R_9)$ $a^\alpha b^{1-\alpha} \ge \alpha a + (1-\alpha)b$, where $\alpha < 0$;

$(R_{10})$ $\left(1+\frac{y}{q}\right)^q \le \left(1+\frac{y}{p}\right)^p$, where $p>q>0$ and $-q<y$, hence $\left(1-\frac{y}{q}\right)^q \le \left(1-\frac{y}{p}\right)^p$, where $p>q>0$ and $q>y$. Thus, $\left(1+\frac{y}{q}\right)^q$ is an increasing function of $q$, where $q>-y$ and $y\in R$;

$(R_{11})$ $\left(1+\frac{y}{q}\right)^q \le \left(1+\frac{y}{p}\right)^p$, where $q<p<0$ and $y<-p$; hence $\left(1-\frac{y}{q}\right)^{-q} \ge \left(1-\frac{y}{p}\right)^{-p}$, where $p>q>0$ and $y<q$;

$(R_{12})$ $\left(1+\frac{y}{q}\right)^q \ge \left(1+\frac{y}{p}\right)^p$, where $q<0<p$ and $y<-q$ or $q<0<p$ and $y>-p$;

$(R_{13})$ $\sum_{i=1}^n q_i a_i^\alpha \le \left(\sum_{i=1}^n q_i a_i\right)^\alpha$, where $0<\alpha<1$;

$(R_{14})$ $\sum_{i=1}^n q_i a_i^\alpha \ge \left(\sum_{i=1}^n q_i a_i\right)^\alpha$, where $\alpha>1$;

$(R_{14}^a)$ $\sum_{i=1}^n a_i b_i \le \left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}$ if $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$ (Hölder's inequality);

$(R_{14}^b)$ $\left(\sum_{i=1}^n a_i b_i\right)^2 \le \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$ (Cauchy's inequality);

$(R_{15})$ $\sum_{i=1}^n q_i a_i^\alpha \ge \left(\sum_{i=1}^n q_i a_i\right)^\alpha$, where $\alpha<0$;

$(R_{16})$ $M_r(a,p) \le M_s(a,p)$, where $r<s$,

$$M_r(a,p) := \begin{cases} \left(\sum_{i=1}^n p_i\right)^{-\frac{1}{r}} \left(\sum_{i=1}^n p_i a_i^r\right)^{\frac{1}{r}}, & 0<|r|<\infty, \\ \left(a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}\right)^{\frac{1}{p_1+p_2+\cdots+p_n}}, & r=0, \\ \min\{a_1,a_2,\cdots,a_n\}, & r=-\infty, \\ \max\{a_1,a_2,\cdots,a_n\}, & r=\infty. \end{cases}$$

 

Here $a=(a_1,a_2,\cdots,a_n), p=(p_1,p_2,\cdots,p_n)$. In particular,

$(R_{16}^a)$ $\left(\sum_{i=1}^n q_i a_i^r\right)^{\frac{1}{r}} \le \left(\sum_{i=1}^n q_i a_i^s\right)^{\frac{1}{s}}$, $r<s$ (Schlömich's inequality);

$(R_{16}^b)$ $\left[\sum_{i=1}^n (a_i+b_i)^p\right]^{\frac{1}{p}} \le \left[\sum_{i=1}^n a_i^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^n b_i^p\right]^{\frac{1}{p}}$ for $p>1$; (Minkowski' inequality)

$(R_{16}^c)$ $\left[\sum_{i=1}^n (a_i+b_i)^p\right]^{\frac{1}{p}} \ge \left[\sum_{i=1}^n a_i^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^n b_i^p\right]^{\frac{1}{p}}$ for $p<1$; (Minkowski' inequality)

$(R_{17})$ $M_1(a,q)=\sum_{i=1}^n q_i a_i \ge M_0(a,q)=a_1^{q_1}a_2^{q_2}\cdots a_n^{q_n}$, hence $a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n} \le \sum_{i=1}^n t_i a_i + \left(1-\sum_{i=1}^n t_i\right)$, where $0<\sum_{i=1}^n t_i \le 1$.

In general, $\prod_{i=1}^n a_i^{\frac{p_i}{\sum_{k=1}^n p_k}} \le \dfrac{\sum_{i=1}^n p_i a_i}{\sum_{k=1}^n p_k}$; (AGM inequality)

$(R_{17}^a)$ $\prod_{i=1}^n \left(\dfrac{p_i}{a_i}\right)^{p_i} \ge \left(\dfrac{\sum_{i=1}^n p_i}{\sum_{i=1}^n a_i}\right)^{\sum_{i=1}^n p_i}$;

$(R_{17}^b)$ Shanon's inequality: $\sum_{i=1}^n p_i \ln\dfrac{p_i}{a_i} \ge \left(\sum_{i=1}^n p_i\right)\ln\dfrac{\sum_{i=1}^n p_i}{\sum_{i=1}^n a_i}$;

$(R_{17}^c)$ (see [7]) $\sum_{i=1}^m \beta_i G_i \le \prod_{j=1}^n A_j^{\alpha_j}$, where $\alpha_j,\beta_i \in (0,\infty)$, $\sum_{j=1}^n \alpha_j = \sum_{i=1}^m \beta_i = 1$, $G_i = a_{i1}^{\alpha_1}a_{i2}^{\alpha_2}\cdots a_{in}^{\alpha_n}$, $A_j=\beta_1 a_{1j}+\beta_2 a_{2j}+\cdots+\beta_m a_{mj}$, $i=1,2,\cdots,m;\ j=1,2,\cdots,n$, see the following figure:

| | | | | |
|---|---|---|---|---|
| | $\alpha_1$ | $\alpha_2$ | $\cdots$ | $\alpha_n$ |
| $\beta_1$ | $a_{11}$ | $a_{12}$ | $\cdots$ | $a_{1n}$ |
| $\beta_2$ | $a_{21}$ | $a_{22}$ | $\cdots$ | $a_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\beta_m$ | $a_{m1}$ | $a_{m2}$ | $\cdots$ | $a_{mn}$ |

$(R_{17}^d)$ $\dfrac{1}{m}\sum_{i=1}^m a_{i1}^{\alpha_1}a_{i2}^{\alpha_2}\cdots a_{in}^{\alpha_n} \le \prod_{j=1}^n \left(a_{1j}+a_{2j}+\cdots+a_{mj}\right)^{\alpha_j}$;

$(R_{17}^e)$ $\sqrt[n]{A_1A_2\cdots A_n} \ge \dfrac{G_1+G_2+\cdots+G_n}{n}$, where $A_j=\dfrac{1}{m}\sum_{i=1}^m a_{ij}, G_i=\left(\prod_{j=1}^n a_{ij}\right)^{\frac{1}{n}}$.

$(R_{18})$ $A_n := a_1+a_2+\cdots+a_n n \ge \sqrt[n]{a_1a_2\cdots a_n}=G_n$, which is equivalent to $G_n \ge H_n$;

$(R_{18}^a)$ $\sqrt{ab} \le \dfrac{a+b}{2}$;

$(R_{18}^b)$ $a^{\frac{m}{n}}b^{\frac{n-m}{n}} \le \dfrac{m}{n}a+\dfrac{n-m}{n}b$, where $m,n$ are positive integers, $m<n$ and $n\ge 2$, that is, $a^r b^{1-r} \le ra+(1-r)b$, where $0<r<1$ is a rational number;

$(R_{18}^c)$ $x-1 \ge n\left(x^{\frac{1}{n}}-1\right)$, where $x\ge 0$ and $n(\text{a positive integer})\ge 2$;

$(R_{18}^d)$ $\dfrac{n}{\sqrt[n]{x}}+x \ge n+1$, where $x>0$;

$(R_{18}^e)$ $e^x > x^e$, $x\ne e$, $x>0$;

$(R_{19}^a)$ $\left(1+\dfrac{1}{x}\right)^x$ is (strictly) increasing on $(0,\infty)$;

$\left(R_{19}^{b}\right)$ $\left(1-\frac{1}{x}\right)^{x}$ is (strictly) increasing on $(1,\infty)$;

$\left(R_{19}^{c}\right)$ $\left(1+\frac{1}{x}\right)^{x+1}$ is (strictly) decreasing on $(0,\infty)$;

$\left(R_{19}^{d}\right)$ $\left(1+\frac{1}{x}\right)^{x} < \mathrm{e} < \left(1+\frac{1}{x}\right)^{x+1}$, $x>0$, where $\mathrm{e}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$;

$\left(R_{20}\right)$ $\mathrm{e}^{x} \ge 1+x$, where $x\in R$, it has following some variants:

$\left(a_{1}\right)$ $\mathrm{e}^{x-1} \ge x$, where $x\in R$

$\left(a_{2}\right)$ $x-1 \ge \ln x$, where $x>0$,

$\left(a_{3}\right)$ $\mathrm{e}^{-x} < \frac{1}{1+x}$, where $x>-1$,

$\left(a_{4}\right)$ $\mathrm{e}^{x}-1 < \frac{x}{1-x}$ or $\mathrm{e}^{x} < \frac{1}{1-x}$, where $x<1$,

