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Mild Solutions of Fractional Semilinear Integro-Differential Equations on an Unbounded Interval

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ABSTRACT

In this paper, we study the existence of mild solutions for fractional semilinear integro-differential equations in an arbitrary Banach space associated with operators generating compact semigroup on the Banach space. The arguments are based on the Schauder fixed point theorem.

Keywords: Semilinear Integrodifferential Equations; Mild Solutions; Schauder Fixed Point Theorem

1. Introduction

The purpose of the present paper is to present an alternative approach to the existence of solution of fractional semilinear integro-differential equations in an arbitrary Banach space X of the form

$$\begin{cases} x^{(\alpha)}(t) = A(t)x(t) + f\left(t, x(t), \int_0^t u(t,s,x(s))\,\mathrm{d}s\right), \\ x(0) = x_0 \in X, \end{cases} \tag{1}$$

where $t > 0, 0 < \alpha < 1$ and $A(t): D_t \subset X \to X$ generates an evolution system $U(t,s)$, satisfying:

- $U(t,s) \in B(X)$, where $B(X)$ denotes the Banach space of bounded linear operators from X into X,
- $U(t,t) = I$ (I is the identity operator in X),
- $U(t,s)U(s,z) = U(t,z)$ for $0 \le z \le s \le t < \infty$,
- the mapping $(t,s) \to U(t,s)x$ is strongly continuous in $\{(t,s): 0 \le s \le t < \infty\}$, and $f: \mathbb{R}_+ \times X \to X$ is a given function.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. This equations also serve as an tool for the description of hereditary properties of various materials and processes. For details, see [1-5]. The most important problem examined up to now is that concerning the existence of solutions of considered equations. In order to solve (1), many different methods have been applied in the literature. Most of these methods use the notion of a measure of noncompactness in Banach spaces, see [6-10]. Such a method can be to apply in this work. The method we are going to use is to reduce the existence of mild solutions of fractional semilinear integro-differential equations of type (0.1) to searching a fixed points of a suitable map on the space $C(\mathbb{R}_+, X, p(t))$ tempered by an arbitrary positive real continuous function $p(t)$ defined on $\mathbb{R}_+$. In order to prove the existence of fixed points, we shall rely on the Schauder theorem. Moreover, an application to fractional differential equations is provided to illustrate the results of this work.

2. Preliminary Tools

In what follows, X will represent a Banach space with norm $\|.\|$. Denote by $C(\mathbb{R}_+, X)$ the space of continuous functions $x: \mathbb{R}_+ \to X$. Now, let us assume thet $\rho = \rho(t)$ is a given function defined and continuous on the interval $\mathbb{R}_+$ with real positive values. Denote by $C(\mathbb{R}_+, \rho(t), X) = C_\rho(\mathbb{R}_+, X)$ the Banach space consisting of all functions $x = x(t)$ defined and continuous on $\mathbb{R}_+$ with values in the Banach space X such that

$$\sup\left\{\rho(t)\|x(t)\| : t \ge 0\right\} < \infty.$$

The space $C_\rho(\mathbb{R}_+, X)$ is furnished with the following standard norm

$$\|x\|_\rho = \sup\left\{\rho(t)\|x(t)\| : t \ge 0\right\}.$$

Let us recall two facts:

- The convergence in $C_\rho(\mathbb{R}_+, X)$ is the uniform convergence in the compact intervals, *i.e.* x_j converge to x in $C_\rho(\mathbb{R}_+, X)$ if and only if (x_j) is uniformly convergent to x on compact subsets of $\mathbb{R}_+$.

- A subset $M \subset C_\rho(\mathbb{R}_+, X)$ is relatively compact if and only if the restrictions to $[0,T]$ of all functions from X form an equicontinuous set for each $T > 0$ and $M(t)$ is relatively compact in X for each $t \in \mathbb{R}_+$,where $M(t) = \{x(t) : x \in X\}$, See [11].

Definition 1 *A nonempty subset* $\mathcal{M} \subset C_\rho(\mathbb{R}_+, X)$ *is said to be bounded if the there is a function* $r : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\|x(t)\| \le r(t)$ *for each* $t \ge 0$ *and* $x \in \mathcal{M}$.

Namely, denote by $L^1_{loc}(\mathbb{R}_+)$ the space of real functions defined and Lebesgue integrable on $\mathbb{R}_+$ and equiped with the standard norm. For $x \in L^1_{loc}(\mathbb{R}_+)$ and for a fixed number $\alpha > 0$ we define the Riemann-Liouville fractional integral of order α of the function $x(t)$ by putting

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}},\ t \ge 0.$$

It may be shown that the fractional integral operator I^α transforms the space $L^1(\mathbb{R}_+)$ into itself and has some other properties (see [6-8], for example). More generally, we can consider the operator I^α on the function space $L^1_{loc}(\mathbb{R}_+)$ consisting of real functions being locally integrable over $\mathbb{R}_+$.

The following result is well known, one can see Michalski [12]

Lemma 1 *For all* $\alpha > 0$ *and* $\beta > -1$.

$$\int_0^t (t-s)^{1-\alpha} s^\beta \mathrm{d}s \le \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \tag{2}$$

Our consideration is based on following Schauder fixed point theorem.

Theorem 1 [13] *Let* C *be a closed convex subset of the Banach space* X. *Suppose* $F : C \to C$ *and* F *is compact* (*i.e., bounded sets in* C *are mapped into relatively compact sets*). *Then,* F *has a fixed point in* C.

3. Existence of Mild Solutions

The following hypotheses well be needed in the sequel.

- (A) $A(t)$ is a bounded linear operator on E for each $t \ge 0$ and generates a uniformly continuous evolution system $U(t,s)$ such that

$$\sup\{\|U(t,s)\| : 0 \le s \le t\} < \infty.$$

- (C_f) (i) $(t,x,y) \in \mathbb{R}_+ \times X \times X \mapsto f(t,x,y) \in X$ satisfies the Caratheodory type conditions, *i.e.* $f(.,x,y)$ is measurable for $(x,y) \in X \times X$ and $f(t,.,.)$ continuous for a.e. $t \ge 0$, (ii) there exists a continuous positive function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,x,y)\| \le p(t)(\|x\| + \|y\|)$$

for a.e. $t \ge 0$ and all $x, y \in X$.

- (C_u) (i) $u(t,s,x) : \mathbb{R}_+ \times \mathbb{R}_+ \times X \to X$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times X$,
- (ii) $k : (t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ being continuous such that

$$\|u(t,s,x)\| \le k(t,s)\phi(\|x\|),$$

where $\phi : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing function with

$$\int_1^\infty \frac{\mathrm{d}s}{s + \phi(s)} = \infty. \tag{3}$$

- (iii) For all positive function ψ there exist $l \in L^1_{loc}([\mathbb{R}_+])$ such that

$$\int_0^t \frac{p(s)\hat{k}(s)}{(t-s)^{1-\alpha}} \psi(s) \mathrm{d}s \le \int_0^t l(s)\psi(s) \mathrm{d}s.$$

where $\hat{k}(s) = \max\left\{1, \int_0^t k(t,s)\mathrm{d}s\right\}$.

Definition 2 [14] *A continuous function* $x : \mathbb{R}_+ \to X$ *is said to be a mild solution of* (0.1) *if* x *satisfies to*

$$x(t) = U(t,0)x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{U(t,s)}{(t-s)^{1-\alpha}} f\left(s, x(s), \int_0^s u(s,\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s.$$

Our main result is given by the following theorem.

Theorem 2 *If the Banach space* X *is separable. Assume that the hypotheses* $((A),(C_f))$ *and* (C_u) *are satisfied. Then for each* $x_0 \in X$, *the problem* (0.1) *has at least one mild solution* x *in* $C_\rho(\mathbb{R}_+, X)$, *for* $t \ge 0$.