$\left(a_{5}\right)$ $\mathrm{e}^{\frac{x}{1+x}} \le 1+x$, where $x>-1$,

$\left(a_{6}\right)$ $\mathrm{e}^{\frac{x}{x-1}} \le 1-x$, where $x<1$,

$\left(a_{7}\right)$ $\frac{x}{1+x} \le \ln(1+x)$, where $x>-1$,

$\left(a_{8}\right)$ $\frac{x}{x-1} \le \ln(1-x)$, where $x<1$,

$\left(a_{9}\right)$ $x\ln x \ge x-1$, where $x>0$,

$\left(a_{10}\right)$ $\frac{1}{1+x} \le \ln\frac{x+1}{x}$, where $x<-1$ or $x>0$,

$\left(a_{11}\right)$ $\mathrm{e}^{x} \ge \left(\frac{x+a}{a}\right)^{a}$, where $x>-a$, $a>0$,

$\left(a_{12}\right)$ $a\ln\frac{x+a}{a} \ge \frac{xa}{x+a}$, where $x>-a$, $a>0$,

$\left(a_{13}\right)$ $\frac{x-1}{x} \le \ln x$, where $x>0$,

$\left(a_{14}\right)$ $x\mathrm{e}^{y} \le \mathrm{e}^{x}+\mathrm{e}^{y}(y-1)$, where $x,y\in R$,

$\left(a_{15}\right)$ $xy \le y\ln y - y + \mathrm{e}^{x}$, where $x\in R$ and $y>0$;

$\left(R_{20}^{a}\right)$ $\ln(x+1) \le x$, where $x>-1$, that is, $1-y \le -\ln y$, where $y>0$;

$\left(R_{20}^{b}\right)$ $\ln x \le n\left(x^{\frac{1}{n}}-1\right)$, where $x>0$;

$\left(R_{20}^{c}\right)$ $\frac{1}{1+x} \le \ln\left(\frac{x+1}{x}\right)$, where $x<-1$ or $x>0$; that is, $\frac{y-1}{y} \le \ln y \le y-1$, $y>0$;

$\left(R_{20}^{d}\right)$ $\mathrm{e}^{x} \ge \mathrm{e}x$, where $x>0$; that is $\mathrm{e}^{x} \ge \left(\frac{\mathrm{e}x}{a}\right)^{a}$, $x>0$ and $a>0$;

$\left(R_{20}^{e}\right)$ $\mathrm{e}^{x} \le \frac{1}{1-x}$, where $x<1$;

$\left(R_{21}\right)$ $(1+x)^{n} \ge 1+nx$, where $x\ge -1$;

$\left(R_{21}^{a}\right)$ $y^{n}+n-1 \ge ny$, where $y>0$, that is, $a^{n}+(n-1)b^{n} \ge nab^{n-1}$ (Jacobsthal's inequality);

$(R_{21}^b)$ $(1+x_1)^{c_1}(1+x_2)^{c_2}\cdots(1+x_n)^{c_n} > 1+c_1x_1+\cdots+c_nx_n$, where $c_1,c_2,\cdots,c_n \ge 1$ and $x_1,x_2,\cdots,x_n \in (-1,0)$ or $x_1,x_2,\cdots,x_n \in (0,\infty)$.

**Proof** Taking $x = ab$ in Corollary 3.2, we see that $(R_7)$, $(R_8)$ and $(R_9)$ are equivalent. Similarly, replacing $x$ by $1+x$ in Corollary 2.2, we get that $(R_1)$, $(R_2)$ and $(R_3)$ are equivalent. Hence, it follows from Theorem 3.1 that $(R_1)$-$(R_{12})$, $(R_1^a)$, $(R_2^a)$, $(R_3^a)$ and $(R_{19}^a)$ are equivalent. If $y_2 = \cdots = y_n = 0$, then, clearly, $(R_{1,n}) \Rightarrow (R_1)$, $(R_{1,n}^a) \Rightarrow (R_1^a)$, $(R_{3,n}) \Rightarrow (R_3)$, $(R_{3,n}^a) \Rightarrow (R_3^a)$, $(R_{4,n}) \Rightarrow (R_4)$.

$(R_i) \Leftrightarrow (R_i^b)$ with $i$ = 1, 2, 3 follows by taking $y = x + 1$.

$(R_1) \Leftrightarrow (R_1^a)$: We see that $\phi(x) = \dfrac{-x}{1+x} > -1$ iff $x > -1$. Hence

$(R_1)$ holds

$$\Leftrightarrow \left(1-\frac{x}{1+x}\right)^{\alpha} \le 1+\frac{-\alpha x}{1+x}, 0<\alpha<1, x>-1$$

$$\Leftrightarrow (1+x)^{-\alpha} \le 1+\frac{-\alpha x}{1+x}, 0<\alpha<1, x>-1$$

$\Leftrightarrow (R_1^a)$ holds

Similarly, we can prove $(R_2) \Leftrightarrow (R_2^a), (R_3) \Leftrightarrow (R_3^a)$.

$(R_1) \Leftrightarrow (R_{1,n})$ follows from $(r_1') \Leftrightarrow (r_{1,n}')$ and $(r_1'') \Leftrightarrow (r_{1,n}'')$ in Theorem 3.1.

$(R_2) \Leftrightarrow (R_{2,n})$ and $(R_3) \Leftrightarrow (R_{3,n})$ follows from Theorem 3.1 too.

$(R_{1,n}) \Rightarrow (R_{4,n})$: Let $c_i \ge 0$, $i = 1,2,\cdots,n$ satisfy $\sum_{i=1}^{n} c_i \le 1$. By $y_i > -1$, we see that $-\dfrac{y_i}{1+y_i} > -1$. Thus, it follows from $(R_{1,n})$ that

$$\prod_{i=1}^{n}(1+y_i)^{-c_i} = \prod_{i=1}^{n}\left(1+\frac{-y_i}{1+y_i}\right)^{c_i} \le 1-\sum_{i=1}^{n}\frac{c_iy_i}{1+y_i}.$$

Hence $(R_{4,n})$ holds.

$(R_{1,n}) \Rightarrow (R_{1,n}^a)$ by replacing $y_i$ and $c_i$ by $-\dfrac{y_i}{1+y_i}, -c_i$, respectively.

Similarly, we can prove $(R_{1,n}^a) \Rightarrow (R_{1,n})$.

$(R_1^b) \Rightarrow (R_{13})$: Let $0<\alpha<1$, $b_i := \dfrac{a_i}{\sum_{j=1}^{n} q_j a_j}$, where $i = 1,2,\cdots,n$, then $\sum_{i=1}^{n} q_i b_i = 1$. It follows from $(R_1^b)$ that

$$\sum_{i=1}^{n} q_i b_i^{\alpha} \le \sum_{i=1}^{n} q_i\left(1+\alpha(b_i-1)\right) = 1.$$

Hence,

$$\sum_{i=1}^{n} q_i\left(\frac{a_i}{\sum_{j=1}^{n} q_j a_j}\right)^{\alpha} \le 1.$$

This completes the proof of $(R_{13})$.

$(R_2) \Rightarrow (R_{16})$, see Hardy etc. ([8], Theorem 9, 11 and 16).

$(R_2) \Rightarrow (R_{18})$: It follows from $(R_2)$ and $\dfrac{A_n}{A_{n-1}} - 1 > -1$ that

$$\left(\frac{A_n}{A_{n-1}}\right)^n = \left[1+\left(\frac{A_n}{A_{n-1}}-1\right)\right]^n$$
$$\ge 1+n\left(\frac{A_n}{A_{n-1}}-1\right) = \frac{a_n}{A_{n-1}}.$$

Thus, $A_n^n \ge a_n A_{n-1}^{n-1}$ (see Maligrands [18] or Rooin [28]). Hence,

$$A_n^n \ge a_n A_{n-1}^{n-1} \ge a_n a_{n-1} A_{n-2}^{n-2} \ge \cdots \ge a_n a_{n-1} \cdots a_2 a_1 = G_n^n.$$

Therefore, $(R_{18})$ holds.

$(R_2) \Rightarrow (R_{18}^d)$: Taking $\alpha = n+1$ and $y = \sqrt[n]{x} - 1$ in $(R_2)$, we see that

$$x^{\frac{n+1}{n}} \ge 1+(n+1)\left(\sqrt[n]{x}-1\right).$$

Hence

$$n + x\sqrt[n]{x} \ge (n+1)\sqrt[n]{x}.$$

Dividing both sides by $\sqrt[n]{x}$, we get $(R_{18}^d)$.

$(R_2) \Rightarrow (R_{21})$ is clear.