Proof. Consider the operator F defined by the formula

$$(Fx)(t) = U(t,0)x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{U(t,s)}{(t-s)^{1-\alpha}} f\left(s, x(s), \int_0^s u(s,\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s, \tag{4}$$

for $t \ge 0$ and $x \in C(\mathbb{R}_+, X)$. Let

$$r(t) = G^{-1}\left(\frac{M}{\Gamma(\alpha)} \int_0^t l(s)\mathrm{d}s\right),$$

where

$$G(t) = \int_{M\|x_0\|}^t \frac{\mathrm{d}s}{s + \phi(s)},$$

and

$$M = \sup_{s \le t} \{U(t,s)\}.$$

The estimate (0.3) guarantee the convergence of the integral $G(t)$. In the other hand, observe that if $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing function, then the function $\int_0^t \frac{\beta(s)}{(t-s)^{1-\alpha}}\mathrm{d}s$ is also nondecreasing on $\mathbb{R}_+$.Therefore, the function $r(t)$ is will defined and nondecreasing on $\mathbb{R}_+$. Next, put

$$\rho(t)=\frac{1}{(t^{\alpha+1}+1)(r(t)+1)}. \tag{5}$$

Obviously, the function $\rho(t)$ is continuous, positive and decreasing. In the space $C_\rho(\mathbb{R}_+,X)$ let us consider the set

$$\Omega_\rho := \{x\in C_\rho(\mathbb{R}_+,X): \|x\|\le r(t), t\ge 0\}. \tag{6}$$

Clearly Ω_ρ is closed convex of $C_\rho(\mathbb{R}_+,X)$. Next, let $x\in\Omega_\rho$. Applying assumptions C_f (1) and C_u (2) we have

$$\begin{aligned}
\|(Fx)(t)\| &\le \|U(t,0)x_0\| + \frac{1}{\Gamma(\alpha)}\left\|\int_0^t \frac{U(t,s)}{(t-s)^{1-\alpha}} f\left(s,x(s),\int_0^s u(s,\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right\| \\
&\le \|U(t,0)x_0\| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\|U(t,s)\|}{(t-s)^{1-\alpha}}\left\| f\left(s,x(s),\int_0^s u(s,\tau,x(\tau))\mathrm{d}\tau\right)\right\|\mathrm{d}s \\
&\le M\|x_0\| + \frac{M}{\Gamma(\alpha)}\int_0^t \frac{p(s)}{(t-s)^{1-\alpha}}\left[\|x(s)\| + \int_0^s k(s,\tau)\phi(\|x(\tau)\|)\mathrm{d}\tau\right]\mathrm{d}s \\
&\le M\|x_0\| + \frac{M}{\Gamma(\alpha)}\int_0^t \frac{p(s)}{(t-s)^{1-\alpha}}\left[r(s) + \left(\int_0^s k(s,\tau)\mathrm{d}\tau\right)\phi(r(s))\right]\mathrm{d}s \qquad (7)\\
&\le M\|x_0\| + \int_0^t \frac{M}{\Gamma(\alpha)}\frac{p(s)\hat{k}(s)}{(t-s)^{1-\alpha}}\left(r(s)+\phi(r(s))\right)\mathrm{d}s \\
&\le M\|x_0\| + \int_0^t \frac{M}{\Gamma(\alpha)} l(s)\left(r(s)+\phi(r(s))\right)\mathrm{d}s \\
&\le M\|x_0\| + \int_0^t r'(s)\mathrm{d}s = r(t),
\end{aligned}$$

From the estimate (7), we deduce that F transforms Ω_ρ into itself. In what follows we show that $F:\Omega_\rho\to\Omega_\rho$ is continuous. To do this, let us fix $x\in\Omega_\rho$ and take arbitrary sequence $(x_n)\in\Omega_\rho$ such that x_n converge to x in $C_\rho(\mathbb{R}_+,X)$. Further, let us fix $\delta>0$. Applying the properties of ρ and F we get

$$\rho(t)\|Fx_n(t)-Fx(t)\| \le \rho(t)\left(\|Fx_n(t)\|+\|Fx(t)\|\right) \le \frac{2r(t)}{(t^{\alpha+1}+1)(r(t)+1)} < \frac{2}{(t^{\alpha+1}+1)}.$$

Then, keeping in mind that $\lim_{t\to\infty}(t^{\alpha+1}+1)=\infty$, we obtain, that there exists T so big that

$$\sup\{\rho(t)\|Fx_n(t)-Fx(t)\| : t\ge T\} < \delta. \tag{8}$$

Next, for $t\le T$, denote S the operator defined by

$$Sx(t) = f\left(t,x(t),\int_0^t u(t,s,x(s))\mathrm{d}s\right).$$

For $t\le T$, we have,

$$\left\|\frac{\rho(t)U(t,s)}{(t-s)^{1-\alpha}}(Sx_n(t)-Sx(t))\right\| \le 2(r(T)+\phi(r(T)))\frac{Mp(s)\hat{k}(s)}{\Gamma(\alpha)(t-s)^{1-\alpha}} \in L^1_{loc}([0,T]).$$

Next, by the Lebesgue dominated convergence theorem and (0.8) we derive that for suitable large n we have $\|Fx_n-Fx\|_\rho \le \delta.$ this fact proves that F is continuous on Ω_ρ.

Next, from (**) we see that to prove the compactness of F, we should prove that $F\Omega_\rho\subset C_\rho(\mathbb{R}_+,X)$ is equicontinuous on $[0,T]$ and $(F\Omega_\rho)$ is relatively compact for each $T>0$ and $t\ge 0$. For any $x\in\Omega_\rho$ and $t_1,t_2\in[0,T]$ $t_2\ge t_1$ we get,

$$\begin{aligned}
&\left\|\rho(t_2)(Fx)(t_2)-\rho(t_1)(Fx)(t_1)\right\| \le \left\|\rho(t_2)U(t_2,s)-\rho(t_1)U(t_1,s)\right\|\left\|x_0\right\| \\
&+\frac{1}{\Gamma(\alpha)}\left\|\rho(t_2)\int_0^{t_2}\frac{U(t_2,s)Sx(s)}{(t_2-s)^{1-\alpha}}\mathrm{d}s-\rho(t_1)\int_0^{t_1}\frac{U(t_1,s)Sx(s)}{(t_1-s)^{1-\alpha}}\mathrm{d}s\right\| \\
&\le \left\|\rho(t_2)U(t_2,s)-\rho(t_1)U(t_1,s)\right\|\left\|x_0\right\|+\frac{1}{\Gamma(\alpha)}\left\|\rho(t_2)\int_0^{t_2}\frac{U(t_2,s)Sx(s)}{(t_2-s)^{1-\alpha}}\mathrm{d}s-\rho(t_2)\int_0^{t_2}\frac{U(t_1,s)Sx(s)}{(t_2-s)^{1-\alpha}}\mathrm{d}s\right\| \\
&+\frac{1}{\Gamma(\alpha)}\left\|\rho(t_2)\int_0^{t_2}\frac{U(t_1,s)Sx(s)}{(t_2-s)^{1-\alpha}}\mathrm{d}s-\rho(t_2)\int_0^{t_1}\frac{U(t_1,s)Sx(s)}{(t_2-s)^{1-\alpha}}\mathrm{d}s\right\| \\
&+\frac{1}{\Gamma(\alpha)}\left\|\rho(t_2)\int_0^{t_1}\frac{U(t_1,s)Sx(s)}{(t_2-s)^{1-\alpha}}\mathrm{d}s-\rho(t_2)\int_0^{t_1}\frac{U(t_1,s)Sx(s)}{(t_1-s)^{1-\alpha}}\mathrm{d}s\right\| \\
&+\frac{1}{\Gamma(\alpha)}\left\|\rho(t_2)\int_0^{t_1}\frac{U(t_1,s)Sx(s)}{(t_1-s)^{1-\alpha}}\mathrm{d}s-\rho(t_1)\int_0^{t_1}\frac{U(t_1,s)Sx(s)}{(t_1-s)^{1-\alpha}}\mathrm{d}s\right\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\|\rho(t_2)(Fx)(t_2)-\rho(t_1)(Fx)(t_1)\right\| \le \left\|\rho(t_2)U(t_2,s)-\rho(t_1)U(t_1,s)\right\|\left\|x_0\right\| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\frac{\rho(t_2)}{(t_2-s)^{1-\alpha}}\left\|U(t_2,s)-U(t_1,s)\right\|\left\|Sx(s)\right\|\mathrm{d}s+\frac{\rho(t_2)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{\left\|U(t_1,s)\right\|\left\|Sx(s)\right\|}{(t_2-s)^{1-\alpha}}\mathrm{d}s \\
&+\frac{\rho(t_2)}{\Gamma(\alpha)}\int_0^{t_1}\left\|U(t_1,s)\right\|\left\|Sx(s)\right\|\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\left|\rho(t_2)-\rho(t_1)\right|\int_0^{t_1}\frac{\left\|U(t_1,s)\right\|\left\|Sx(s)\right\|}{(t_2-s)^{1-\alpha}}\mathrm{d}s.
\end{aligned} \tag{9}$$