$(R_2) \Rightarrow (R_{21}^b)$: We show $(R_{21}^b)$ by mathematical induction on $n$. If $n = 1$, then $(R_{21}^b)$ is obvious by $(R_2)$. Suppose $(R_{21}^b)$ holds for $n = 1,2,\cdots,m$ with $(m \ge 1)$. Set $n = m + 1$. If $1+\sum_{k=1}^{m} c_k x_k < 0$, it is easy to see that each $x_k \in (-1,0)$ by the assumption, and hence $1+\sum_{k=1}^{n} c_k x_k < 0$. Therefore $(R_{21}^b)$ holds. Assume $1+\sum_{k=1}^{m} c_k x_k \ge 0$. Since $x_j x_k > 0$, we have

$$\prod_{k=1}^{n}(1+y_k)\ge\left(1+\sum_{k=1}^{m}c_kx_k\right)(1+c_nx_n)$$
$$=1+\sum_{k=1}^{n}c_kx_k+c_nx_n\sum_{k=1}^{m}c_kx_k$$
$$>1+\sum_{k=1}^{n}c_kx_k.$$

Thus $(R_{21}^b)$ holds.

$(R_3)\Rightarrow(R_{18}^c)$ : Taking $\alpha=-\frac{1}{n}$ and $1+y=x$ in $(R_3)$, we see that $(R_{18}^c)$ holds.

$(R_3^a)\Rightarrow(R_{3,n}^a)$: Clearly, $(R_3)\Leftrightarrow(R_3^a)$. Let $c_i\ge 0$, $i=1,2,\cdots,n$ such that $\gamma:=\sum_{i=1}^{n}c_i>0$. If $\alpha_i:=\frac{c_i}{\gamma}$, then it follows from $(R_{1,n}^a)$ that

$$\prod_{i=1}^{n}(1+y_i)^{-\alpha_i}\le 1+\sum_{i=1}^{n}\frac{-\alpha_iy_i}{1+y_i}. \quad (5)$$

By $y_i>-1$ and $-\frac{y_i}{1+y_i}=\frac{1}{1+y_i}-1>-1$, $i=1,2,\cdots,n$, $\sum_{i=1}^{n}\frac{-\alpha_iy_i}{1+y_i}>-1$. Clearly, $-\gamma\le 0$. By (5) and $(R_3)$,

$$\prod_{i=1}^{n}(1+y_i)^{c_i}=\left[\prod_{i=1}^{n}(1+y_i)^{-\alpha_i}\right]^{-\gamma}\ge\left[1+\sum_{i=1}^{n}\frac{(-\alpha_i)y_i}{1+y_i}\right]^{-\gamma}$$
$$\ge 1+\sum_{i=1}^{n}\frac{c_iy_i}{1+y_i}.$$

Thus, $(R_3^a)\Rightarrow(R_{3,n}^a)$ is proved.

$(R_{3,n}^a)\Rightarrow(R_{3,n})$: Let $c_i\le 0, i=1,2,\cdots,n$. If $y_i>-1$, then $x_i:=\frac{-y_i}{1+y_i}>-1$. By $(R_{3,n}^a)$,

$$\prod_{i=1}^{n}(1+y_i)^{c_i}=\prod_{i=1}^{n}(1+x_i)^{-c_i}\ge 1+\sum_{i=1}^{n}\frac{(-c_i)x_i}{1+x_i}$$
$$=1+\sum_{i=1}^{n}c_iy_i.$$

This completes the proof of $(R_{3,n})$.

$(R_7)\Leftrightarrow(R_7^a)$: Without loss of generality, we my assume that $x, y>0$ and $t\in[0,1]$. Since $\ln(x)$ is strictly increasing,

$$x^ty^{1-t}\le tx+(1-t)y$$
$$\Leftrightarrow t\ln x+(1-t)\ln y\le\ln(tx+(1-t)y)$$

By the definition of the convex function, $(R_7)\Leftrightarrow(R_7^a)$.

$(R_7)\Rightarrow(R_1^b)$: Taking $a=x$ and $b=1$ in $(R_7)$, we see that $(R_1^b)$ holds.

$(R_7)\Rightarrow(R_{19}^a)$: Let $y>x>0$. Then, by $(R_7)$,

$$\left(1+\frac{1}{x}\right)^{\frac{x}{y}}=\left(1+\frac{1}{x}\right)^{\frac{x}{y}}\cdot 1^{1-\frac{x}{y}}<\frac{x}{y}\left(1+\frac{1}{x}\right)+1-\frac{x}{y}=1+\frac{1}{y}.$$

$$\left(1+\frac{1}{x}\right)^{x}<\left(1+\frac{1}{y}\right)^{y}.$$

$(R_7)\Rightarrow(R_{19}^b)$: Let $y>x>0$. Then, by $(R_7)$,

$$\left(1-\frac{1}{x}\right)^{\frac{x}{y}}=\left(1-\frac{1}{x}\right)^{\frac{x}{y}}\cdot 1^{1-\frac{x}{y}}<\frac{x}{y}\left(1-\frac{1}{x}\right)+1-\frac{x}{y}=1-\frac{1}{y}.$$

$$\left(1-\frac{1}{x}\right)^{x}<\left(1-\frac{1}{y}\right)^{y}.$$

$(R_7)\Rightarrow(R_{19}^c)$: Let $y>x>0$. Then, by $(R_7)$,

$$\left(1-\frac{1}{x+1}\right)^{\frac{x+1}{y+1}}=\left(1-\frac{1}{x+1}\right)^{\frac{x+1}{y+1}}1^{1-\frac{x+1}{y+1}}$$
$$<\frac{x+1}{y+1}\left(1-\frac{1}{x+1}\right)+\left(1-\frac{x+1}{y+1}\right)\cdot 1$$
$$=1-\frac{1}{y+1}.$$

Hence

$$\left(1-\frac{1}{x+1}\right)^{x+1}<\left(1-\frac{1}{y+1}\right)^{y+1}$$

and so

$$\left(1+\frac{1}{x}\right)^{x+1}>\left(1+\frac{1}{y}\right)^{y+1}.$$

$(R_{13})\Rightarrow(R_{14})$: If $\alpha>1$, then, by $(R_{13})$,

$$\sum_{i=1}^{n}q_ia_i^{\frac{1}{\alpha}}\le\left(\sum_{i=1}^{n}q_ia_i\right)^{\frac{1}{\alpha}}, \alpha>1.$$

Replacing $a_i$ by $a_i^{\alpha}$,

$$\sum_{i=1}^{n}q_ia_i\le\left(\sum_{i=1}^{n}q_ia_i^{\alpha}\right)^{\frac{1}{\alpha}}, \alpha>1.$$

Thus, $(R_{14})$ is proved.
Similarly, we can prove $(R_{14})\Rightarrow(R_{13})$.

$(R_{13})\Rightarrow(R_{16})$: Let $0<\frac{r}{s}<1$, then, by $(R_{13})$,

$$\left(\sum_{i=1}^{n}q_ia_i^{\frac{r}{s}}\right)^{\frac{s}{r}}\le\sum_{i=1}^{n}q_ia_i.$$

Replacing $a_i$ by $a_i^{s}$,

$$\left(\sum_{i=1}^{n} q_i a_i^r\right)^{\frac{s}{r}} \le \sum_{i=1}^{n} q_i a_i^s.$$

Hence

$$M_r(a,q) \le M_s(a,q) \text{ for } 0 < r < s$$

and

$$M_r(a,q) \ge M_s(a,q) \text{ for } 0 > r > s.$$

f $r \to 0^+$, then $M_0(a,q) \le M_s(a,q)$, $0 < s$. If $r \to 0^-$, then $M_0(a,q) \ge M_s(a,q)$, $s < 0$. Hence,

$$M_r(a,q) \le M_s(a,q) \text{ for } r < s.$$

It follows by taking $q_i = p_i \sum_{k=1}^{n} p_k$ that $(R_{16})$ holds.

Similarly, we can prove $(R_{14}) \Rightarrow (R_{16})$, $(R_{15}) \Rightarrow (R_{16})$.

$(R_{14}) \Rightarrow (R_{14}^a)$: Let $A = \sum_{i=1}^{n} a_i^p$. Replacing $a_i$ and $q_i$ by $b_i a_i^{1p-1}$ and $a_i^q A$ in $(R_{14})$, respectively, for $i = 1,2,\cdots,n$, we obtain $(R_{14}^a)$, thus, we complete the proof.

$(R_{14}^a) \Rightarrow (R_{14}^b)$ follows by taking $p = q = 2$ in $(R_{14}^a)$.

$(R_{14}^b) \Rightarrow (R_{14}^a)$: Let $x_1, x_2 \in (0,1)$ and $f(x) = \sum_{i=1}^{n} b_i^q \left(\frac{a_i^p}{b_i^q}\right)^x$ for $x \in (0,1)$. Then, it follows from $(R_{14}^b)$ that

$$F\left(\frac{x_1}{2} + \frac{x_2}{2}\right) = \sum_{i=1}^{n} \left[b_i^q \left(\frac{a_i^p}{b_i^q}\right)^{x_1}\right]^{\frac{1}{2}} \left[b_i^q \left(\frac{a_i^p}{b_i^q}\right)^{x_2}\right]^{\frac{1}{2}}$$

$$\le \left[\sum_{i=1}^{n} b_i^q \left(\frac{a_i^p}{b_i^q}\right)^{x_1}\right]^{\frac{1}{2}} \cdot \left[\sum_{i=1}^{n} b_i^q \left(\frac{a_i^p}{b_i^q}\right)^{x_2}\right]^{\frac{1}{2}}$$

$$= F(x_1)^{\frac{1}{2}} F(x_2)^{\frac{1}{2}}.$$

Thus, $\ln F$ is midconvex on $(0,1)$, and hence $\ln F$ is convex on $(0,1)$. Hence, for any $r \in (0,1)$,

$$\ln F\left(\frac{r}{p} + \frac{1-r}{q}\right) \le \frac{1}{p} \ln F(r) + \frac{1}{q} \ln F(1-r),$$

which implies

$$F\left(\frac{r}{p} + \frac{1-r}{q}\right) \le F^{\frac{1}{p}}(r) F^{\frac{1}{q}}(1-r).$$

Letting $r \to 1^-$ in the both sides of the above inequality,

$$\sum_{i=1}^{n} a_i b_i \le F^{\frac{1}{p}}(1) F^{\frac{1}{q}}(0) = \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}.$$

This shows that $(R_{14}^a)$ holds, see Li and Shaw [15].

$(R_{14}) \Rightarrow (R_{15})$: Let $\alpha > 1$, $a_i > 0$, $q_i \ge 0$, $q_i' := \frac{q_i a_i}{\sum_{j=1}^{n} q_j a_j}$ and $b_i := a_i^{-1}$, $i = 1,2,\cdots,n$, where $\sum_{i=1}^{n} q_i = 1$. Thus $\sum_{i=1}^{n} q_i' = 1$. By $(R_{14})$,

$$\sum_{i=1}^{n} q_i' b_i^\alpha \ge \left(\sum_{i=1}^{n} q_i' b_i\right)^\alpha = \frac{1}{\left(\sum_{i=1}^{n} q_i' a_i\right)^\alpha}. \quad (6)$$

It follows from (6) and

$$\sum_{i=1}^{n} q_i' b_i^\alpha = \sum_{i=1}^{n} q_i a_i^{1-\alpha} \Big/ \sum_{j=1}^{n} q_j a_j,$$

that

$$\sum_{i=1}^{n} q_i a_i^{1-\alpha} \le \left(\sum_{i=1}^{n} q_i a_i\right)^{1-\alpha}.$$

Hence, $(R_{15})$ holds.

$(R_{15}) \Rightarrow (R_{14})$: Let $\alpha > 1$. Then, by $(R_{15})$,

$$\sum_{i=1}^{n} q_i' b_i^{1-\alpha} \ge \left(\sum_{i=1}^{n} q_i' b_i\right)^{1-\alpha},$$

where $q'$ and $a_i$ are defined as above. Hence,

$$\frac{\sum_{i=1}^{n} q_i a_i^\alpha}{\left(\sum_{j=1}^{n} q_j a_j\right)} \ge \left(\frac{\sum_{i=1}^{n} q_i}{\sum_{j=1}^{n} q_i a_i}\right)^{1-\alpha}.$$

Thus,

$$\sum_{j=1}^{n} q_j a_j^\alpha \ge \left(\sum_{j=1}^{n} q_j a_j\right)^\alpha.$$

This completes the proof of $(R_{14})$.

$(R_{16}) \Rightarrow (R_{14}^a)$: Taking $r = 1, s = p, p_i = b_i^q$ and replacing $a_i$ by $\left(\frac{a_i^p}{b_i^q}\right)^{\frac{1}{p}}$ in $(R_{16})$, for $i = 1,2,\cdots,n$, we obtain $(R_{14}^a)$.

$(R_{16}) \Rightarrow (R_{16}^a)$ follows by taking $p_i = q_i$ in $(R_{16})$ for $i = 1,2,\cdots,n$.

$(R_{16}) \Rightarrow (R_{16}^b)$: Taking $r = 1, s = p, p_i = (a_i + b_i)^p$ and replacing $a_i$ by $\frac{a_i}{a_i + b_i}$ in $(R_{16})$, for $i = 1,2,\cdots,n$, thus we complete the proof.

$(R_{14}^a) \Rightarrow (R_{16}^a)$: see p. 55 of Mitrinovic [19]. Similarly, we can prove $(R_{14}^a) \Rightarrow (R_{16}^b)$.

$(R_{16}^a) \Rightarrow (R_{13})$ follows by taking $0 < r = \alpha < s = 1$ in $(R_{16}^a)$.

$(R_{16}^a) \Rightarrow (R_{14})$ follows by taking $r = 1 < s = \alpha$ in $(R_{16}^a)$.

$(R_{16}^a) \Rightarrow (R_{15})$ follows by taking $r = \alpha < s = 0$ in $(R_{16}^a)$.

$(R_{16}^b) \Rightarrow (R_{18}^a)$ follows by taking $p = n = 2$ in $(R_{16}^b)$.

$(R_{16}^c) \Rightarrow (R_{18}^a)$ follows by taking $n = 2$ and $p = 1/2$ in $(R_{16}^c)$.

$(R_{16}) \Rightarrow (R_{17})$, $(R_{17}) \Rightarrow (R_7)$ (with $n = 2$) and $(R_{17}) \Rightarrow (R_{18})$ (with $q_i = \frac{1}{n}$) are clear.

$(R_{17}) \Leftrightarrow (R_{17}^a)$: Replacing $a_i$ by $\frac{a_i}{p_i}$ in $(R_{17})$, we proved $(R_{17}^a)$. Similarly, we can prove $(R_{17}^a) \Rightarrow (R_{17})$.

$(R_{17}) \Leftrightarrow (R_{3,n})$: Taking $a_i = 1 + y_i$, $c_i = t_i$, $i = 1,2,\cdots,n$, $a_{n+1} = 1$, $y_{n+1} = 0$, and $c_{n+1} = t_{n+1} = 1 - \sum_{i=1}^{n} t_i$, we see easily that both $(R_{17})$ and $(R_{3,n})$ are equivalent.

$(R_{17}) \Rightarrow (R_{17}^c)$: Without loss of generality, we may assume that $A_j > 0$, $j = 1,2,3,\cdots,n$. Thus, by $(R_{17})$,

$$\alpha_1 \frac{a_{11}}{A_1} + \alpha_2 \frac{a_{12}}{A_2} + \cdots + \alpha_n \frac{a_{1n}}{A_n} \ge \left(\frac{a_{11}}{A_1}\right)^{\alpha_1} \left(\frac{a_{12}}{A_2}\right)^{\alpha_2} \cdots \left(\frac{a_{1n}}{A_n}\right)^{\alpha_n},$$

$$\alpha_1 \frac{a_{21}}{A_1} + \alpha_2 \frac{a_{22}}{A_2} + \cdots + \alpha_n \frac{a_{2n}}{A_n} \ge \left(\frac{a_{21}}{A_1}\right)^{\alpha_1} \left(\frac{a_{22}}{A_2}\right)^{\alpha_2} \cdots \left(\frac{a_{2n}}{A_n}\right)^{\alpha_n},$$

$$\alpha_1 \frac{a_{m1}}{A_1} + \alpha_2 \frac{a_{m2}}{A_2} + \cdots + \alpha_n \frac{a_{mn}}{A_n} \ge \left(\frac{a_{m1}}{A_1}\right)^{\alpha_1} \left(\frac{a_{m2}}{A_2}\right)^{\alpha_2} \cdots \left(\frac{a_{mn}}{A_n}\right)^{\alpha_n}.$$

Hence,

$$1 = \sum_{j=1}^{n} \alpha_j \ge \frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}} \sum_{i=1}^{m} \beta_i \left(a_{i1}^{\alpha_1} a_{i2}^{\alpha_2} \cdots a_{in}^{\alpha_n}\right),$$

This completes the proof of $(R_{17}^c)$.

$(R_{17}^a) \Leftrightarrow (R_{17}^b)$: Taking the natural logarithm in the both sides of $(R_{17}^a)$, we get $(R_{17}^b)$. Conversely, deleting the natural logarithm of the both sides of $(R_{17}^b)$, we get $(R_{17}^a)$.

$(R_{17}^a) \Leftrightarrow (R_{18})$: Taking $p_i = 1$ in $(R_{17}^a)$, we get $G_n \ge H_n$.

$(R_{17}^c) \Rightarrow (R_{17}^d)$: Taking $\beta_1 = \beta_2 = \cdots = \beta_m = \frac{1}{m}$ in $(R_{17}^c)$, we see that $(R_{17}^d)$ holds.