Observe that for any $\epsilon>0$ there exists $\delta>0$ such that $\left\|U(t_2,s)-U(t_1,s)\right\|\left\|Sx(s)\right\|<\epsilon$ for all $x\in\Omega_\rho$, $s\in[0,T]$ and t_2, $t_1\in[0,T]$ such that $\left|t_2-t_1\right|\le\delta$. Then, by the monotonicity of ρ and for all $t\ge0$ $\rho(t)\le1$, we get

$$\begin{aligned}
&\left\|\rho(t_2)(Fx)(t_2)-\rho(t_1)(Fx)(t_1)\right\| \le \left\|\rho(t_2)U(t_2,s)-\rho(t_1)U(t_1,s)\right\|\left\|x_0\right\| \\
&+\frac{\epsilon B\left(r(T)+\phi(r(T))\right)}{\Gamma(\alpha)}\int_0^{t_2}\frac{\mathrm{d}s}{(t_2-s)^{1-\alpha}}+\frac{BM\left(r(T)+\phi(r(T))\right)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{\mathrm{d}s}{(t_2-s)^{1-\alpha}} \\
&+\frac{BM\left(r(T)+\phi(r(T))\right)}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\mathrm{d}s \\
&+\frac{BM\left(r(T)+\phi(r(T))\right)}{\Gamma(\alpha)}\left|\rho(t_2)-\rho(t_1)\right|\int_0^{t_1}\frac{\mathrm{d}s}{(t_2-s)^{1-\alpha}},
\end{aligned} \tag{10}$$

where $B=\sup_{s\in[0,T]}p(s)\hat{k}(s)$. Keeping in mind the continuity of ρ, the right-hand side of the above inequality tends to zero as $t_2\to t_1$.

- If $t_1,t_2\ge T$, then we have

$$\begin{aligned}
&\left\|\rho(t_2)(Fx)(t_2)-\rho(t_1)(Fx)(t_1)\right\| \\
&\le\left\|\rho(t_2)(Fx)(t_2)\right\|+\left\|\rho(t_1)(Fx)(t_1)\right\| \\
&\le\frac{1}{t_2^{\alpha+1}}+\frac{1}{t_1^{\alpha+1}}\le\epsilon
\end{aligned} \tag{11}$$

- If $0<t_1<T<t_2$, note that $t_2\to t_1$ implies that $t_2\to T$ and $T\to t_1$. According to the above results, we have

$$\begin{aligned}
&\left\|\rho(t_2)(Fx)(t_2)-\rho(t_1)(Fx)(t_1)\right\| \\
&\le\left\|\rho(t_2)(Fx)(t_2)-\rho(T)(Fx)(T)\right\| \\
&+\left\|\rho(T)(Fx)(T)-\rho(t_1)(Fx)(t_1)\right\|
\end{aligned} \tag{12}$$

converging to 0 as $t_1\to t_2$.