$(R_{17}^d) \Rightarrow (R_{17}^e)$: Taking $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$ in $(R_{17}^d)$, we get $(R_{17}^e)$.

$(R_{17}^e) \Rightarrow (R_{18})$: Taking $m = n$, $\alpha_i = \beta_i = \frac{1}{n}$ in $(R_{17}^e)$ and using the following figure, we get $(R_{18})$.

| | | | |
|---|---|---|---|
| $a_1$ | $a_2$ | $\cdots$ | $a_n$ |
| $a_2$ | $a_3$ | $\cdots$ | $a_1$ |
| $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $a_n$ | $a_1$ | $\cdots$ | $a_{n-1}$ |

$(R_{18}) \Rightarrow (R_{18}^a)$ is clear.

$(R_{18}) \Rightarrow (R_{18}^b)$: Taking $a_1 = a_2 = \cdots = a_m = a$ and $a_{m+1} = a_{m+2} = \cdots = a_n = b$ in $(R_{18})$, we get $(R_{18}^b)$.

$(R_{18}) \Rightarrow (R_{21})$ (see [33]): It follows from $(R_{18})$ that $(1+nx)^{\frac{1}{n}} \le \frac{1}{n}\left[(1+nx)+1+\cdots+1\right] = \frac{1}{n}(n+nx) = 1+x.$

$(R_{18}^b) \Rightarrow (R_{18}^c)$ follows by taking $m = n-1$ and $b = ax$ in $(R_{18}^b)$.

$(R_{18}^c) \Rightarrow (R_{21})$: Let $\sqrt[n]{x} = 1+y$. Then, by $(R_{18}^c)$, $\frac{n}{1+y} + (1+y)^n \ge n+1.$

Thus, $n + (1+y)^{n+1} \ge (n+1)(1+y)$. Clearly, if $n = 0$, then the above inequality holds too.

$(R_{18}^d) \Rightarrow (R_{18})$: Clearly, if $n = 1$, then $(R_{18})$ holds.

Suppose that $(R_{18})$ holds for $n = k$. Thus, for $n = k+1$, if $a_1 a_2 \cdots a_{k+1} = 1$, then $a_1 a_2 \cdots a_n = \frac{1}{a_{k+1}}$. Hence $a_1 + a_2 \cdots + a_{k+1} \ge k\sqrt[k]{a_1 a_2 \cdots a_k} + a_{k+1} \ge k\sqrt[k]{a_{k+1}}$. This and $(R_{18}^d)$ complete the proof of $(R_{18})$.

$(R_{18}^e) \Rightarrow (R_{17})$: If $a_1^{q_1} a_2^{q_2} \cdots a_n^{q_n} = \mathrm{e}\alpha$, then $\prod_{i=1}^{n} \left(\frac{a_i}{\alpha}\right)^{q_i} = \mathrm{e}$. Therefore, by $(R_{18}^e)$,

$$\frac{a_i}{\alpha} \ge \mathrm{e} \ln \frac{a_i}{\alpha}, i = 1, 2, \cdots, n.$$

Thus,

$$\sum_{i=1}^{n} q_i \frac{a_i}{\alpha} \ge \mathrm{e} \sum_{i=1}^{n} \left[ q_i \ln \frac{a_i}{\alpha} \right] = \mathrm{e}.$$

Thus, $(R_{17})$ holds.

$(R_{19}^a) \Leftrightarrow (R_2)$: Let $\alpha > 1$. Then $\alpha x > x$, $x > 0$. Thus,

$(R_{19}^a)$ holds, that is, $\left(1+\frac{1}{x}\right)^x$ is a strictly increasing function on $(0,\infty)$.

$\Leftrightarrow \left(1+\frac{1}{\alpha x}\right)^{\alpha x} > \left(1+\frac{1}{x}\right)^x$, where $\alpha > 1$ and $x > 0$

$\Leftrightarrow \left(1+\frac{1}{\alpha x}\right)^{\alpha} > 1+\frac{1}{x} = 1+\alpha\frac{1}{\alpha x}$, $\alpha > 1$ $x > 0$

$\Leftrightarrow (R_2)$ holds by Theorem 1.

$(R_{19}^b) \Leftrightarrow (R_2)$: $(R_{19}^b)$ holds, that is, $\left(1-\frac{1}{x}\right)^x$ is a strictly increasing function on $(1,\infty)$.

$\Leftrightarrow \left(1-\frac{1}{\alpha x}\right)^{\alpha x} > \left(1-\frac{1}{x}\right)^x$, where $\alpha > 1$ and $x > 1$

$\Leftrightarrow \left(1-\frac{1}{\alpha x}\right)^{\alpha} > 1-\frac{1}{x} = 1-\alpha\frac{1}{\alpha x}$, $\alpha > 1$, $x > 1$

$\Leftrightarrow (1+y)^{\alpha} > 1+\alpha y$, $\alpha > 1$, $y := -\frac{1}{\alpha x} \in (-1,0)$

$\Leftrightarrow (R_2)$ holds by Theorem 3.1.

$(R_{19}^b) \Leftrightarrow (R_{19}^c)$:

$(R_{19}^b)$ holds, that is, $\left(1-\frac{1}{x}\right)^x$ is a strictly increasing function on $(1,\infty)$.

$\Leftrightarrow \left(\frac{x}{x-1}\right)^x$ is a strictly increasing function on $(1,\infty)$.

$\Leftrightarrow \left(\frac{x+1}{x}\right)^{x+1}$ is a strictly increasing function on $(0,\infty)$.

$\Leftrightarrow (R_{19}^c)$ holds.

Thus, $(R_{19}^a)$, $(R_{19}^b)$ and $(R_{19}^c)$ are equivalent.

$(R_{19}^a) \Rightarrow (R_{19}^d)$: For all $y > x > 0$, since $(R_{19}^a)$ and $(R_{19}^c)$ are equivalent,

$$\left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{y}\right)^y < \left(1+\frac{1}{y}\right)^{y+1} < \left(1+\frac{1}{x}\right)^{x+1}, 0 < x < y < \infty.$$

In particular, for all $x > 0$, $\left(1+\frac{1}{x}\right)^x < (1+1)^2 = 4$ and

$$0 \le \left(1+\frac{1}{x}\right)^{x+1} - \left(1+\frac{1}{x}\right)^x = \left(1+\frac{1}{x}\right)^x \frac{1}{x} \le \frac{4}{x}$$

approaches to 0 as $x \to \infty$. Thus, by $\mathrm{e} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$, $(R_{19}^a)$ and $(R_{19}^c)$, we get $(R_{19}^d)$, see [2].

$(R_{19}^d) \Rightarrow (R_{20})$: By the first inequality of $(R_{19}^d)$,

$$1+\frac{1}{x} < \mathrm{e}^{\frac{1}{x}},\ x > 0.$$

Hence,

$$\mathrm{e}^x > x+1,\ x > 0.$$

By the second inequality of $(R_{19}^d)$,

$$\mathrm{e}^{\frac{1}{x+1}} < \frac{x+1}{x} = \frac{1}{1-\frac{1}{x+1}},\ x > 0.$$

Hence,

$$\mathrm{e}^{\frac{-1}{x+1}} > 1 - \frac{1}{x+1}, x > 0.$$

It follows from $\frac{-1}{1+x} \in (-1,0)$ for $x > 0$ that, for each $x \in (-1,0)$,

$$\mathrm{e}^x > 1 + x.$$

If $x = 1$ or $x \le -1$, then, clearly, $(R_{20})$ holds. This completes the proof of $(R_{20})$.

$(R_{20}) \Rightarrow (R_{19}^d)$: By $(R_{20})$,

$$\mathrm{e}^{\frac{1}{x}} > 1 + \frac{1}{x}, x > 0.$$

Hence, $\mathrm{e} > \left(1+\frac{1}{x}\right)^x$, $x > 0$, that is, the first inequality of $(R_{19}^d)$ holds. Next, by $(R_{20})$,

$$\mathrm{e}^{-\frac{1}{x}} > 1 - \frac{1}{x}, x > 0.$$

Hence, if $x > 1$, then

$$\mathrm{e} < \left(1-\frac{1}{x}\right)^{-x} = \left(\frac{x}{x-1}\right)^x = \left(1+\frac{1}{x-1}\right)^x = \left(1+\frac{1}{y}\right)^{y+1},$$

where $y := x-1 > 0$. Thus, the second inequality of $(R_{19}^d)$ holds.

$(R_{20}) \Rightarrow (R_{20}^a)$ is clear.

$(R_{20}) \Rightarrow (R_{18})$: Taking $x = \frac{a_i}{A_n}$ in $(a_2)$ of $(R_{20})$,

$$\ln\frac{a_i}{A_n} \le \frac{a_i}{A_n} - 1,$$

where $i = 1,2,\cdots,n$. Summing this $n$ inequalities, we get $(R_{18})$ holds.

$(R_{20}) \Rightarrow (R_{18})$:

$$1 = \exp\left\{\sum_{k=1}^n\left(\frac{a_k}{A_n}-1\right)\right\} = \prod\nolimits_{k=1}^n \exp\left(\frac{a_k}{A_n}-1\right)$$

$$\ge \prod\nolimits_{k=1}^n\left(\frac{a_k}{A_n}\right) = \left(\frac{G_n}{A_n}\right)^n,$$

see Bullen ([3], p. 117) or Kuang ([14], p. 33).

$(R_{20}^a) \Leftrightarrow (R_{20}^b)$ follows by taking $y = x^{\frac{1}{n}}$.

$(R_{20}^a) \Leftrightarrow (R_{20}^c)$ follows by taking $y = \frac{x}{x+1}$.

$(R_{20}^a) \Leftrightarrow (R_{20}^d)$: By $(R_{20}^a)$, $x \ge 1 + \ln x = \ln \mathrm{e}x$. Thus $\mathrm{e}^x > \mathrm{e}x$. Replacing $x$ by $\frac{x}{a}$, we completes the proof, see Cloud and Dranchman ([6], p. 32).

$(R_{20}^c) \Leftrightarrow (R_{20}^e)$ is clear.

$(R_{20}^d) \Rightarrow (R_{18})$: By $(R_{20}^d)$, $\mathrm{e}^{\frac{a_i}{G_n}} \ge \frac{ea_i}{G_n}, i = 1,2,\cdots,n.$

Hence, $e^{\frac{nA_n}{G_n}} = \prod_{i=1}^{n} e^{\frac{a_i}{G_n}} \geq e^n$. Thus, $(R_{18})$ holds, see [3] and [29].

$(R_{18})$, $(R_{19}^a)$, $(R_{20})$ and $(R_{20}^c)$ are equivalent, we can also refer to [12].

$(R_{20}^a) \Rightarrow (R_{18})$: Without loss of generality, we may assume that $\prod_{i=1}^{n} a_i = 1$. By $(R_{20}^a)$,

$$\sum_{i=1}^{n} \ln a_i - \sum_{i=1}^{n} a_i + n \leq 0.$$

Hence

$$\sum_{i=1}^{n} a_i \geq n + \ln \prod_{i=1}^{n} a_i = n.$$

Thus, $(R_{18})$ holds.

$(R_{20}^d) \Rightarrow (R_{18}^e)$: For any $x$, $a > 0$, it follows from $(R_{20}^d)$ that $e^x = \left(e^{x/a}\right)^a \geq \left(e\frac{x}{a}\right)^a$. Taking $a = e$, we get $(R_{18}^e)$.

$(R_{21}) \Rightarrow (R_2)$ see Hardy etc. ([8], pp. 40-41) or Wang, Su, Wang [33].

$(R_{21}) \Rightarrow (R_{21}^a)$: Taking $1 + x = y(\geq 0)$ in $(R_{21})$, $y^n \geq 1 + n(y-1)$.

Hence, $(R_{21}^a)$.

$(R_{21}^a) \Rightarrow (R_{18})$: Taking $y = \left(\frac{a_n}{A_n}\right)^{\frac{1}{n-1}}$ in $(R_{21}^a)$, $A_n^n \geq a_n A_{n-1}^{n-1}$.

Hence,

$$A_n^n \geq a_n A_{n-1}^{n-1} \geq a_n a_{n-1} A_{n-2}^{n-2} \geq \cdots \geq a_n a_{n-1} \cdots a_1 = G_n^n.$$

This completes the proof of $(R_{18})$, see Bullen ([3], p. 98) or Kuang ([14], p. 33).

$(R_{21}^b) \Rightarrow (R_{21}^a)$ follows by taking $c_k = 1$ for $k = 1, 2, \cdots, n$.

$(R_{18}^a) \Rightarrow (R_{17})$: By $(R_{18}^a)$, $\frac{1}{2}[\ln a + \ln b] \leq \ln \frac{a+b}{2}$. Hence, $-\ln(x)$ is midpoint convex on $(0,\infty)$. Since $\ln(x)$ is continuous on $(0,\infty)$, $-\ln(x)$ is a convex function on $(0,\infty)$. Thus, $(R_{17})$ holds.

We can also prove $(R_{18}^a) \Rightarrow (R_{18})$ by using the mathematical induction.

Thus, our proof is complete.

## 5. Other Equivalent Forms of Bernoulli's Inequality

In this section, we shall collect some variants of Young' inequality which is equivalent to the Bernoulli's inequality.

**Theorem 5.1** *Let* $a, b, a_i, r, s$ *be positive numbers for* $i = 1, 2, \cdots, n$. *If* $\frac{1}{p} + \frac{1}{q} = 1$, *where the real numbers p, q satisfy* $p \neq 0$, $p \neq 1$, *then the following some inequalities are equivalent*:

$(Y_0^a)$ $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$, where $p > 1$;

$(Y_0^b)$ $ab \geq \frac{1}{p}a^p + \frac{1}{q}b^q$, where $p < 0$;

$(Y_0^c)$ $ab \geq \frac{1}{p}a^p + \frac{1}{q}b^q$, where $0 < p < 1$;

$(Y_1^a)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$, where $p > 1$;

$(Y_1^b)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \geq \frac{a}{p} + \frac{b}{q}$, where $p < 0$;

$(Y_1^c)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \geq \frac{a}{p} + \frac{b}{q}$, where $0 < p < 1$;

$(Y_2^a)$ $ab \leq \frac{(ta)^p}{p} + \frac{1}{q}\left(\frac{b}{t}\right)^q$, where $t > 0$ and $p > 1$;

$(Y_2^b)$ $ab \geq \frac{(ta)^p}{p} + \frac{1}{q}\left(\frac{b}{t}\right)^q$, where $t > 0$ and $p < 0$;

$(Y_2^c)$ $ab \geq \frac{(ta)^p}{p} + \frac{1}{q}\left(\frac{b}{t}\right)^q$, where $t > 0$ and $p \in (0,1)$;

$\left(Y_3^a\right)$ $ab \le \frac{ta^p}{p} + t^{-\frac{q}{p}} \frac{b^q}{q}$, where $t > 0$ and $p > 1$;

$\left(Y_3^b\right)$ $ab \ge \frac{ta^p}{p} + t^{-\frac{q}{p}} \frac{b^q}{q}$, where $t > 0$ and $p < 0$;

$\left(Y_3^c\right)$ $ab \ge \frac{ta^p}{p} + t^{-\frac{q}{p}} \frac{b^q}{q}$, where $t > 0$ and $p \in (0,1)$;

$\left(Y_4^a\right)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{1}{p} at^{-\frac{1}{q}} + \frac{1}{q} bt^{\frac{1}{p}}$, where $t > 0$ and $p > 1$;

$\left(Y_4^b\right)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \ge \frac{1}{p} at^{-\frac{1}{q}} + \frac{1}{q} bt^{\frac{1}{p}}$, where $t > 0$ and $p < 0$;

$\left(Y_4^c\right)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \ge \frac{1}{p} at^{-\frac{1}{q}} + \frac{1}{q} bt^{\frac{1}{p}}$, where $t > 0$ and $p \in (0,1)$;

$\left(Y_5^a\right)$ $ab \le ta^{\frac{1}{\alpha}} + \left(\frac{\alpha}{t}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) b^{\frac{1}{1-\alpha}}$, where $t > 0$ and $0 < \alpha < 1$;

$\left(Y_5^b\right)$ $ab \ge ta^{\frac{1}{\alpha}} + \left(\frac{\alpha}{t}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) b^{\frac{1}{1-\alpha}}$, where $t < 0$ and $\alpha < 0$;

$\left(Y_5^c\right)$ $ab \ge ta^{\frac{1}{\alpha}} + \left(\frac{\alpha}{t}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) b^{\frac{1}{1-\alpha}}$, where $t > 0$ and $\alpha > 1$;

$\left(Y_6^a\right)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \le \left(\frac{1}{pr}\right)^{\frac{1}{p}} \left(\frac{1}{qs}\right)^{\frac{1}{q}} (ar + bs)$, where $p > 1$;

$\left(Y_6^b\right)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \ge \left(\frac{-1}{pr}\right)^{\frac{1}{p}} \left(\frac{1}{qs}\right)^{\frac{1}{q}} (-ar + bs)$, where $p < 0$;

$\left(Y_6^c\right)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \ge \left(\frac{1}{pr}\right)^{\frac{1}{p}} \left(\frac{-1}{qs}\right)^{\frac{1}{q}} (ar - bs)$, where $p \in (0,1)$;

$\left(Y_7^a\right)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{at^p}{p} + \frac{bt^{-q}}{q}$, where $p > 1$;

$\left(Y_7^b\right)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \ge \frac{at^p}{p} + \frac{bt^{-q}}{q}$, where $p < 0$;