So for $t_1,t_2\ge0$, $\left\{\rho(t)F\Omega_\rho\right\}$ is equicontinuous.

Meanwhile, $F(\Omega_\rho)$ is relatively compact because that $F(\Omega_\rho)\subset\Omega_\rho$ is uniformly bounded. Thus F is completely continuous on Ω_ρ. By Schauder fixed point theorem, we deduce that F has a fixed point x in Ω_ρ.

The last result in this article is to prove the existence of solutions to (0.1) but with the following conditions.

- $(t,x,y)\in\mathbb{R}_+\times X\times X\mapsto f(t,x,y)\in X$ satisfies the Caratheodory type conditions, *i.e.* $f(.,x,y)$ is measurable for $(x,y)\in X\times X$ and $f(t,.,.)$ continuous for a.e. $t\geq 0$,
- $\|f(t,x,y)\|\leq t^{-\frac{\alpha}{2}}$ for a.e. $t\geq 0$ and all $x,y\in X$.

Theorem 3 *If the Banach space X is separable. Assume that the hypotheses A and C'_f are satisfied. Then for each $x_0\in X$, the problem* (0.1) *has at least one mild solution x in $C(\mathbb{R}_+,E)$, for $t\geq 0$.*

Proof. Define the operator H by:

$$(Hx)(t)=U(t,0)x_0+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{U(t,s)}{(t-s)^{1-\alpha}}f\left(s,x(s),\int_0^s u(s,\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s, \tag{13}$$

for $t\geq 0$. Kipping in mind the result of lemma (0.2), we get, for $x\in C(\mathbb{R}_+,E)$.

$$\begin{aligned}\|(Hx)(t)\|&\leq\|U(t,0)x_0\|+\frac{1}{\Gamma(\alpha)}\left\|\int_0^t\frac{U(t,s)}{(t-s)^{1-\alpha}}f\left(s,x(s),\int_0^s u(s,\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right\|\\&\leq\|U(t,0)x_0\|+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\|U(t,s)\|}{(t-s)^{1-\alpha}}\left\|f\left(s,x(s),\int_0^s u(s,\tau,x(\tau))\mathrm{d}\tau\right)\right\|\mathrm{d}s\\&\leq M\|x_0\|+\frac{M}{\Gamma(\alpha)}\int_0^t\frac{s^{-\frac{\alpha}{2}}}{(t-s)^{1-\alpha}}\leq M\|x_0\|+\frac{M\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)}t^{\frac{\alpha}{2}}.\end{aligned} \tag{14}$$

Put $m(t)=M\|x_0\|+\frac{M\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)}t^{\frac{\alpha}{2}}$ and

$$\rho'(t)=\frac{1}{\left(t^{\alpha+1}+1\right)\left(m(t)+1\right)}$$

Next, define the set

$$B_{\rho'}:=\left\{x\in C_{\rho'}(\mathbb{R}_+,X):\|x\|\leq m(t),t\geq 0\right\}. \tag{15}$$

Then, we have that H is a self-mapping of $B_{\rho'}$. We omit the proof of continuity H and $H(B_{\rho'})$ is relatively compact, because are similar to that in Theorem 2.