$\left(Y_7^c\right)$ $a^{\frac{1}{p}} b^{\frac{1}{q}} \ge \frac{at^p}{p} + \frac{bt^{-q}}{q}$, where $p \in (0,1)$;

$\left(Y_8^a\right)$ $xy \le \frac{x\left(x^r - 1\right)}{r} + \left(\frac{1+ry}{1+r}\right)^{\frac{1+r}{r}}$, where $r > 0$, $x > 0$ and $y > -\frac{1}{r}$;

$\left(Y_9^a\right)$ $xy \le x \ln x + \mathrm{e}^{y-1}$, where $x > 0$ and $y \in \mathbb{R}$;

$\left(Y_9^b\right)$ $y \le \mathrm{e}^{y-1}$, where $y \in R$;

$\left(Y_{10}^a\right)$ $xy \le ax^p + by^q$, where $p > 1$ and $(pa)^q (qb)^p \ge 1, x, y > 0$;

$\left(Y_{10}^b\right)$ $xy \ge -ax^p + by^q$, where $p < 0$ and $(pa)^q (qb)^p \ge 1, x, y > 0$;

$\left(Y_{10}^c\right)$ $xy \ge ax^p - by^q$, where $p \in (0,1)$ and $(pa)^q (qb)^p \ge 1, x, y > 0$;

$\left(Y_{11}^{a}\right)$ $a^h b^k \le \dfrac{ha^{h+k}+kb^{h+k}}{h+k}$, where $h>0$ and $k>0$;

$\left(Y_{11}^{b}\right)$ $a^h b^k \ge \dfrac{ha^{h+k}+kb^{h+k}}{h+k}$, where $\dfrac{h+k}{h}<0$;

$\left(Y_{11}^{c}\right)$ $a^h b^k \ge \dfrac{ha^{h+k}+kb^{h+k}}{h+k}$, where $0<\dfrac{h+k}{h}<1$;

$\left(Y_{12}^{a}\right)$ $a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n} \le p_1a_1+p_2a_2+\cdots+p_na_n$, where $p_i>0, i=1,2,\cdots,n,\ p_1+p_2+\cdots+p_n=1$;

$\left(Y_{12}^{b}\right)$ $a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n} \ge p_1a_1+p_2a_2+\cdots+p_na_n$, where $p_1+p_2+\cdots+p_n=1$ and there exists exactly one of $p_1,p_2,\cdots,p_n$ is positive, the other are negative;

$\left(Y_{13}^{a}\right)$ $\dfrac{1}{r}\left(a_1a_2\cdots a_m\right)^r \le \dfrac{1}{r_1}a_1^{r_1}+\dfrac{1}{r_2}a_2^{r_2}+\cdots+\dfrac{1}{r_m}a_m^{r_m}$, where $r_i>0$ and $a_i>0, i=1,2,\cdots,m,$ satisfying $\sum_{i=1}^m \dfrac{1}{r_i}=\dfrac{1}{r}>0$ and $m\ge 2$;

$\left(Y_{13}^{b}\right)$ $\dfrac{1}{r}\left(a_1a_2\cdots a_m\right)^r \ge \dfrac{1}{r_1}a_1^{r_1}+\dfrac{1}{r_2}a_2^{r_2}+\cdots+\dfrac{1}{r_m}a_m^{r_m}$, where $r_i<0, i=2,\cdots,m, r_m>0$ satisfying $\sum_{i=1}^m \dfrac{1}{r_i}=\dfrac{1}{r}, r>0$ and $m\ge 2$;

**Proof** Clearly, $\left(Y_0^a\right),\left(Y_0^b\right),\left(Y_0^c\right)$ are variant of $\left(R_7\right),\left(R_9\right),\left(R_8\right)$, respectively. Hence $\left(Y_0^a\right),\left(Y_0^b\right)$ and $\left(Y_0^c\right)$ are equivalent.

$\left(Y_0^i\right)\Leftrightarrow\left(Y_1^i\right)$, $i=a,b,c$ is clear.

$\left(Y_0^i\right)\Leftrightarrow\left(Y_2^i\right)$: Replacing $a,b$ by $ta,\dfrac{b}{t}$ in $\left(Y_0^i\right)$, respectively, we get $\left(Y_2^i\right)$, where $i=a,b,c$.

$\left(Y_0^a\right)\Rightarrow\left(Y_8^a\right)$: Let $a=x$, $b=\dfrac{1+ry}{1+r}$, $p=1+r$, where $r<0$. Then, by $\left(Y_0^a\right)$,

$$x\frac{1+ry}{1+r}\le\frac{1}{1+r}x^{1+r}+\frac{r}{1+r}\left(\frac{1+ry}{1+r}\right)^{\frac{1+r}{r}}.$$

Hence

$$r\frac{xy}{1+r}\le\frac{1}{1+r}x\left(x^r-1\right)+\frac{r}{1+r}\left(\frac{1+ry}{1+r}\right)^{\frac{1+r}{r}}.$$

Thus,

$$xy\le x\frac{x^r-1}{r}+\left(\frac{1+ry}{1+r}\right)^{\frac{1+r}{r}}.$$

This completes our proof.

$\left(Y_0^a\right)\Rightarrow\left(Y_{10}^a\right)$: For all $x,y,a,b>0$, $1<p,q<\infty$ satisfying $\dfrac{1}{p}+\dfrac{1}{q}=1$ and $\left(pa\right)^q\left(qb\right)^p\ge 1$, by $\left(Y_0^i\right)$,

$$ax^p+by^q=\frac{apx^p}{p}+\frac{bqy^q}{q}\ge\left[\left(ap\right)^{\frac{1}{p}}x\right]\cdot\left[\left(bq\right)^{\frac{1}{q}}y\right]=\left[\left(ap\right)^q\left(bq\right)^p\right]^{\frac{1}{pq}}xy.$$

This prove the proof of $\left(Y_{10}^a\right)$.

$\left(Y_{10}^a\right)\Rightarrow\left(Y_0^a\right)$ follows by taking $a=\dfrac{1}{p}$, $b=\dfrac{1}{q}$ in $\left(Y_{10}^a\right)$.

$\left(Y_0^a\right)\Leftrightarrow\left(Y_{10}^a\right)$ and $\left(Y_0^c\right)\Leftrightarrow\left(Y_{11}^b\right)$, see also Sun [31].

$\left(Y_0^a\right)\Rightarrow\left(Y_{11}^b\right)$: Let $p>1$, $a,b>0$ satisfy $\left(pa\right)^q\left(qb\right)^p\ge 1$. Then, for $x,y>0$,

$$ax^p+by^q=\frac{1}{p}\left(pax^p\right)+\frac{1}{q}\left(bqy^q\right)\ge\left(pax^p\right)^{1/p}\left(bqy^q\right)^{1/q}=\left(pa\right)^{1/p}\left(bq\right)^{1/q}xy\ge xy.$$

Thus, $\left(Y_0^a\right)$ holds.

$\left(Y_0^a\right)\Rightarrow\left(Y_{11}^a\right)$ follows by replacing $a,b,p,q$ by $a^h,b^k,\dfrac{h+k}{h},\dfrac{h+k}{k}$ in $\left(Y_0^a\right)$, respectively. Conversely, $\left(Y_{11}^a\right)\Rightarrow\left(Y_0^a\right)$ is clear.

Similarly, we can prove $\left(Y_0^b\right)\Rightarrow\left(Y_{11}^b\right)$, $\left(Y_0^c\right)\Rightarrow\left(Y_{11}^c\right)$.

$\left(Y_0^b\right)\Leftrightarrow\left(Y_{10}^b\right)$ and $\left(Y_0^c\right)\Leftrightarrow\left(Y_{10}^c\right)$ can be proved similarly.

$\left(Y_0^a\right)\Rightarrow\left(Y_{12}^a\right),\left(Y_0^b\right)\Rightarrow\left(Y_{12}^b\right)$ follows by using the mathematical induction.

If $n=2$, then $\left(Y_{12}^a\right)\Rightarrow\left(Y_0^a\right)$ and $\left(Y_{12}^b\right)\Rightarrow\left(Y_0^b\right)$.

$\left(Y_1^i\right)\Rightarrow\left(Y_2^i\right)$, $i=a,b,c$: Replacing *a*, *b* by $(ta)^p,\left(\dfrac{b}{t}\right)^q$ in $\left(Y_1^i\right)$, respectively, we get $\left(Y_2^i\right)$.

Similarly, $\left(Y_2^i\right)\Rightarrow\left(Y_1^i\right)$, where $i=a,b,c$ follows by replacing *a*, *b* by $\dfrac{a^p}{t}$, $tb^{\frac{1}{q}}$ in $\left(Y_2^i\right)$, respectively.

$\left(Y_1^i\right)\Rightarrow\left(Y_4^i\right)$: Replacing *a*, *b* by $at^{-\frac{1}{q}},bt^{\frac{1}{p}}$ in $\left(Y_1^i\right)$, respectively, $\left(Y_4^i\right)$, where $i=a,b,c$. Similarly, we can prove $\left(Y_4^i\right)\Rightarrow\left(Y_1^i\right)$, where $i=a,b,c$.

$\left(Y_1^a\right)\Rightarrow\left(Y_6^a\right)$: Replacing *a*, *b* by $apr,bqs$ in $\left(Y_1^a\right)$, respectively, we get $\left(Y_6^a\right)$.

$\left(Y_1^b\right)\Rightarrow\left(Y_6^b\right)$: Replacing *a*, *b* by $-apr,bqs$ in $\left(Y_1^a\right)$, respectively, we get $\left(Y_6^b\right)$.

$\left(Y_1^i\right)\Rightarrow\left(Y_7^i\right)$: Replacing *a*, *b* by $at^p,bt^{-q}$ in $\left(Y_1^i\right)$, respectively, we get $\left(Y_7^i\right)$, Similarly, we can prove $\left(Y_7^i\right)\Rightarrow$ $\left(Y_1^i\right)$, where $i=a,b,c$.

$\left(Y_1^i\right)\Leftrightarrow\left(Y_{11}^i\right)$: Replacing $\dfrac{h}{h+k},\dfrac{k}{h+k},a,b$ by $p,q,a^{\frac{1}{h+k}},b^{h+k}$ in $\left(Y_{11}^i\right)$, respectively, we get $\left(Y_1^i\right)$, where $i=a,b,c$.

$\left(Y_1^a\right)\Rightarrow\left(Y_{12}^b\right)$: It follows from Theorem 3.1 that $\left(Y_1^a\right)$, $\left(Y_0^a\right)$ and $\left(R_{3,n}\right)$ are equivalent. Without loss of generality, we may assume that $p_1>0$ in $\left(Y_{12}^b\right)$. Replacing $c_i,1+y_i$ by $p_i,a_i$ in $\left(R_{3,n}\right)$, respectively, where $i=2,3,\cdots,n$ $\left(c_i\le 0\right)$. It follows from $p_1=1-\sum_{i=2}^n p_i>0$ and $\left(R_{3,n}\right)$ that

$$a^{p_1}\prod_{i=2}^n a_i^{p_i}=a_1\prod_{i=2}^n\left(\frac{a_i}{a_1}\right)^{p_i}\ge a_1\left[1+\sum_{i=2}^n p_i\left(\frac{a_i}{a_1}-1\right)\right]=a_1\left[1-\sum_{i=2}^n p_i+\sum_{i=2}^n p_i\left(\frac{a_i}{a_1}\right)\right]=a_1\left[p_1+\sum_{i=2}^n p_i\left(\frac{a_i}{a_1}\right)\right]=\sum_{i=1}^n p_i a_i.$$

$\left(Y_{12}^b\right)$ holds.

$\left(Y_2^i\right)\Leftrightarrow\left(Y_3^i\right)$: Replacing $t$ by $t^{\frac{1}{p}}$ in $\left(Y_2^i\right)$, we get $\left(Y_3^i\right)$, where $i=a,b,c$.

$\left(Y_3^i\right)\Leftrightarrow\left(Y_5^i\right)$: Replacing $t,\dfrac{1}{p},\dfrac{1}{q}$ by $tp,\alpha,1-\alpha$ in $\left(Y_3^i\right)$, respectively, we get $\left(Y_5^i\right)$, where $i=a,b,c$.

$\left(Y_2^i\right)\Rightarrow\left(Y_3^i\right)$: Replacing $t$ by $t^{\frac{1}{p}}$ in $\left(Y_2^i\right)$, we get $\left(Y_3^i\right)$, where $i=a,b,c$. Similarly, we can prove $\left(Y_3^i\right)\Rightarrow\left(Y_2^i\right)$, where $i=a,b,c$.

$\left(Y_5^i\right)\Rightarrow\left(Y_3^i\right)$ follows by replacing $t,\alpha,1-\alpha$ by $\dfrac{t}{p},\dfrac{1}{p},\dfrac{1}{q}$ in $\left(Y_5^i\right)$, respectively.

$\left(Y_6^i\right)\Rightarrow\left(Y_1^i\right)$ follows by taking by $pr=qs=1$ in $\left(Y_6^i\right)$, where $i=a,b,c$.

 

$\left(Y_6^b\right) \Leftrightarrow \left(Y_6^c\right)$ follows by replacing $a,b,p,q,r,s$ by $b,a,q,p,s,r$, respectively.

$\left(Y_8^a\right) \Rightarrow \left(Y_9^a\right)$: Since $\lim_{r\to 0}\frac{x^r-1}{r}=\ln x$, $\lim_{r\to 0}\left(1+\frac{y-1}{\frac{1+r}{r}}\right)^{\frac{1+r}{r}}=\mathrm{e}^{y-1}$, therefore, by $\left(Y_8^a\right)$, we get $\left(Y_9^a\right)$.

$\left(Y_9^a\right) \Rightarrow \left(Y_9^b\right)$: Taking $x=1$ in $\left(Y_9^a\right)$, we get $\left(Y_9^b\right)$.

$\left(Y_9^b\right) \Rightarrow \left(Y_9^a\right)$: Letting $u=r-s, r,s\in R$ in $\left(Y_9^b\right)$, $r\mathrm{e}^s \le \mathrm{e}^{r-1}+s\mathrm{e}^s$. And then, by taking $r=y, s=\ln x, x>0$, we get $xy\le \mathrm{e}^{y-1}+x\ln x$, This completes the proof of $\left(Y_9^a\right)$.

$\left(Y_9^b\right)$ is a variant of $\left(R_{20}\right)$.

$\left(Y_{10}^a\right) \Rightarrow \left(Y_{13}^a\right), \left(Y_{10}^b\right) \Rightarrow \left(Y_{13}^b\right), \left(Y_{13}^a\right) \Rightarrow \left(Y_0^a\right)$ and $\left(Y_{13}^b\right) \Rightarrow \left(Y_0^a\right)$, see Sun [31].

$\left(Y_{11}^i\right) \Rightarrow \left(Y_0^i\right)$ follows by taking $h=\frac{1}{p}, k=\frac{1}{k}$, where $i=a,b,c$.

$\left(Y_{12}^b\right) \Rightarrow \left(Y_1^a\right)$: By Theorem 3.1, $\left(Y_1^a\right)$ and $\left(R_{3,n-1}\right)$ are equivalent. It suffices to show $\left(Y_{12}^b\right) \Rightarrow \left(R_{3,n-1}\right)$. We assume $p_1>0$ and $x_1,x_2,\cdots,x_{n-1}\in(-1,0)$. Replacing $a_1,a_2,\cdots,a_n$ by $1,1+x_1,1+x_2,\cdots,1+x_{n-1}$ in $\left(Y_{12}^b\right)$, respectively, and then, replacing $p_2,p_3,\cdots,p_n$ by $c_1,c_2,\cdots,c_{n-1}$ and taking $p_1=1-\sum_{i=1}^{n-1}c_i$, it follows from $\left(Y_{12}^b\right)$ that $\left(R_{3,n-1}\right)$ holds.

$\left(Y_{12}^a\right) \Rightarrow \left(Y_{13}^a\right)$: It follows from $\left(Y_{12}^a\right)$ and $\sum_{i=1}^m \frac{r}{r_i}=1$ that $a_1^{\frac{r}{r_1}}a_2^{\frac{r}{r_2}}\cdots a_m^{\frac{r}{r_m}} \le \frac{r}{r_1}a_1+\frac{r}{r_2}a_2+\cdots+\frac{r}{r_m}a_m$.

Replacing $a_i$ by $a_i^{r_i}$, we complete our proof.

Similarly, we can prove $\left(Y_{12}^b\right) \Rightarrow \left(Y_{13}^b\right)$.

$\left(Y_{13}^a\right) \Rightarrow \left(Y_{12}^a\right)$ and $\left(Y_{13}^b\right) \Rightarrow \left(Y_{12}^b\right)$ are clear.

**Remark 5.2** For inequality $\left(Y_6^a\right)$, we refer to Isumino and Tominaga [13]. For inequality $\left(Y_8^a\right)$, we refer to [3]. For inequality $\left(Y_9^a\right)$ and $\left(Y_9^b\right)$, we refer to Sun [31]. For inequality $\left(Y_{10}^a\right), \left(Y_{10}^b\right)$ and $\left(Y_{10}^c\right)$, we refer to Kuang [14]. For inequality $\left(Y_{11}^i\right)$, $i=a,b,c$, we refer to [34]. For inequality $\left(Y_{13}^i\right)$, we refer to Sun [31].

**Remark 5.3** There are many variants of Hölder's inequality, Schlömich's inequality, AGM inequality, Minkowski's inequality, and so on, we omit the detail.

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