4. Example

In this section, we illustrate the main result contained in Theorem 2 by the following quadratic fractional differential equation

$$\begin{cases}\dfrac{\partial^{\frac{1}{2}}x(t,\xi)}{\partial t^{\frac{1}{3}}}=\dfrac{\partial^2 x(t;\xi)}{\partial\xi^2}+\dfrac{\sqrt{|\sin(t)|}}{1+t^2}x(t,\xi)+\sqrt{|x(t;\xi)|}\displaystyle\int_0^t\frac{\sqrt{|\sin(t)|}\left(\mathrm{e}^{-t}+1\right)x(s,\xi)}{\left(1+t^2+s^2\right)\left(1+|x(s,\xi)|\right)}\mathrm{d}s,\\ x(0,\xi)=\theta,\end{cases} \tag{16}$$

for $t\geq 0$. Let $([0,1],\mathcal{A},P)$ be a complete probability measure space. Let $X=L^2([0,1],\mathcal{A},P)$ the space of $\mathcal{A}$-measurable maps $v(x;t)$ with

$$\|x(t)\|_2=\left(\int_0^1|x(t;s)|\mathrm{d}P(s)\right)^{\frac{1}{2}}.$$

Consider the operator

$$A:D(A)\subseteq X\to X$$

defined by

$$D(A)=\left\{v\in X:\frac{\partial v}{\partial x},\frac{\partial^2 v}{\partial x^2}\in X\text{ and }x(0)=x(1)=0\right\}.$$

Put $A(x)=\frac{\partial^2 x}{\partial t^2}$.

Clearly A is densely defined in X and is the infinitesimal generator of a strongly continuous semigroup

$\left(U(t,s)_{\{0<s\le t<\infty\}}\right)$ in X. Observe that the above equation is a special case of Equation (1) if we put $\alpha = \frac{1}{3}$ and

$$f\left(t, x(t,\xi), \int_0^t u\left(t,s,x(s,\xi)\right)\mathrm{d}s\right) = \frac{\sqrt{|\sin(t)|}}{1+t^2}x(t,\xi)$$
$$+\sqrt{|x(t;\xi)|}\int_0^t \frac{\sqrt{|\sin(t)|}\left(\mathrm{e}^{-t}+1\right)x(s,\xi)}{\left(1+t^2+s^2\right)\left(1+|x(t;\xi)|\right)}\mathrm{d}s$$

Bay using the Jensen's inequality it is not difficult to see that

$$\left\| f\left(t, x(t,\xi), \int_0^t u\left(t,s,x(s,\xi)\right)\mathrm{d}s\right)\right\|_2$$
$$\le \frac{\sqrt{|\sin(t)|}}{1+t^2}\left[\|x(t)\|_2 + \left(\mathrm{e}^{-t}\right)\sqrt{\|x(t)\|_2}\right].$$

To check conditions (H_f) and (H_u) it is enough to take

$$\begin{cases} p(t) = \sqrt{|\sin(t)|}\Big/\left(1+t^2\right), \\ k(t,s) = \mathrm{e}^{-t}+1, \quad \text{and} \quad \max\left(1, \mathrm{e}^{-t}+1\right) = \mathrm{e}^{-t}+1 \\ \phi(s) = \sqrt{s}, \quad s \ge 0. \end{cases}$$

Let be a positive function ψ defined on $\mathbb{R}_+$.

$$\int_0^t \frac{m(s)\left(1+q(s)\right)}{(t-s)^{1-\alpha}}\psi(s)\mathrm{d}s = \int_0^t \frac{2|\sin(t)|^{1-\alpha}}{t^2(t-s)^{1-\alpha}}\psi(s)\mathrm{d}s$$
$$= \int_0^t \frac{2|\sin(t-u)|^{1-\alpha}}{\left(1+(t-u)^2\right)u^{1-\alpha}}\psi(t-u)\mathrm{d}s$$
$$\le \int_0^t \frac{2|t-u|^{1-\alpha}}{\left(1+(t-u)^2\right)u^{1-\alpha}}\psi(t-u)\mathrm{d}s$$
$$\le \int_0^t \frac{2}{\left(1+(t-u)^2\right)}\psi(t-u)\mathrm{d}s$$
$$\le \int_0^t \frac{2}{\left(1+s^2\right)}\psi(s)\mathrm{d}s$$

Thus, on the basis of Theorem 2, we conclude that Equation (4.1) has at least one mild solution in the space $C_\rho\left(\mathbb{R}_+, L^2([0,1])\right)$.

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